SIMULTANEOUS NONVANISHING
OF GL(2) × GL(2) AND GL(2) L-FUNCTIONS

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(Communicated by Ken Ono)

ABSTRACT. Let $f$ be a fixed holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$. We prove that for each $K$ large enough, there exists a holomorphic Hecke cusp form $g$ of weight $k$ with $K \leq k \leq 2K$ such that $L\left(\frac{1}{2}, g\right) L\left(\frac{1}{2}, f \times g\right) \neq 0$.

1. INTRODUCTION

The vanishing or nonvanishing of an automorphic $L$-function is an interesting problem in number theory. In many cases, it can encode a deep arithmetic property. For example, the Birch and Swinnerton-Dyer conjecture relates the order of vanishing of the Hasse-Weil $L$-function at the central point to the rank of an elliptic curve.

Let $\pi$ be a unitary cuspidal automorphic representation of $GL(n)$ and let $s_0 \in \mathbb{C}$. The nonvanishing of $L(s_0, \pi)$ twists by a Dirichlet character $\chi$ has been studied by Rohrlich [Ro], Barthel-Ramakrishnan [BR], Luo [Lu], and Luo-Rudnick-Sarnak [LRS]. Especially, this problem is related to the Ramanujan conjecture (see [LRS]). The simultaneous nonvanishing of twists of automorphic $L$-functions was studied by many authors ([MV], [RR], [Li], [Kh], [Xu]). In her paper [Li], Li proved a simultaneous nonvanishing result for $GL(3) \times GL(2)$ and $GL(2)$ $L$-functions at the central point. More precisely, let $F$ be a fixed $GL(3)$ Maass cusp form. She showed that there are infinitely many $GL(2)$ Maass-Hecke cusp forms $u_j$ such that $L\left(\frac{1}{2}, F \times u_j\right) L\left(\frac{1}{2}, u_j\right) \neq 0$. Later Khan [Kh] proved a similar result by replacing the twists with $GL(2)$ holomorphic forms of prime level. In this paper, we will establish a simultaneous nonvanishing result for $GL(2) \times GL(2)$ and $GL(2)$ $L$-functions (Theorem 1).

Let $H_k$ denote the set of holomorphic Hecke cusp forms $g$ of weight $k$ for $SL(2, \mathbb{Z})$, where $g(z)$ has the Fourier expansion

$$g(z) = \sum_{n=1}^{\infty} \lambda_g(n)n^{\frac{k-1}{2}}e^{2\pi inz}$$

with $\lambda_g(1) = 1$.

Our main theorem is the following:

**Theorem 1.** Let $f$ be a fixed holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$. For each $K$ large enough, there exists $g \in H_k$ with $K \leq k \leq 2K$ such that $L\left(\frac{1}{2}, f \times g\right) L\left(\frac{1}{2}, g\right) \neq 0$. 

Received by the editors July 11, 2012. 
2010 Mathematics Subject Classification. Primary 11F11, 11M99.
Note that a similar result in the Maass form case was obtained in [Xu].

**Theorem 1.** An immediate corollary of the following asymptotic formula for the average of \( L \)-functions.

**Theorem 2.** Let \( f \) be a fixed holomorphic Hecke cusp form for \( SL(2, \mathbb{Z}) \). Let \( u(\xi) \in C_c^\infty(0, \infty) \). Then for \( K \) large, we have

\[
\sum_{k \geq 2, \ 4 | k} u\left(\frac{k-1}{K}\right) \sum_{g \in H_k} \omega_g^{-1} L\left(\frac{1}{2}, f \times g\right) L\left(\frac{1}{2}, g\right) = K (\log K) L(1, f) \int_0^\infty u(\xi) d\xi + O(K^\varepsilon),
\]

where \( \gamma \) is the Euler constant and the implied constant depends on \( f \) and \( \varepsilon \).

**Remarks.**

(1) \( L(1, f) \neq 0 \) (see [JS]).

(2) We only consider \( k \equiv 0 \pmod{4} \), since for \( k \equiv 2 \pmod{4} \), the root number of \( L(s, g) \) is \(-1\) and hence \( L(\frac{1}{2}, g) = 0 \).

2. **Preliminaries**

2.1. **GL(2) holomorphic cusp forms.** The following Petersson trace formula is well-known and can be found in Iwaniec’s book [Iw].

**Proposition 1.** (Petersson trace formula).

\[
\sum_{g \in H_k} \omega_g^{-1} \lambda_g(m) \lambda_g(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c=1}^\infty \frac{S(m, n; c)}{c} J_{k-1}\left(\frac{4\pi \sqrt{mn}}{c}\right),
\]

where \( \omega_g = \frac{(4\pi)^{k-1}}{1(k-1)^2} \|g\|^2 \), \( \delta_{m,n} \) equals \( 1 \) if \( m = n \) and \( 0 \) otherwise, \( S(m, n; c) \) is the Kloosterman sum defined below, and \( J_{k-1} \) is the \( J \)-Bessel function.

The Kloosterman sum is defined as

\[
S(m, n; c) = \sum_{a\equiv 1 (\text{mod } c)} e\left(\frac{ma + n\bar{a}}{c}\right),
\]

and A. Weil proved that

\[
|S(m, n; c)| \leq (m, n, c)^{\frac{1}{2}} e^{\frac{1}{2}\tau(c)},
\]

where \( \tau(c) \) is the divisor function.

**Lemma 1.** For any \( \varepsilon > 0 \), we have

\[
(2.1) \quad \sum_{n \leq x} |S(m, n; c)| \ll xe^{1/2+\varepsilon},
\]

where the implied constant depends only on \( \varepsilon \).

**Proof.** By Weil’s bound, we have

\[
\sum_{n \leq x} |S(m, n; c)| \ll e^{1/2+\varepsilon} \sum_{n \leq x} (n, c)^{1/2}
\]

\[
\ll e^{1/2+\varepsilon} \sum_{d|c} \sum_{d|n \leq x} d^{1/2} \ll xe^{1/2+\varepsilon}. \quad \square
\]
For each \( g \in H_k \), the \( L \)-function associated to \( g \) is defined by

\[
L(s, g) = \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^s}.
\]

It is entire and satisfies the functional equation

\[
\Lambda(s, g) = i^k \Lambda(1 - s, g),
\]

where

\[
\Lambda(s, g) = \pi^{-s} \Gamma\left(\frac{s + \frac{k-1}{2}}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2}}{2}\right) L(s, g).
\]

**Lemma 2** (Approximate functional equation). Let \( k \equiv 0 \pmod{4} \) and let \( G(u) = e^{u^2} \). We have

\[
L\left(\frac{1}{2}, g\right) = 2 \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n, k),
\]

where

\[
V(y, k) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma_1(\frac{1}{2} + u, k)}{\gamma_1(\frac{1}{2}, k)} du
\]

and

\[
\gamma_1(s, k) = \pi^{-s} \Gamma\left(\frac{s + \frac{k-1}{2}}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2}}{2}\right).
\]

**Proof.** See [IK, Theorem 5.3]. \( \square \)

**Lemma 3.** 1) The derivatives of \( V(y, k) \) with respect to \( y \) satisfy

\[
y^a \frac{\partial^a}{\partial y^a} V(y, k) \ll \left(1 + \frac{y}{k}\right)^{-A}.
\]

The implied constant depends only on \( a \) and \( A \).

2) If \( 1 \leq y \ll k^{1+\epsilon} \), then as \( k \to \infty \)

\[
V(y, k) = V_1\left(\frac{y}{k}\right) + O\left(\frac{1}{k} V_1^*\left(\frac{y}{k}\right)\right),
\]

where

\[
V_1(x) = \frac{1}{2\pi i} \int_{(3)} x^{-u} G_0(u) \frac{du}{u}, \quad V_1^*(x) = \frac{1}{2\pi i} \int_{(3)} x^{-u} G_1(u) \frac{du}{u}
\]

and \( G_0(u), G_1(u) \) are holomorphic functions with exponential decay as \( |\Im u| \to \infty \).

**Proof.** 1) See [IK, Proposition 5.4].

2) By Stirling’s formula

\[
G(u) \frac{\gamma_1(\frac{1}{2} + u, k)}{\gamma_1(\frac{1}{2}, k)} = G(u) \left(\frac{k}{4\pi}\right)^u \left(1 + O\left(\frac{p(u)}{k}\right)\right)
= k^u G_0(u) + O\left(\frac{1}{k} k^u G_1(u)\right),
\]

where \( p(u) \) is a polynomial in \( u \), \( G_0(u) = G(u) (4\pi)^{-u} \), and \( G_1(u) = G(u) p(u) (4\pi)^{-u} \). The result follows by inserting this into (2.3). \( \square \)
2.2. Rankin-Selberg $L$-functions. For $f \in H_l$ and $g \in H_k$, the Rankin-Selberg $L$-function of $f$ and $g$ is defined by

$$L(s, f \times g) = \zeta(2s) \sum_{m=1}^{\infty} \frac{\lambda_f(m)\lambda_g(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\lambda_{f \times g}(m)}{m^s},$$

where

$$\lambda_{f \times g}(m) = \sum_{a b^2 = m} \lambda_f(a)\lambda_g(a).$$

It is entire and satisfies the functional equation

$$\Lambda(s, f \times g) = \Lambda(1-s, f \times g),$$

where

$$\Lambda(s, f \times g) = (2\pi)^{-s} \Gamma\left(s + \frac{k+l}{2} - 1\right) \Gamma\left(s + \frac{k-l}{2}\right) L(s, f \times g).$$

**Lemma 4** (Approximate functional equation). Let $G(u) = e^{u^2}$. We have

$$L\left(\frac{1}{2}, f \times g\right) = 2 \sum_{m=1}^{\infty} \frac{\lambda_{f \times g}(m)}{m^{1/2}} U(m, k),$$

where

$$U(y, k) = \frac{1}{2\pi i} \int_{(3)} y^{-r} G(u) \frac{\gamma_2(\frac{1}{2} + u, k)}{\gamma_2(\frac{1}{2}, k)} \frac{du}{u}$$

and

$$\gamma_2(s, k) = (2\pi)^{-s} \Gamma\left(s + \frac{k+l}{2} - 1\right) \Gamma\left(s + \frac{k-l}{2}\right).$$

**Proof.** See [IK, Theorem 5.3].

**Lemma 5.** 1) The derivatives of $U(y, k)$ with respect to $y$ satisfy

$$y^a \frac{\partial^a}{\partial y^a} U(y, k) \ll \left(1 + \frac{y}{k^2}\right)^{-A}.$$  

The implied constant depends only on $a$ and $A$.

2) If $1 \leq y \ll k^{2+\varepsilon}$, then as $k \to \infty$

$$U(y, k) = U_1\left(\frac{y}{k^2}\right) + O\left(\frac{1}{k} U_1^*\left(\frac{y}{k^2}\right)\right),$$

where

$$U_1(x) = \frac{1}{2\pi i} \int_{(3)} x^{-r} G_2(u) \frac{du}{u}, \quad U_1^*(x) = \frac{1}{2\pi i} \int_{(3)} x^{-r} G_3(u) \frac{du}{u}$$

and $G_0(u), G_1(u)$ are holomorphic functions with exponential decay as $|\Im u| \to \infty$.

**Proof.** 1) See [IK, Proposition 5.4].

2) By Stirling's formula

$$G(u) \frac{\gamma_2(\frac{1}{2} + u, k)}{\gamma_2(\frac{1}{2}, k)} = G(u) \left(\frac{k^2}{16\pi^2}\right)^u \left(1 + O\left(\frac{p(u)}{k}\right)\right)$$

$$= k^{2u} G_2(u) + O\left(\frac{1}{k} k^{2u} G_3(u)\right),$$
where \( p(u) \) is a polynomial in \( u \), \( G_2(u) = G(u)(16\pi^2)^{-u} \), and \( G_3(u) = G(u)p(u)(16\pi^2)^{-u} \). The result follows by inserting this into (2.6).

3. Proof of Theorem 2

By the approximate functional equations (2.2), (2.5) and the Petersson trace formula (Proposition 1), we have

\[
\sum_{g \in H_k} \omega_g^{-1} L \left( \frac{1}{2}, f \times g \right) L \left( \frac{1}{2}, g \right)
= 4 \sum_{g \in H_k} \omega_g^{-1} \sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m^{1/2}} U(m, k) \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n, k)
= 4 \sum_{g \in H_k} \omega_g^{-1} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\lambda_f(a)\lambda_g(a)}{(ab^2)^{1/2}} U(ab^2, k) \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n, k)
= 4 \sum_{n=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n, k) \frac{\lambda_f(n)}{(nb^2)^{1/2}} U(nb^2, k)
+ 8\pi \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n, k) \frac{\lambda_f(a)}{(ab^2)^{1/2}} U(ab^2, k) \sum_{c=1}^{\infty} S(n, a; c) \frac{J_{k-1}}{c} \left( \frac{4\pi \sqrt{n\alpha}}{c} \right)
= 4D_k + 8\pi E_k.
\]

3.1. Diagonal terms. We first deal with the diagonal terms, and these will contribute to the main term. Let

\[
D := 4 \sum_{k \geq 2, 4 \mid k} u \left( \frac{k-1}{K} \right) D_k.
\]

**Lemma 6.** We have

(3.1) \( D_k = (\log k + \gamma - \log 4\pi) L(1, f) + \frac{1}{2} L'(1, f) + O(k^{-1}) \)

and

\[
D = K (\log K) L(1, f) \int_{0}^{\infty} u(\xi) d\xi
+ K \int_{0}^{\infty} u(\xi) \left( (\log \xi + \gamma - \log 4\pi) L(1, f) + \frac{1}{2} L'(1, f) \right) d\xi + O(1).
\]

**Proof.**

\[
D_k = \sum_{n=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n, k) \frac{\lambda_f(n)}{(nb^2)^{1/2}} U(nb^2, k)
= \frac{1}{(2\pi)^2} \int_{(3)} \int_{(3)} \sum_{n=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{n^{1/2} + u + w} \frac{1}{b^{1/2} + 2w} \frac{\gamma_1(\frac{1}{2} + u, k)}{\gamma_1(\frac{1}{2}, k)} \frac{\gamma_2(\frac{1}{2} + w, k)}{\gamma_2(\frac{1}{2}, k)}
\]

\[
\times G(u) G(w) \frac{du \, dw}{u \, w}
= \frac{1}{(2\pi)^2} \int_{(3)} \int_{(3)} L(1 + u + w, f) \zeta(1 + 2w) \frac{\gamma_1(\frac{1}{2} + u, k)}{\gamma_1(\frac{1}{2}, k)} \frac{\gamma_2(\frac{1}{2} + w, k)}{\gamma_2(\frac{1}{2}, k)}
\]

\[
\times G(u) G(w) \frac{du \, dw}{u \, w}.
\]
Moving the line of integration to \( \Re u = -2 \) and \( \Re w = -\frac{1}{2} \), we pick up a simple pole at \( u = 0 \) and a double pole at \( w = 0 \). By the residue theorem, we have

\[
D_k = \left( \frac{1}{2} \frac{\gamma_k'}{\gamma_k} + \gamma \right) L(1, f) + \frac{1}{2} L'(1, f) + \frac{1}{2\pi i} \int_{(-2)} \left( \frac{1}{2} \frac{\gamma_k'}{\gamma_k} + \gamma \right) L(1 + u, f) + \frac{1}{2} L'(1 + u, f) \right) \times \frac{\gamma_1}{\gamma_1} G(u) \frac{du}{u}
\]

\[
+ \frac{1}{2\pi i} \int_{(-\frac{1}{2})} L(1 + w, f) \zeta(1 + 2w) \frac{\gamma_2}{\gamma_2} \frac{G(w) dw}{w}
\]

\[
+ \frac{1}{(2\pi i)^2} \int_{(-2)} \int_{(-\frac{1}{2})} L(1 + u + w, f) \zeta(1 + 2w) \frac{\gamma_1}{\gamma_1} \frac{\gamma_2}{\gamma_2} \frac{G(u) G(w) du dw}{u w}.
\]

By Stirling’s formula, we have

\[
\frac{\gamma_k'}{\gamma_k} = 2 \log k - 4 \log 2 - 2 \log \pi + O(k^{-1}),
\]

\[
\frac{\gamma_1}{\gamma_1} \ll k^{-2}
\]

for \( \Re u = -2 \), and

\[
\frac{\gamma_2}{\gamma_2} \ll k^{-1}
\]

for \( \Re w = -\frac{1}{2} \). (3.1) follows from using the above estimates. For (3.2), by the Poisson summation formula, we have

\[
\sum_{k \geq 2, 4 \mid k} u \left( \frac{k - 1}{K} \right) = K \int_0^\infty u(\xi) d\xi + O(K^{-A})
\]

for any \( A > 0 \). By writing \( \log k = \log \left( \frac{k - 1}{K} \right) + O(k^{-1}) \) and using (3.3) for \( u(\xi) \log \xi \), we obtain (3.2). □

3.2. Off-diagonal terms. We will prove that the off-diagonal terms only contribute to the error terms. Let

\[
E := \sum_{4 \mid k} u \left( \frac{k - 1}{K} \right) E_k
\]

\[
= \sum_{n=1}^\infty \sum_{a=1}^\infty \sum_{b=1}^\infty \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{ab^2} \sum_{c=1}^\infty S(n, a; c) \sum_{k \mid k} u \left( \frac{k - 1}{K} \right) \times V(n, k) U(ab^2, k) J_{k-1} \left( \frac{4\pi \sqrt{nab}}{c} \right).
\]

Lemma 7. For any \( \varepsilon > 0 \), we have

\[
E \ll K^\varepsilon.
\]
Proof. By taking a smooth dyadic subdivision, it suffices to estimate

\[ E_{N,M} = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{c=1}^{\infty} S(n,a;c) \frac{1}{c} \]

\[ \times \sum_{4|k} u \left( \frac{k-1}{K} \right) V(n,k)U(ab^2,k)J_{k-1} \left( \frac{4\pi\sqrt{na}}{c} \right) \]

for \( N \ll K^{1+\varepsilon} \) and \( M \ll K^{2+\varepsilon} \), where \( w(\xi) \) and \( h(\xi) \) are fixed smooth functions with support contained in \([1,2]\).

Let

\[ R_1 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{c\geq 16\pi\sqrt{Na}} S(n,a;c) \frac{1}{c} \]

\[ \times \sum_{4|k} u \left( \frac{k-1}{K} \right) V(n,k)U(ab^2,k)J_{k-1} \left( \frac{4\pi\sqrt{na}}{c} \right) \]

and

\[ R_2 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{c< 16\pi\sqrt{Na}} S(n,a;c) \frac{1}{c} \]

\[ \times \sum_{4|k} u \left( \frac{k-1}{K} \right) V(n,k)U(ab^2,k)J_{k-1} \left( \frac{4\pi\sqrt{na}}{c} \right). \]

Then \( E_{N,M} = R_1 + R_2 \). We will show \( R_1 \) is negligible and \( R_2 \ll K^{\varepsilon} \) in the next two sections.

3.3. Estimate of \( R_1 \). For \( c \geq 16\pi\sqrt{Na} \) and \( N \leq n \leq 2N \), we have

\[ x = \frac{4\pi\sqrt{na}}{c} \leq \frac{1}{4} \left( \frac{n}{N} \right)^{1/2} < e^{-1}. \]

Using Weil’s bound for Kloosterman sums, the bound \( J_{k-1}(x) \ll x^{k-1} \) for \( 0 < x < 1 \) and

\[ \sum_{n\leq x} |\lambda_f(n)| \ll x, \]

we have

\[ R_1 \ll \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \left| w \left( \frac{n}{N} \right) \right| \left| h \left( \frac{ab^2}{M} \right) \right| \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{c\geq 16\pi\sqrt{Na}} c^{-1/2+\varepsilon} (n,a,c)^{1/2} \]

\[ \times \sum_{4|k} \left| u \left( \frac{k-1}{K} \right) \right| \left( 4\pi\sqrt{na} \right)^{k-1} \]

\[ \ll \sum_{N \leq n \leq 2N} \frac{1}{n^{1/2}} \sum_{M \leq ab^2 \leq 2M} \frac{|\lambda_f(a)|}{(ab^2)^{1/2}} \sum_{c\geq 16\pi\sqrt{Na}} c^{\varepsilon} \left( 4\pi\sqrt{na} \right)^2 \sum_{4|k, k \sim K} e^{-\left( k-3 \right)} \]

\[ \ll K^{-A} \]

for any \( A > 0 \).
4. Estimate of \( R_2 \)

**Lemma 8.** Fix a function \( g(\xi) \in C_c^\infty(0, \infty) \). We have
\[
\sum_{4|k} g(k-1)J_{k-1}(x) = -\frac{1}{2} F_1(x) - \frac{i}{2} F_2(x),
\]
where
\[
F_1(x) = \int_{-\infty}^{\infty} \hat{g}(t) \sin(x \cos(2\pi t)) dt,
\]
\[
F_2(x) = \int_{-\infty}^{\infty} \hat{g}(t) \sin(x \sin(2\pi t)) dt
\]
and
\[
\hat{g}(t) = \int_{-\infty}^{\infty} g(\xi) e(t\xi) d\xi
\]
is the Fourier transform of \( g(\xi) \).

**Proof.** See [Iw, pp. 85-87]. □

Let \( g(\xi) = u(\frac{\xi}{K})V(n, \xi + 1)U(ab^2, \xi + 1) \). By Lemma 9, \( R_2 = -\frac{1}{2} H_1 - \frac{i}{2} H_2 \), where
\[
H_1 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w\left( \frac{n}{N} \right) h\left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{(ab^2)^{1/2}} \sum_{c<16\pi\sqrt{Na}} S(n, a; c) \frac{F_1\left( \frac{4\pi\sqrt{na}}{c} \right)}{c}
\]
and
\[
H_2 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w\left( \frac{n}{N} \right) h\left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{(ab^2)^{1/2}} \sum_{c<16\pi\sqrt{Na}} S(n, a; c) \frac{F_2\left( \frac{4\pi\sqrt{na}}{c} \right)}{c}.
\]

Let \( g_0(\xi) = u(\xi)V(n, K\xi + 1)U(ab^2, K\xi + 1) \).

An easy change of variables gives \( \hat{g}(t) = K \hat{g}_0(Kt) \) and
\[
F_1(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) \sin \left( x \cos \left( \frac{2\pi t}{K} \right) \right) dt,
\]
\[
F_2(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) \sin \left( x \sin \left( \frac{2\pi t}{K} \right) \right) dt.
\]

4.1. Estimate of \( H_1 \).

**Lemma 9.** For any \( \varepsilon > 0 \), we have
\[
H_1 \ll K^{-1+\varepsilon}.
\]

**Proof.** Let
\[
W_1(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) e\left( \frac{x}{2\pi} \left( 1 - \cos \frac{2\pi t}{K} \right) \right) dt;
\]
then
\[
F_1(x) = -i e\left( \frac{x}{2\pi} \right) W_1(x) + e\left( \frac{x}{2\pi} \right) W_1(-x).
\]
Since the contribution to $W_1(x)$ from $|t| \geq K^\varepsilon$ is negligible, we only need to consider $|t| \leq K^\varepsilon$. Now we expand $\cos(\frac{2\pi x t}{K})$ into the Taylor series to get

$$W_1(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) e \left( \frac{\pi x t^2}{K^2} \right) dt + O \left( \int_{-\infty}^{\infty} |\hat{g}_0(t)| \frac{|x||t|^4}{K^4} dt \right).$$

Recall that $\hat{g}_0^{(a)}(t) = (-2\pi i t)^a \hat{g}_0(t)$. Hence

(4.1) $W_1(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) e \left( \frac{\pi x t^2}{K^2} \right) dt + O \left( \frac{|x|}{K^4} \right).$

Using (2.1) and (3.4), the $O$-term in (4.1) contributes at most

$$\sum_{N \leq n \leq 2N} \sum_{M \leq ab^2 \leq 2M} \frac{1}{n^{1/2}} \frac{1}{(ab^2)^{1/2}} \sum_{c < 16\sqrt{NaNa}} \frac{|\lambda_f(a)|}{c} \frac{|S(n,a;c)|}{\sqrt{Na}} K^{-4} \ll K^{-4} \sum_{M \leq ab^2 \leq 2M} \frac{|\lambda_f(a)|}{b} \sum_{c < 16\sqrt{NaNa}} \frac{1}{c^2} \sum_{N \leq n \leq 2N} |S(n,a;c)| \ll NMK^{-4} \ll K^{-1+\varepsilon}.$$

By Parseval and [GR 3.691, part 1],

$$\int_{-\infty}^{\infty} \hat{g}_0(t) e \left( \frac{\pi x t^2}{K^2} \right) dt = \frac{1 + i}{2} \int_{0}^{\infty} g_0(t) \frac{1}{\sqrt{\pi x t}} e \left( \frac{-t^2}{4\pi x} \right) dt$$

$$= \frac{1 + i}{2\sqrt{\pi}} \frac{K}{\sqrt{|x|}} \int_{0}^{\infty} g_0(t) e \left( \frac{-K^2 t^2}{4\pi x} \right) dt$$

$$\ll \left( \frac{K^2}{|x|} \right)^{-A},$$

for any $A > 0$ (by using integration by parts).

Note that for $|x| = \frac{4\pi \sqrt{NaNa}}{c} \ll N^{1/2} M^{1/2}$,

$$\frac{K^2}{|x|} \gg K^2 N^{-1/2} M^{-1/2} \gg K^{1/2-\varepsilon}.$$

Hence the contribution from the integral in (4.1) to $H_1$ is negligible.

4.2. Estimate of $H_2$. To estimate $H_2$, we need the following Voronoi formula on $GL(2)$ ([12]).

**Proposition 2.** Let $F(\xi)$ be a smooth and compactly supported function on $\mathbb{R}^+$. For any integers $c \geq 1$ and $(c, d) = 1$ we have

$$\sum_{n} \lambda_f(n) e \left( \frac{dn}{c} \right) F(n) = \sum_{r} \lambda_f(r) e \left( \frac{-d \bar{r}}{c} \right) \tilde{F}(r),$$

where $d \bar{d} \equiv 1 \pmod{c}$ and $\tilde{F}(r)$ is the Hankel-type transform

$$\tilde{F}(y) = 2\pi c^{-1} \int_{0}^{\infty} F(\xi) J_{1-1} \left( \frac{4\pi \sqrt{\xi y}}{c} \right) d\xi,$$

and where $J_{\nu}(z)$ is the usual Bessel function.

**Lemma 10.** For any $\varepsilon > 0$, we have

$$H_2 \ll K^{\varepsilon}. $$
Proof. Let

\[ W_2(x) = \int_{-\infty}^{\infty} \hat{g}_0(t)e \left( \frac{xt}{2\pi} \sin \frac{2\pi t}{K} \right) dt; \]

then

\[ F_2(x) = \frac{W_2(x) - W_2(-x)}{2i}. \]

Since the contribution to \( W_2(x) \) from \(|t| \geq K^\varepsilon\) is negligible, we only need to consider \(|t| \leq K^\varepsilon\). Now expanding \( \sin(\frac{2\pi t}{K}) \) into the Taylor series and using \( \hat{g}_0(x) = g_0(-x) \), we get

\[
W_2(x) = \int_{-\infty}^{\infty} \hat{g}_0(t)e \left( \frac{xt}{K} \right) dt + O \left( \int_{-\infty}^{\infty} |\hat{g}_0(t)| \frac{|x|}{K^2} |t|^3 dt \right)
\]

(4.2)

\[
= g_0 \left( \frac{-x}{K} \right) + O \left( \frac{|x|}{K^3} \right).
\]

Using (2.1) and (3.1), the \( O \)-term in (4.2) contributes at most

\[
K^{-3}NM \ll K^\varepsilon.
\]

Since \( g_0(\xi) \) has compact support, \( g_0 \left( \frac{-x}{R} \right) \) is nonzero only when \(|x| \sim K\). For \(|x| = \frac{4\pi \sqrt{na}}{K}\), \( g_0 \left( \frac{-x}{K} \right) \) is nonzero only when \( c \sim \frac{4\pi \sqrt{na}}{K} \sim \frac{\sqrt{NM}}{K^\varepsilon} \) and the contribution to \( H_2 \) is

\[
\sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{c \sim \frac{\sqrt{NM}}{K^\varepsilon}} S(n, a; c) g_0 \left( \frac{4\pi \sqrt{na}}{Kc} \right).
\]

Note that

\[
g_0 \left( \frac{4\pi \sqrt{na}}{Kc} \right) = u \left( \frac{4\pi \sqrt{na}}{Kc} \right) V \left( n, \frac{4\pi \sqrt{na}}{c} + 1 \right) U \left( \frac{ab^2}{c}, \frac{4\pi \sqrt{na}}{c} + 1 \right).
\]

By Lemma 2 and Lemma 3, we only need to consider the contribution from

\[
u \left( \frac{4\pi \sqrt{na}}{Kc} \right) V_1 \left( \frac{n}{4\pi \sqrt{na}c + 1} \right) U_1 \left( \frac{ab^2}{c}, \frac{4\pi \sqrt{na}}{c} + 1 \right)^{1/2}
\]

in (4.3) since the \( O \)-term is smaller. Let

\[
\bar{H}_2 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{c \sim \frac{\sqrt{NM}}{K^\varepsilon}} S(n, a; c) u \left( \frac{4\pi \sqrt{na}}{Kc} \right) V_1 \left( \frac{n}{4\pi \sqrt{na}c + 1} \right) U_1 \left( \frac{ab^2}{c}, \frac{4\pi \sqrt{na}}{c} + 1 \right)^{1/2}.
\]

After opening the Kloosterman sum and applying Proposition 2 to sum over \( a \) with

\[
F(\xi) = h \left( \frac{\xi b^2}{M} \right) u \left( \frac{4\pi \sqrt{n\xi}}{Kc} \right) V_1 \left( \frac{n}{4\pi \sqrt{n\xi}c + 1} \right) U_1 \left( \frac{\xi b^2}{c}, \frac{4\pi \sqrt{n\xi}}{c} + 1 \right)^{1/2} \xi^{-1/2},
\]

we have

\[
\bar{H}_2 = \sum_{n=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c \sim \frac{\sqrt{NM}}{K^\varepsilon} (\text{mod } c)} w \left( \frac{n}{N} \right) \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{r=1}^{\infty} \lambda_f(r) e \left( \frac{nd}{c} \right) \sum_{r=1}^{\infty} \lambda_f(r) e \left( \frac{-rd}{c} \right) \bar{F}(r).
\]
Note that $F(\xi)$ has support contained in $[\frac{M}{16}, \frac{2M}{16}]$ and
\[(4.3) \quad F^{(i)}(\xi) \ll \left(\frac{M}{b^2}\right)^{-1/2-i}.
\]
Using the recurrence formula
\[
\frac{d}{dz}(z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z)
\]
and the bound $J_\nu(z) \ll (1+z)^{-1/2}$, we have
\[(4.4) \quad \hat{F}(y) \ll c^{-1}y^{-7/4}\frac{N^{7/4}M^{1/2}}{K^{7/2}b} \ll y^{-7/4}\frac{N^{3/4}}{K^{5/2}}
\]
by integration by parts three times and (4.3). By (4.4) and Deligne’s bound $\lambda(r) \ll r^{\varepsilon}$, we have
\[
\tilde{H}_2 \ll \sum_{N \leq n \leq 2N} \sum_{b=1}^{\infty} \sum_{c \equiv \pm 1 (mod \ b)} \frac{1}{n^{1/2}bc} \sum_{r=1}^{\infty} r^{-7/4+\varepsilon} \frac{N^{3/4}}{K^{5/2}}
\ll N^{7/4}M^{1/2}K^{-7/2} \ll K^{-3/4+\varepsilon}.
\]

5. A NOTE ON SIMULTANEOUS NONVANISHING OF $GL(3) \times GL(2)$ AND $GL(2)$ $L$-FUNCTIONS

Let $F$ be a fixed $GL(3)$ Maass-Hecke cusp form. Li [Li] and Khan [Kh] proved that there are infinitely many $GL(2)$ cusp forms $g$ such that $L(\frac{1}{2}, F \times g)L(\frac{1}{2}, g) \neq 0$. Their proofs were to establish an asymptotic formula like Theorem 2 which is very delicate in their cases. However if we just want to prove that there exists one $g$ such that $L(\frac{1}{2}, F \times g)L(\frac{1}{2}, g) \neq 0$ for the special case $F = \text{sym}^2 f$, where $f$ is a $GL(2)$ cusp form, it is much easier. Actually, it is a consequence of Watson’s formula [Wa]. More precisely, we have the following result.

**Theorem 3.** Let $f$ be a fixed holomorphic Hecke cusp form of weight $k$ for $SL(2, \mathbb{Z})$. Then there exists $g \in H_{2k}$ such that
\[
(5.1) \quad L(\frac{1}{2}, \text{sym}^2 f \times g)L(\frac{1}{2}, g) \neq 0.
\]

**Proof.** Since $f^2$ is a holomorphic cusp of weight $2k$, we can express it as
\[
f^2 = \sum_{g \in H_{2k}} \left\langle f^2, g \| g \right\rangle \frac{g}{\| g \|}.
\]
Using Watson’s formula [Wa] Theorem 3]
\[(5.2) \quad \frac{|\left\langle f^2, g \right\rangle|^2}{\langle f, f \rangle^2 \langle g, g \rangle} = \frac{\Lambda(\frac{1}{2}, f \times f \times g)}{4\Lambda(1, \text{sym}^2 f)^2\Lambda(1, \text{sym}^2 g)}
\]
(here $\Lambda(s, f \times f \times g), \Lambda(s, \text{sym}^2 f)$ and $\Lambda(s, \text{sym}^2 g)$ are completed $L$-functions) and the factorization $L(s, f \times f \times g) = L(s, \text{sym}^2 f \times g)L(s, g)$, we deduce that
\[(5.3) \quad 0 \neq \left\langle f^2, f^2 \right\rangle = \sum_{g \in H_{2k}} \frac{|\left\langle f^2, g \right\rangle|^2}{\langle g, g \rangle} = C_{k,f} \sum_{g \in H_{2k}} \frac{L(\frac{1}{2}, \text{sym}^2 f \times g)L(\frac{1}{2}, g)}{L(1, \text{sym}^2 g)},
\]
where $C_{k,f}$ is a nonzero constant which depends on $k$ and $f$. The theorem follows from (5.3). \qed
ACKNOWLEDGEMENT

The author would like to thank the referee for valuable comments.

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