

A NOTE ON JOINT REDUCTIONS AND MIXED MULTIPLICITIES

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ABSTRACT. Let (A, \mathfrak{m}) be a noetherian local ring with maximal ideal \mathfrak{m} and infinite residue field $k = A/\mathfrak{m}$. Let J be an \mathfrak{m} -primary ideal, I_1, \dots, I_s ideals of A , and M a finitely generated A -module. In this paper, we interpret mixed multiplicities of (I_1, \dots, I_s, J) with respect to M as multiplicities of joint reductions of them. This generalizes Rees's theorem on mixed multiplicity. As an application we show that mixed multiplicities are also multiplicities of Rees superficial sequences.

1. INTRODUCTION

Let (A, \mathfrak{m}) be a noetherian local ring with maximal ideal \mathfrak{m} and infinite residue field $k = A/\mathfrak{m}$. Let M be a finitely generated A -module, and I_1, \dots, I_s ideals of A such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\text{Ann}_A M}$. Set $\dim M/0_M : I^\infty = q$. Let J be an \mathfrak{m} -primary ideal. By [18, Proposition 3.1] (see also [9]),

$$\ell_A \left(\frac{J^{n_0} I_1^{n_1} \cdots I_s^{n_s} M}{J^{n_0+1} I_1^{n_1} \cdots I_s^{n_s} M} \right)$$

is a polynomial of total degree $q - 1$ for all large n_0, n_1, \dots, n_s . Write the terms of total degree $q - 1$ in this polynomial in the form

$$\sum_{k_0+k_1+\cdots+k_s=q-1} e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) \frac{n_0^{k_0} n_1^{k_1} \cdots n_s^{k_s}}{k_0! k_1! \cdots k_s!}.$$

Then $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M)$ are non-negative integers not all zero. We call (see [9] and [17])

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M)$$

the *mixed multiplicity of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$* .

Risler and Teissier in 1973 [15] defined mixed multiplicities of \mathfrak{m} -primary ideals and interpreted them as Hilbert-Samuel multiplicities of ideals generated by general elements. Katz and Verma in 1989 [6] started the investigation of mixed multiplicities of ideals of positive height. For the case of arbitrary ideals, the first author in 2000 [18] described mixed multiplicities as Hilbert-Samuel multiplicities via (FC)-sequences. Moreover, Trung and Verma in 2007 [16] interpreted mixed volumes of polytopes as mixed multiplicities of ideals. In past years, the positivity and the

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relationship between mixed multiplicities and Hilbert-Samuel multiplicity of ideals have attracted much attention (see e.g. [2–4, 7–9, 13, 14, 17, 19–23]).

We turn now to Rees’s work in 1984 [12]. The author of this work built joint reductions of \mathfrak{m} -primary ideals and showed that each mixed multiplicity of \mathfrak{m} -primary ideals is the multiplicity of a joint reduction of them. O’Carroll in 1987 [11] proved the existence of joint reductions in the general case.

Definition (See Definition 2.4). Let \mathfrak{R} be a subset of $\bigcup_{i=1}^s I_i$ consisting of k_1 elements of I_1, \dots, k_s elements of I_s with $k_1, \dots, k_s \geq 0$. Set $\mathfrak{J}_i = \mathfrak{R} \cap I_i$ for $i = 1, \dots, s$ and $(\emptyset) = 0_A$. Then \mathfrak{R} is called a *joint reduction* of (I_1, \dots, I_s) with respect to M of the type (k_1, \dots, k_s) if $I_1^{n_1+1} \dots I_s^{n_s+1} M = \sum_{i=1}^s (\mathfrak{J}_i) I_1^{n_1+1} \dots I_i^{n_i} \dots I_s^{n_s+1} M$ for all large integers n_1, n_2, \dots, n_s .

Although the relationship between mixed multiplicities and Hilbert-Samuel multiplicity of arbitrary ideals was solved, whether there is a similar result to Rees’s theorem [12, Theorem 2.4] for arbitrary ideals, i.e., whether there is a relationship between mixed multiplicities of arbitrary ideals and multiplicities of their joint reductions, is not yet known. This problem became an open question in the theory of mixed multiplicities. The aim of this paper is to extend Rees’s theorem to arbitrary ideals. As one might expect, we obtain the following result.

Main Theorem (See Theorem 3.1). *Let M be a finitely generated A -module of Krull dimension $d > 0$. Let J be an \mathfrak{m} -primary ideal, and I_1, \dots, I_s ideals of A . Set $I = I_1 \dots I_s$. Assume that $\text{ht}\left(\frac{I + \text{Ann}_A M}{\text{Ann}_A M}\right) = h > 0$ and k_0, k_1, \dots, k_s are non-negative integers such that $k_0 + k_1 + \dots + k_s = d - 1$ and $k_1 + \dots + k_s < h$. Let $\mathfrak{R} = \{x_1, \dots, x_d\}$ be a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ such that \mathfrak{R} is a system of parameters for M . Then*

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) = e_A(\mathfrak{R}, M).$$

It should be noted that this theorem does not hold in general if one omits the assumption $k_1 + \dots + k_s < h$ (see Remark 3.5). Our approach, which is based on the results in [18] and [24], links the multiplicity of (FC)-sequences in [18] and the multiplicity of joint reductions via our studies on these sequences (see Proposition 2.3 and Lemma 2.5). As an application of the Main Theorem, we interpret mixed multiplicities as Hilbert-Samuel multiplicities of Rees superficial sequences (see Remark 3.4) and recover Rees’s theorem in [12, Theorem 2.4] (see Corollary 3.6).

The paper is divided into three sections. Section 2 deals with the existence of Rees superficial sequences, weak-(FC)-sequences, and joint reductions. Apart from the proof of the Main Theorem, Section 3 also contains a brief treatment of the relationship between mixed multiplicities and Rees superficial sequences.

2. (FC)-SEQUENCES AND JOINT REDUCTIONS

In general, the relationship between mixed multiplicities and Hilbert-Samuel multiplicity of arbitrary ideals was solved in [18] by using (FC)-sequences. In this section, we give some results concerning Rees superficial sequences, (FC)-sequences, and joint reductions which will be used in this paper.

Definition 2.1 ([18]). Let (A, \mathfrak{m}) be a noetherian local ring with maximal ideal \mathfrak{m} and infinite residue field $k = A/\mathfrak{m}$. Let M be a finitely generated A -module, and I_1, \dots, I_s ideals of A such that $I_1 \dots I_s$ is not contained in $\sqrt{\text{Ann}_A M}$. Set

$I = I_1 \cdots I_s$. An element $x \in A$ is called an (FC)-element of (I_1, \dots, I_s) with respect to M if there exists $i \in \{1, \dots, s\}$ such that $x \in I_i$ and the following conditions are satisfied:

- (FC1): $xM \cap I_1^{n_1} \cdots I_i^{n_i+1} \cdots I_s^{n_s} M = xI_1^{n_1} \cdots I_i^{n_i} \cdots I_s^{n_s} M$ for all $n_i \gg 0$ and all $n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s \geq 0$.
- (FC2): x is an I -filter-regular element with respect to M , i.e., $0_M : x \subseteq 0_M : I^\infty$.
- (FC3): $\dim M/(xM : I^\infty) = \dim M/(0_M : I^\infty) - 1$.

We call x a weak-(FC)-element of (I_1, \dots, I_s) with respect to M if x satisfies the conditions (FC1) and (FC2). One can also call an element satisfying the condition (FC1) a Rees superficial element of (I_1, \dots, I_s) with respect to M .

Let x_1, \dots, x_t be elements of A . For any $0 \leq i \leq t$, set $M_i = \frac{M}{(x_1, \dots, x_i)M}$. Then x_1, \dots, x_t is called an (FC)-sequence (respectively, a weak-(FC)-sequence, a Rees superficial sequence) of (I_1, \dots, I_s) with respect to M if x_{i+1} is an (FC)-element (respectively, a weak-(FC)-element, a Rees superficial element) of (I_1, \dots, I_s) with respect to M_i for all $i = 0, \dots, t - 1$. If an (FC)-sequence (respectively, a weak-(FC)-sequence, a Rees superficial sequence) of (I_1, \dots, I_s) with respect to M consists of k_1 elements of I_1, \dots, k_s elements of I_s ($k_1, \dots, k_s \geq 0$) then it is called an (FC)-sequence (respectively, a weak-(FC)-sequence, a Rees superficial sequence) of (I_1, \dots, I_s) with respect to M of the type (k_1, \dots, k_s) . A weak-(FC)-sequence x_1, \dots, x_t is called a maximal weak-(FC)-sequence if $I \not\subseteq \sqrt{\text{Ann}_A M_{t-1}}$ and $I \subseteq \sqrt{\text{Ann}_A M_t}$.

The existence of Rees superficial elements and weak-(FC)-elements is established in Lemma 2.2 and Proposition 2.3, which are the improvements of [12, Lemma 1.2] and [18, Remark 1], respectively.

Lemma 2.2 (See [12, Lemma 1.2]). *Let (A, \mathfrak{m}) be a noetherian local ring with maximal ideal \mathfrak{m} and infinite residue field $k = A/\mathfrak{m}$, I_1, \dots, I_s ideals of A , and Σ a finite set of prime ideals of A . Let M be a finitely generated A -module. If for some $i \in \{1, \dots, s\}$, I_i is contained in no prime ideal of Σ , then there exists a non-empty Zariski open subset U of $I_i/\mathfrak{m}I_i$ such that for any element $x \in I_i$ with image $x + \mathfrak{m}I_i \in U$, x is not contained in any prime ideal of Σ and x is a Rees superficial element of (I_1, \dots, I_s) with respect to M , i.e.,*

$$xM \cap I_1^{n_1} \cdots I_i^{n_i} \cdots I_s^{n_s} M = xI_1^{n_1} \cdots I_i^{n_i-1} \cdots I_s^{n_s} M$$

for all $n_i \gg 0$ and $n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s \geq 0$.

Proof. Set $\mathcal{R} = \bigoplus_{n_1, \dots, n_s \in \mathbb{Z}} I_1^{n_1} \cdots I_s^{n_s} t_1^{n_1} \cdots t_s^{n_s}$ and

$$\mathcal{M} = \bigoplus_{n_1, \dots, n_s \in \mathbb{Z}} I_1^{n_1} \cdots I_s^{n_s} t_1^{n_1} \cdots t_s^{n_s} M,$$

where t_1, \dots, t_s are indeterminates and $I_j^{n_j} = A$ for $n_j \leq 0$. Then \mathcal{R} is a noetherian \mathbb{Z}^s -graded ring and \mathcal{M} is a noetherian graded \mathcal{R} -module. Set $u_1 = t_1^{-1}, \dots, u_s = t_s^{-1}$. Since $u = u_1 \cdots u_s$ is a non-zero-divisor on \mathcal{M} , the set of prime ideals associated with $u^T \mathcal{M}$ is the same for all T and so is finite (see [12, Lemma 2.7]). Let Q_1, \dots, Q_t be all the associated prime ideals of $\mathcal{M}/u^T \mathcal{M}$ that do not contain $I_i t_i$. For each $l = 1, \dots, t$, set

$$Q'_l = \{a \in A \mid at_i \in Q_l\}.$$

Then Q'_l is an ideal of A that does not contain I_i . Let W_l be the image of $Q'_l \cap I_i$ in $I_i/\mathfrak{m}I_i$. Assume that $\Sigma = \{P_1, \dots, P_r\}$. Also, for each $h = 1, \dots, r$, let V_h be the image of $P_h \cap I_i$ in $I_i/\mathfrak{m}I_i$. By Nakayama's lemma, W_l and V_h are proper k -vector subspaces of $I_i/\mathfrak{m}I_i$. Let U be the complement of the union of all these W_l and V_h . Since k is an infinite field, U is a non-empty Zariski open subset of $I_i/\mathfrak{m}I_i$.

Now, let $x \in I_i$ be such that its image $x + \mathfrak{m}I_i$ lies in U . Then x is not contained in any prime ideal in Σ and xt_i is not contained in Q_l for all $l = 1, \dots, t$. Using the same arguments as in the proof of [12, Lemma 1.2] we have

$$xM \cap I_1^{n_1} \dots I_i^{n_i} \dots I_s^{n_s}M = xI_1^{n_1} \dots I_i^{n_i-1} \dots I_s^{n_s}M$$

for all $n_i \gg 0$ and $n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s \geq 0$. Hence x is a Rees superficial element of (I_1, \dots, I_s) with respect to M . □

Proposition 2.3 (See [18, Remark 1]). *Let (A, \mathfrak{m}) be a noetherian local ring with maximal ideal \mathfrak{m} and infinite residue field $k = A/\mathfrak{m}$, and I_1, \dots, I_s ideals of A . Let M be a finitely generated A -module. Assume that $I = I_1 \dots I_s \not\subseteq \sqrt{\text{Ann}_A M}$. Then for any $1 \leq i \leq s$, there exists a non-empty Zariski open subset U of $I_i/\mathfrak{m}I_i$ such that if $x \in I_i$ with image $x + \mathfrak{m}I_i \in U$, then x is a weak-(FC)-element of (I_1, \dots, I_s) with respect to M .*

Proof. Set $\Sigma = \text{Ass}_A(M/(0_M : I^\infty))$. Since $I \not\subseteq \sqrt{\text{Ann}_A M}$, $\Sigma \neq \emptyset$. It is easily seen that Σ is a finite set and $\Sigma = \{P \in \text{Ass}_A M \mid P \not\supseteq I\}$. By Lemma 2.2, for any $1 \leq i \leq s$, there exists a non-empty Zariski open subset U of $I_i/\mathfrak{m}I_i$ such that if $x \in I_i$ with image $x + \mathfrak{m}I_i \in U$, then x is not contained in any prime ideal in Σ and x is a Rees superficial element of (I_1, \dots, I_s) with respect to M . Since $x \notin P$ for all $P \in \Sigma$, x is also an I -filter-regular element with respect to M . Hence, for any element $x \in I_i$ with image $x + \mathfrak{m}I_i \in U$, x is a weak-(FC)-element of (I_1, \dots, I_s) with respect to M . □

Recall that the concept of joint reductions of \mathfrak{m} -primary ideals was given by Rees in 1984 [12]. Here he proved that mixed multiplicities of \mathfrak{m} -primary ideals are multiplicities of ideals generated by joint reductions. This concept was extended to the set of arbitrary ideals by [9, 11, 21].

Definition 2.4. Let \mathfrak{R} be a subset of $\bigcup_{i=1}^s I_i$ consisting of k_1 elements of I_1, \dots, k_s elements of I_s with $k_1, \dots, k_s \geq 0$. Set $\mathfrak{J}_i = \mathfrak{R} \cap I_i$ for $i = 1, \dots, s$ and $(\emptyset) = 0_A$. Then \mathfrak{R} is called a *joint reduction* of (I_1, \dots, I_s) with respect to M of the type (k_1, \dots, k_s) if

$$I_1^{n_1+1} \dots I_s^{n_s+1}M = \sum_{i=1}^s (\mathfrak{J}_i)I_1^{n_1+1} \dots I_i^{n_i} \dots I_s^{n_s+1}M$$

for all large integers n_1, n_2, \dots, n_s .

Lemma 2.5. *Let M be a finitely generated A -module with $\dim M = d > 0$, J an \mathfrak{m} -primary ideal, and I_1, \dots, I_s ideals of A . Assume that k_0, k_1, \dots, k_s are non-negative integers such that $k_0 + k_1 + \dots + k_s = d - 1$. Let $\mathfrak{R} = \{x_1, \dots, x_d\}$ be a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ such that \mathfrak{R} is a system of parameters for M and $x_1 \in I_1$. Then there exists a non-empty Zariski open subset U of $I_1/\mathfrak{m}I_1$ such that for any $x \in I_1$ with image $x + \mathfrak{m}I_1 \in U$, $\{x, x_2, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ and $\{x, x_2, \dots, x_d\}$ is a system of parameters for M .*

Proof. By [5, Lemma 17.1.4], a set of elements of A is a joint reduction of (I_1, \dots, I_s, J) with respect to M if and only if it is so with respect to $A/\text{Ann}_A M$. Thus we may assume that $M = A = A/\text{Ann}_A M$. Set

$$T = (I_1/\mathfrak{m}I_1)^{k_1} \oplus \dots \oplus (I_s/\mathfrak{m}I_s)^{k_s} \oplus (J/\mathfrak{m}J)^{k_0+1}.$$

By [5, Lemma 17.3.2], there exists a Zariski open subset $V \subseteq T$ such that $\{y_1, \dots, y_d\}$ is a joint reduction of (I_1, \dots, I_s, J) of the type $(k_1, \dots, k_s, k_0 + 1)$ if and only if the image $(\bar{y}_1, \dots, \bar{y}_d)$ of (y_1, \dots, y_d) in T lies in V . Since $\{x_1, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) of the type $(k_1, \dots, k_s, k_0 + 1)$, V is non-empty. Let U_1 be the subset of $I_1/\mathfrak{m}I_1$ consisting of elements \bar{x} ($x \in I_1$) such that $(\bar{x}, \bar{x}_2, \dots, \bar{x}_d) \in V$. By [5, Lemma 8.5.12], U_1 is a non-empty Zariski open subset of $I_1/\mathfrak{m}I_1$. Now, assume that P_1, P_2, \dots, P_t are all the prime ideals of $\text{Ass}_A \frac{A}{(x_2, \dots, x_d)}$ such that

$$\dim_A \frac{A}{(x_2, \dots, x_d)} = \text{Coht} P_j \quad (1 \leq j \leq t).$$

For each $j = 1, \dots, t$, let W_j be the image of $P_j \cap I_1$ in $I_1/\mathfrak{m}I_1$. Since $x_1 \in I_1 \setminus \bigcup_{j=1}^t P_j$, W_1, \dots, W_t are proper k -vector subspaces of $I_1/\mathfrak{m}I_1$ by Nakayama's lemma. Since k is an infinite field, $U_2 = (I_1/\mathfrak{m}I_1) \setminus \bigcup_{j=1}^t W_j$ is a non-empty Zariski open subset of $I_1/\mathfrak{m}I_1$. Observe that for any $y \in I_1$ with image $y + \mathfrak{m}I_1 \in U_2$, $\{y, x_2, \dots, x_d\}$ is a system of parameters for M . Set $U = U_1 \cap U_2$. Then for any $x \in I_1$ with image $x + \mathfrak{m}I_1 \in U$, $\{x, x_2, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ and $\{x, x_2, \dots, x_d\}$ is a system of parameters for M . The lemma has been proved. \square

The following proposition shows that one can build a joint reduction \mathfrak{R} from a Rees superficial sequence which satisfies the assumption of the Main Theorem.

Proposition 2.6. *Let M be a finitely generated A -module with $\dim M = d > 0$ and I_1, \dots, I_s ideals of A . Set*

$$h = \max \left\{ \text{ht} \left(\frac{I_j + \text{Ann}_A M}{\text{Ann}_A M} \right) \mid j = 1, \dots, s \right\}.$$

Then the following statements hold:

- (i) *There exists a Rees superficial sequence x_1, \dots, x_h of (I_1, \dots, I_s) with respect to M which is a part of a system of parameters for M .*
- (ii) *If y_1, \dots, y_m is a Rees superficial sequence of (I_1, \dots, I_s) with respect to M and $I_1 \cdots I_s \subseteq \sqrt{(y_1, \dots, y_m) + \text{Ann}_A M}$ (in particular, if y_1, \dots, y_m is a system of parameters for M), then $\{y_1, \dots, y_m\}$ is a joint reduction of (I_1, \dots, I_s) with respect to M .*

Proof. To prove (i), we may assume that $h > 0$. Let x_1, \dots, x_l be a Rees superficial sequence of maximal length of (I_1, \dots, I_s) with respect to M , which is a part of a system of parameters for M . Set $M' = M/(x_1, \dots, x_l)M$ and denote by Σ the set of minimal prime ideals of $\text{Ann}_A M'$ such that for any $P \in \Sigma$, $\dim M' = \text{Coht} P$.

Since

$$\begin{aligned} \text{ht}\left(\frac{I_i + \text{Ann}_A M'}{\text{Ann}_A M'}\right) &\geq \text{ht}\left(\frac{I_i + \text{Ann}_A M}{\text{Ann}_A M}\right) - l, \\ \max\left\{\text{ht}\left(\frac{I_i + \text{Ann}_A M'}{\text{Ann}_A M'}\right) \mid i = 1, \dots, s\right\} \\ &\geq \max\left\{\text{ht}\left(\frac{I_i + \text{Ann}_A M}{\text{Ann}_A M}\right) \mid i = 1, \dots, s\right\} - l = h - l. \end{aligned}$$

We now need to show that $l = h$. Indeed, if $0 \leq l < h$, then $h - l > 0$. Hence there exists I_j that is contained in no prime ideal belonging to Σ . By Lemma 2.2, there is a Rees superficial element $x_{l+1} \in I_j$ of (I_1, \dots, I_s) with respect to M' which does not belong to any element of Σ . It is easily seen that x_1, \dots, x_l, x_{l+1} is a Rees superficial sequence of (I_1, \dots, I_s) with respect to M which is also a part of a system of parameters for M . This contradicts the fact that x_1, \dots, x_l is a sequence of maximal length. Hence $l = h$. We obtain (i).

The proof of (ii) is based on the idea in the proof of [21, Theorem 3.4]. We first show by induction on m that

$$(1) \quad (y_1, \dots, y_m)M \cap I_1^{n_1+1} \dots I_s^{n_s+1}M = \sum_{i=1}^s (\mathfrak{J}_i)I_1^{n_1+1} \dots I_i^{n_i} \dots I_s^{n_s+1}M$$

for all large n_1, \dots, n_s , where $\mathfrak{J}_i = \{y_1, \dots, y_m\} \cap I_i$, $i = 1, \dots, s$. The case $m = 1$ is obvious. Assume now that $m > 1$, that (1) has been proved for $m - 1$, and that $y_m \in I_s$. Set $\overline{M} = M/(y_1, \dots, y_{m-1})M$. Then y_m is a Rees superficial element of (I_1, \dots, I_s) with respect to \overline{M} , so

$$y_m \overline{M} \cap I_1^{n_1+1} \dots I_s^{n_s+1} \overline{M} = y_m I_1^{n_1+1} \dots I_{s-1}^{n_{s-1}+1} I_s^{n_s} \overline{M}$$

for all large n_1, \dots, n_s . This implies that

$$(y_1, \dots, y_m)M \cap I_1^{n_1+1} \dots I_s^{n_s+1}M \subseteq y_m I_1^{n_1+1} \dots I_{s-1}^{n_{s-1}+1} I_s^{n_s}M + (y_1, \dots, y_{m-1})M,$$

and hence

$$\begin{aligned} (y_1, \dots, y_m)M \cap I_1^{n_1+1} \dots I_s^{n_s+1}M \\ \subseteq (y_m I_1^{n_1+1} \dots I_{s-1}^{n_{s-1}+1} I_s^{n_s}M + (y_1, \dots, y_{m-1})M) \cap I_1^{n_1+1} \dots I_s^{n_s+1}M \\ = y_m I_1^{n_1+1} \dots I_{s-1}^{n_{s-1}+1} I_s^{n_s}M + (y_1, \dots, y_{m-1})M \cap I_1^{n_1+1} \dots I_s^{n_s+1}M \end{aligned}$$

for all large n_1, \dots, n_s . By the induction hypothesis,

$$(y_1, \dots, y_{m-1})M \cap I_1^{n_1+1} \dots I_s^{n_s+1}M = \sum_{i=1}^s (\mathfrak{J}'_i)I_1^{n_1+1} \dots I_i^{n_i} \dots I_s^{n_s+1}M$$

for all large n_1, \dots, n_s , where $\mathfrak{J}'_i = \{y_1, \dots, y_{m-1}\} \cap I_i$, $i = 1, \dots, s$. Note that $\mathfrak{J}'_i \subseteq \mathfrak{J}_i$ for $i = 1, \dots, s - 1$ and $\mathfrak{J}_s = \mathfrak{J}'_s \cup \{y_m\}$. Hence

$$\begin{aligned} (y_1, \dots, y_m)M \cap I_1^{n_1+1} \dots I_s^{n_s+1}M \\ \subseteq y_m I_1^{n_1+1} \dots I_{s-1}^{n_{s-1}+1} I_s^{n_s}M + \sum_{i=1}^s (\mathfrak{J}'_i)I_1^{n_1+1} \dots I_i^{n_i} \dots I_s^{n_s+1}M \\ \subseteq \sum_{i=1}^s (\mathfrak{J}_i)I_1^{n_1+1} \dots I_i^{n_i} \dots I_s^{n_s+1}M \end{aligned}$$

for all large n_1, \dots, n_s . Since the reverse inclusion is obvious, we get (1).

Now if

$$I_1 \cdots I_s \subseteq \sqrt{(y_1, \dots, y_m) + \text{Ann}_A M},$$

then

$$I_1^{n_1+1} \cdots I_s^{n_s+1} M \subseteq (y_1, \dots, y_m) M$$

for all large n_1, \dots, n_s . It therefore follows from (1) that $\{y_1, \dots, y_m\}$ is a joint reduction of (I_1, \dots, I_s) with respect to M . □

As a consequence of Proposition 2.6, we obtain the following corollary.

Corollary 2.7. *Let M be a finitely generated A -module with $\dim M = d > 0$, J an \mathfrak{m} -primary ideal, and I_1, \dots, I_s ideals of A . Set $I = I_1 \cdots I_s$. Assume that*

$$\text{ht}\left(\frac{I + \text{Ann}_A M}{\text{Ann}_A M}\right) = h > 0.$$

Let k_0, k_1, \dots, k_s be non-negative integers such that

$$k_0 + k_1 + \cdots + k_s = d - 1 \quad \text{and} \quad k_1 + \cdots + k_s < h.$$

Then there exists a Rees superficial sequence x_1, \dots, x_d of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ which is a system of parameters for M , and in this case, $\{x_1, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to M .

Proof. Set $t = k_1 + \cdots + k_s$. We first show that there exists a Rees superficial sequence x_1, \dots, x_t of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, 0)$ which is a part of a system of parameters for M . Assume that we have built a Rees superficial sequence of maximal length x_1, \dots, x_l of (I_1, \dots, I_s, J) with respect to M of the type $(l_1, \dots, l_s, 0)$ which is a part of a system of parameters for M , where $0 \leq l_i \leq k_i$ for $i = 1, \dots, s$ and $l = l_1 + \cdots + l_s$. Set $M' = M/(x_1, \dots, x_l)M$. Note that

$$\begin{aligned} \text{ht}\left(\frac{I_i + \text{Ann}_A M'}{\text{Ann}_A M'}\right) &\geq \text{ht}\left(\frac{I_i + \text{Ann}_A M}{\text{Ann}_A M}\right) - l \\ &\geq \text{ht}\left(\frac{I + \text{Ann}_A M}{\text{Ann}_A M}\right) - l \\ &= h - l \geq h - t > 0 \end{aligned}$$

for every $i = 1, \dots, s$. We need to show that $l_i = k_i$ for $i = 1, \dots, s$, and hence $l = h$. Indeed, if there is $l_j < k_j$, then with the same argument as in the proof of Proposition 2.6(i), we can find an element $x_{l+1} \in I_j$ such that x_1, \dots, x_l, x_{l+1} is a Rees superficial sequence of (I_1, \dots, I_s, J) with respect to M of the type $(l_1, \dots, l_j + 1, \dots, l_s, 0)$ which is also a part of a system of parameters for M . This contradicts the maximum of sequence x_1, \dots, x_l . Hence $l_i = k_i$ for $i = 1, \dots, s$.

We obtain a sequence x_1, \dots, x_t as required. Put $\overline{M} = \frac{M}{(x_1, \dots, x_t)M}$. Since

$\text{ht}\left(\frac{J + \text{Ann}_A \overline{M}}{\text{Ann}_A \overline{M}}\right) = d - t = k_0 + 1$, it follows from Proposition 2.6(i) that there exists a Rees superficial sequence $x_{t+1}, \dots, x_d \in J$ of (I_1, \dots, I_s, J) with respect to \overline{M} of the type $(0, \dots, 0, k_0 + 1)$ which is a system of parameters for \overline{M} . Thus we get a sequence x_1, \dots, x_d that is a Rees superficial sequence of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$, and this sequence is also a system of parameters for M . By Proposition 2.6(ii), $\{x_1, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to M . □

3. MIXED MULTIPLICITIES OF IDEALS

Let M be a finitely generated A -module of dimension $d > 0$. Let J be an \mathfrak{m} -primary ideal and I_1, \dots, I_s ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\text{Ann}_A M}$. Set $\dim M/0_M : I^\infty = q$. Recall that by [18, Proposition 3.1] (see also [9]),

$$\ell_A\left(\frac{J^{n_0} I_1^{n_1} \cdots I_s^{n_s} M}{J^{n_0+1} I_1^{n_1} \cdots I_s^{n_s} M}\right)$$

is a polynomial of total degree $q - 1$ for all large n_0, n_1, \dots, n_s . Write the terms of total degree $q - 1$ in this polynomial in the form

$$\sum_{k_0+k_1+\dots+k_s=q-1} e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) \frac{n_0^{k_0} n_1^{k_1} \cdots n_s^{k_s}}{k_0! k_1! \cdots k_s!}.$$

Then $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M)$ are non-negative integers not all zero. One calls $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M)$ the *mixed multiplicity of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$* . It is easily seen that if $\text{ht}\left(\frac{I + \text{Ann}_A M}{\text{Ann}_A M}\right) > 0$, then $\dim M/0_M : I^\infty = \dim M = d$.

In this section, we show that mixed multiplicities of (I_1, \dots, I_s, J) with respect to M can be expressed as the multiplicities of ideals generated by joint reductions of them. From this we get Rees’s theorem on mixed multiplicity as a consequence. Moreover, we also obtain a formula that allows us to compute mixed multiplicities in terms of the multiplicities of ideals generated by Rees superficial sequences.

The main result of this paper is the following theorem. This was established by Rees for the case of \mathfrak{m} -primary ideals in [12, Theorem 2.4].

Theorem 3.1. *Let M be a finitely generated A -module of Krull dimension $d > 0$. Let J be an \mathfrak{m} -primary ideal and I_1, \dots, I_s ideals of A . Set $I = I_1 \cdots I_s$. Assume that $\text{ht}\left(\frac{I + \text{Ann}_A M}{\text{Ann}_A M}\right) = h > 0$ and k_0, k_1, \dots, k_s are non-negative integers such that $k_0 + k_1 + \dots + k_s = d - 1$ and $k_1 + \dots + k_s < h$. Let $\mathfrak{R} = \{x_1, \dots, x_d\}$ be a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ such that \mathfrak{R} is a system of parameters for M . Then*

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) = e_A(\mathfrak{R}, M).$$

Proof. First, we consider the following fact.

Remark 3.2. If $k_1 = \dots = k_s = 0$, then $\mathfrak{R} \subset J$ and \mathfrak{R} is a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(0, \dots, 0, d)$. Hence there exists an integer u such that $J^{n+1} I^u M = J^{n+1} I_1^u \cdots I_s^u M = (\mathfrak{R}) J^n I_1^u \cdots I_s^u M = (\mathfrak{R}) J^n I^u M$ for all $n \geq u$. This means that (\mathfrak{R}) is a reduction of J with respect to $I^u M$ by [10]. So by [10, Theorem 1] we have

$$(2) \quad e_A(J, I^u M) = e_A((\mathfrak{R}), I^u M).$$

Since $\text{ht}\left(\frac{I + \text{Ann}_A M}{\text{Ann}_A M}\right) > 0$, it follows that $\dim(M/I^u M) < \dim M$. Hence

$$(3) \quad e_A(J, M) = e_A(J, I^u M) \text{ and } e_A((\mathfrak{R}), M) = e_A((\mathfrak{R}), I^u M).$$

By (2) and (3) we obtain $e_A(J, M) = e_A((\mathfrak{R}), M)$. Remember that \mathfrak{R} is a system of parameters for M , $e_A((\mathfrak{R}), M) = e_A(\mathfrak{R}, M)$. Consequently $e_A(J, M) = e_A(\mathfrak{R}, M)$.

Since $\text{ht}\left(\frac{I + \text{Ann}_A M}{\text{Ann}_A M}\right) > 0$, by [18, Lemma 3.2(ii)] (see also [9, Lemma 3.2(ii)]) we get $e_A(J^{[d]}, I_1^{[0]}, \dots, I_s^{[0]}, M) = e_A(J, M)$. Therefore, $e_A(J^{[d]}, I_1^{[0]}, \dots, I_s^{[0]}, M) = e_A(\mathfrak{R}, M)$.

We now prove the theorem by induction on d that if \mathfrak{R} is a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ such that \mathfrak{R} is a system of parameters for M , then $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) = e_A(\mathfrak{R}, M)$. If $d = 1$, then $k_1 = \dots = k_s = 0$. It follows from Remark 3.2 that $e_A(J^{[1]}, I_1^{[0]}, \dots, I_s^{[0]}, M) = e_A(\mathfrak{R}, M)$.

Next, consider the case $d > 1$. Since \mathfrak{R} is a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$, it is also a joint reduction of (I_1, \dots, I_s, J) with respect to A/\mathfrak{p} of the type $(k_1, \dots, k_s, k_0 + 1)$, where \mathfrak{p} is a minimal prime ideal of $\text{Ann}_A M$ (see e.g. [5, Lemma 17.1.4]). Denote by Λ the set of minimal prime ideals \mathfrak{p} of $\text{Ann}_A M$ such that $\dim A/\mathfrak{p} = d$. For any $\mathfrak{p} \in \Lambda$, set $B = A/\mathfrak{p}$; then B is an A -module with $\dim B = d$ and \mathfrak{R} is a system of parameters for B . We show that the theorem is true for B , i.e.,

$$(4) \quad e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, B) = e_A(\mathfrak{R}, B).$$

We need the following comment.

Remark 3.3. If $\mathfrak{p} \in \Lambda$, then $\text{ht}\left(\frac{I + \mathfrak{p}}{\mathfrak{p}}\right) \geq h$. Indeed, without loss of generality, we may assume that $\text{Ann}_A M = 0$. It is enough to show that there are h elements a_1, \dots, a_h in I such that

$$(5) \quad \text{ht}(a_1, \dots, a_h) = \text{ht}\left(\frac{(a_1, \dots, a_h) + \mathfrak{p}}{\mathfrak{p}}\right) = h.$$

If $h = 1$, this is trivial since $I \not\subseteq \mathfrak{p}$. Assume that $h > 1$ and we have found $a_1, \dots, a_{h-1} \in I$ satisfying $\text{ht}(a_1, \dots, a_{h-1}) = \text{ht}\left(\frac{(a_1, \dots, a_{h-1}) + \mathfrak{p}}{\mathfrak{p}}\right) = h - 1$. We claim that $\text{ht}((a_1, \dots, a_{h-1}) + \mathfrak{p}) = h - 1$. Indeed, if $\text{ht}((a_1, \dots, a_{h-1}) + \mathfrak{p}) > h - 1$, then we can choose $b \in \mathfrak{p}$ such that $\text{ht}(a_1, \dots, a_{h-1}, b) = h$, and hence there exist a_{h+1}, \dots, a_d in A such that $\mathfrak{q} = (a_1, \dots, a_{h-1}, b, a_{h+1}, \dots, a_d)$ is an \mathfrak{m} -primary ideal. In this case, the image of \mathfrak{q} in A/\mathfrak{p} is primary to the maximal ideal $\mathfrak{m}/\mathfrak{p}$. But this contradicts the fact that $\dim A/\mathfrak{p} = d$ since

$$\text{ht}\left(\frac{\mathfrak{q} + \mathfrak{p}}{\mathfrak{p}}\right) = \text{ht}\left(\frac{(a_1, \dots, a_{h-1}, a_{h+1}, \dots, a_d) + \mathfrak{p}}{\mathfrak{p}}\right) \leq d - 1.$$

So $\text{ht}((a_1, \dots, a_{h-1}) + \mathfrak{p}) = h - 1$. Thus we can choose $a_h \in I$ that avoids all the minimal prime ideals of (a_1, \dots, a_{h-1}) and of $(a_1, \dots, a_{h-1}) + \mathfrak{p}$. Observe that the elements a_1, \dots, a_h satisfy (5).

Return to the proof of the theorem. Since $\text{Ann}_A B = \mathfrak{p}$,

$$k_1 + \dots + k_s < h \leq \text{ht}\left(\frac{I + \text{Ann}_A B}{\text{Ann}_A B}\right)$$

by Remark 3.3. Consequently, B satisfies the assumptions of the theorem. By [20, Prop. 3.1(vi)] (see also [9, Theorem 3.6(ii)]), $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, B) \neq 0$. If $k_1 = \dots = k_s = 0$, then (4) is true by Remark 3.2. If k_1, \dots, k_s are not all zero, we may assume that $k_1 > 0$ and $x_1 \in I_1$. By Proposition 2.3, there is a non-empty

Zariski open subset U_1 of $I_1/\mathfrak{m}I_1$ such that if $y \in I_1$ with image $y + \mathfrak{m}I_1 \in U_1$, then y is a weak-(FC)-element of (I_1, \dots, I_s, J) with respect to B . Moreover, by Lemma 2.5, there also exists a non-empty Zariski open subset U_2 of $I_1/\mathfrak{m}I_1$ such that whenever $z \in I_1$ with image $z + \mathfrak{m}I_1 \in U_2$, then $\{z, x_2, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to B and $\{z, x_2, \dots, x_d\}$ is a system of parameters for B . Now choose $x \in I_1$ such that $x + \mathfrak{m}I_1 \in U_1 \cap U_2$. Then x is a weak-(FC)-element of (I_1, \dots, I_s, J) with respect to B and $\{x, x_2, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to B of the type $(k_1, \dots, k_s, k_0 + 1)$ and $\{x, x_2, \dots, x_d\}$ is also a system of parameters for B . Set $\overline{B} = B/xB$. Since x is a weak-(FC)-element and $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, B) \neq 0$, x is an (FC)-element by [20, Proposition 3.1(i)]. Hence by [18, Proposition 3.3] (see also [3] and [9, Proposition 3.3]) we have

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, B) = e_A(J^{[k_0+1]}, I_1^{[k_1-1]}, \dots, I_s^{[k_s]}, \overline{B}).$$

Note that $\{x, x_2, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to B of the type $(k_1, \dots, k_s, k_0 + 1)$, so $\{x_2, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to \overline{B} of the type $(k_1 - 1, \dots, k_s, k_0 + 1)$. Moreover, since $\{x, x_2, \dots, x_d\}$ is a system of parameters for B , $\dim \overline{B} = d - 1$ and $\{x_2, \dots, x_d\}$ is a system of parameters for \overline{B} . Note also that $\text{ht}\left(\frac{I + \text{Ann}_A \overline{B}}{\text{Ann}_A \overline{B}}\right) \geq \text{ht}\left(\frac{I + \text{Ann}_A B}{\text{Ann}_A B}\right) - 1 > (k_1 - 1) + k_2 + \dots + k_s$. Hence by the induction hypothesis,

$$e_A(J^{[k_0+1]}, I_1^{[k_1-1]}, \dots, I_s^{[k_s]}, \overline{B}) = e_A(x_2, \dots, x_d, \overline{B}).$$

Since $x \notin \mathfrak{p}$, x is not a zero divisor on B . Therefore,

$$e_A(x_2, \dots, x_d, \overline{B}) = e_A(x, x_2, \dots, x_d, B)$$

by [1]. Thus, $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, B) = e_A(x, x_2, \dots, x_d, B)$. We now show that

$$e_A(x, x_2, \dots, x_d, B) = e_A(x_1, x_2, \dots, x_d, B).$$

Put $t = \max\{k_0, k_1 - 1, k_2, \dots, k_s\}$. If $t > 0$ we may assume that $k_i > 0$ (or $k_i > 1$ if $i = 1$) and $x_d \in I_i$. Set $B' = B/x_d B$. Then $\{x, x_2, \dots, x_{d-1}\}$ and $\{x_1, x_2, \dots, x_{d-1}\}$ are both systems of parameters for B' . Moreover, they are both joint reductions of (I_1, \dots, I_s, J) with respect to B' of the type $(k_1, \dots, k_i - 1, \dots, k_s, k_0 + 1)$. Similarly as above, we also have $\text{ht}\left(\frac{I + \text{Ann}_A B'}{\text{Ann}_A B'}\right) > k_1 + \dots + k_i - 1 + \dots + k_s$. So by the induction hypothesis,

$$\begin{aligned} e_A(x, x_2, \dots, x_{d-1}, B') &= e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_i^{[k_i-1]}, \dots, I_s^{[k_s]}, B') \\ &= e_A(x_1, x_2, \dots, x_{d-1}, B'). \end{aligned}$$

Since x_d is also not a zero divisor of B , by [1] we have

$$\begin{aligned} e_A(x, x_2, \dots, x_{d-1}, B') &= e_A(x, x_2, \dots, x_d, B), \\ e_A(x_1, x_2, \dots, x_{d-1}, B') &= e_A(x_1, x_2, \dots, x_d, B). \end{aligned}$$

Therefore, $e_A(x, x_2, \dots, x_d, B) = e_A(x_1, x_2, \dots, x_d, B)$.

If $t = 0$, then $k_0 = k_2 = \dots = k_s = 0$, $k_1 = 1$, and $d = 2$. In this case, $x_2 \in J$ and $\dim B/x_2 B = 1$. Since $\text{ht}\left(\frac{I + \mathfrak{p}}{\mathfrak{p}}\right) \geq h > k_1 + \dots + k_s = 1$, I_1, \dots, I_s are ideals of definition of B . It follows that I_1, \dots, I_s are ideals of definition of $B/x_2 B$. As

$\dim B/x_2B = 1$, applying Remark 3.2 for modules of dimension 1 with I_1 playing the role of J , we get

$$e_A(x, B/x_2B) = e_A(J^{[0]}, I_1^{[1]}, I_2^{[0]}, \dots, I_s^{[0]}, B/x_2B) = e_A(x_1, B/x_2B),$$

and hence $e_A(x, x_2, B) = e_A(x_1, x_2, B)$. By the above results, we obtain

$$e_A(x, x_2, \dots, x_d, B) = e_A(x_1, x_2, \dots, x_d, B).$$

Hence $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, B) = e_A(x_1, x_2, \dots, x_d, B)$. This proves (4). Thus for any $\mathfrak{p} \in \Lambda$, we have $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, A/\mathfrak{p}) = e_A(x_1, x_2, \dots, x_d, A/\mathfrak{p})$. Consequently,

$$\sum_{\mathfrak{p} \in \Lambda} \ell_A(M_{\mathfrak{p}}) e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, A/\mathfrak{p}) = \sum_{\mathfrak{p} \in \Lambda} \ell_A(M_{\mathfrak{p}}) e_A(x_1, x_2, \dots, x_d, A/\mathfrak{p}).$$

Remember that $e_A(x_1, \dots, x_d, M) = \sum_{\mathfrak{p} \in \Lambda} \ell_A(M_{\mathfrak{p}}) e_A(x_1, \dots, x_d, A/\mathfrak{p})$ (see [5, Theorem 11.2.4]). Now by [24, Corollary 3.6],

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) = \sum_{\mathfrak{p} \in \Lambda} \ell_A(M_{\mathfrak{p}}) e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, A/\mathfrak{p}).$$

Therefore, $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) = e_A(x_1, \dots, x_d, M)$. The proof is complete. □

Remark 3.4. Keep the notation of Theorem 3.1. Now if x_1, \dots, x_d is a Rees superficial sequence of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ that is a system of parameters for M , then $\{x_1, \dots, x_d\}$ is a joint reduction of (I_1, \dots, I_s, J) with respect to M of the type $(k_1, \dots, k_s, k_0 + 1)$ by Corollary 2.7. Hence $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) = e_A(x_1, \dots, x_d, M)$ by Theorem 3.1.

Remark 3.5. From Theorem 3.1, one may raise a question: Does the theorem hold if $k_1 + \dots + k_s \geq h$? Consider the cases $s = 1$ and $M = A$. Let $I_1 = I$ be an equimultiple ideal of A , i.e., an ideal such that $\text{ht}I = s(I)$, where $s(I) = \dim \bigoplus_{n \geq 0} I^n / \mathfrak{m}I^n$. If $\text{ht}I = h > 0$, then by [19, Theorem 3.1(iii)], $e_A(J^{[d-i]}, I^{[i]}) = 0$ for all $i \geq h$. Therefore, when $i \geq h$, $e_A(J^{[d-i]}, I^{[i]})$ cannot be the multiplicity of any system of parameters. This example shows that Theorem 3.1 does not hold in general if one omits the assumption $k_1 + \dots + k_s < h$.

Finally, we recover Rees’s theorem, which is a motivation for this paper.

Corollary 3.6 ([12, Theorem 2.4(i), (ii)]). *Let M be a finitely generated A -module with $\dim M = d > 0$. Let I_1, \dots, I_s be \mathfrak{m} -primary ideals of A . Assume that k_1, \dots, k_s are non-negative integers such that $k_1 + \dots + k_s = d$ and \mathfrak{R} is a joint reduction of (I_1, \dots, I_s) with respect to M of the type (k_1, \dots, k_s) . Then $e_A(I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) = e_A(\mathfrak{R}, M)$.*

Proof. If $s = 1$, then (\mathfrak{R}) is a reduction of I_1 with respect to M , i.e., $(\mathfrak{R})I_1^n M = I_1^{n+1}M$ for all $n \gg 0$ [10]. So by [10, Theorem 1], we have $e_A(\mathfrak{R}, M) = e_A(I_1, M) = e_A(I_1^{[d]}, M)$. Now assume that $s > 1$. Since I_1, \dots, I_s are \mathfrak{m} -primary ideals, \mathfrak{R} is a system of parameters for M . Moreover, from $k_1 + \dots + k_s = d > 0$, we may assume $k_1 > 0$. Then $k_2 + \dots + k_s < d$. So by Theorem 3.1, we get the proof of this corollary. □

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