PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 7, July 2014, Pages 2517–2527
S 0002-9939(2014)11919-8
Article electronically published on March 27, 2014

ASYMMETRIC L_p-DIFFERENCE BODIES

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(Communicated by Michael Wolf)

Abstract. Lutwak introduced the L_p-difference body of a convex body as the Firey L_p-combination of the body and its reflection at the origin. In this paper, we define the notion of asymmetric L_p-difference bodies and study some of their properties. In particular, we determine the extremal values of the volumes of asymmetric L_p-difference bodies and their polars, respectively.

1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}^n_o$ and $\mathcal{K}^n_{os}$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and denote by $V(K)$ the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we write $\omega_n = V(B)$.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \longrightarrow (-\infty, +\infty)$, is defined by (see [6])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

From (1.1), it follows that if $\phi \in GL(n)$, then for all $u \in S^{n-1}$ (see [6]),

$$h(\phi K, u) = h(K, \phi' u), \tag{1.2}$$

where $GL(n)$ denotes the group of general (non-singular) linear transformations and ϕ' denotes the transpose of ϕ. Moreover, for all $u \in S^{n-1}$, we have that $h(-K, u) = h(K, -u)$ and $h(K + x, u) = h(K, u) + x \cdot u$ with $x \in \mathbb{R}^n$.

For $K, L \in \mathcal{K}^n$, and $\lambda, \mu \geq 0$ (not both zero), the Minkowski linear combination, $\lambda K + \mu L \in \mathcal{K}^n$, of K and L is defined by (see [6], [23])

$$h(\lambda K + \mu L, \cdot\,) = \lambda h(K, \cdot) + \mu h(L, \cdot),$$

where $\lambda K = \{\lambda x : x \in K\}$.

For $K, L \in \mathcal{K}^n_o$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the Firey L_p-combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}^n_o$, of K and L is defined by (see [4])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot\,)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p, \tag{1.3}$$

where the "$\cdot$" in $\lambda \cdot K$ denotes the Firey scalar multiplication.

Received by the editors April 4, 2011 and, in revised form, August 1, 2011 and July 2, 2012.
2010 *Mathematics Subject Classification.* Primary 52A40, 52A20.
Key words and phrases. Asymmetric L_p-difference body, Firey L_p-combination, extremum.
The authors' research was supported in part by the Natural Science Foundation of China (grants No. 11371224, 11161019) and Science Foundation of China Three Gorges University.

For the Firey L_p-combination, Firey ([5]) proved the following Brunn-Minkowski inequality.

Theorem A. *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), then*

$$V(\lambda \cdot K +_p \mu \cdot L)^{\frac{p}{n}} \geq \lambda V(K)^{\frac{p}{n}} + \mu V(L)^{\frac{p}{n}}, \tag{1.4}$$

with equality for $p = 1$ if and only if K and L are homothetic, and equality for $p > 1$ if and only if K and L are dilates.

For $K \in \mathcal{K}^n$, the difference body, ΔK, of K is defined by (see [6])

$$\Delta K = \frac{1}{2}K + \frac{1}{2}(-K).$$

The notion of L_p-difference bodies was introduced by Lutwak (see [15]). For $K \in \mathcal{K}_o^n$, $p \geq 1$, the L_p-difference body, $\Delta_p K$, of K is defined by

$$\Delta_p K = \frac{1}{2} \cdot K +_p \frac{1}{2} \cdot (-K). \tag{1.5}$$

Clearly, for every $K \in \mathcal{K}_o^n$ and $p \geq 1$, $\Delta_p K \in \mathcal{K}_{os}^n$ and $\Delta_1 K = \Delta K$.

In [15], Lutwak obtained the following inequality for L_p-difference bodies:

Theorem B. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, then*

$$V(\Delta_p K) \geq V(K), \tag{1.6}$$

with equality if and only if $K \in \mathcal{K}_{os}^n$.

For $p = 1$ the reverse inequality for the volume of difference bodies is the well-known Rogers-Shephard inequality (see [20]). For the case $p > 1$ an L_p-Rogers-Shephard inequality was recently established in dimension $n = 2$ by Bianchini and Colesanti (see [3]).

In this article, we extend the notion of L_p-difference bodies to a one parameter family of L_p-difference bodies which we call asymmetric L_p-difference bodies and study some of their properties and determine the extremum values of their volume and the volume of their polars, respectively.

For $K \in \mathcal{K}_o^n$, $p \geq 1$ and real $\tau \in [-1, 1]$, the asymmetric L_p-difference body $\Delta_p^\tau K$ of K is defined by

$$h^p(\Delta_p^\tau K, \cdot) = f_1(\tau) h^p(K, \cdot) + f_2(\tau) h^p(-K, \cdot), \tag{1.7}$$

i.e.

$$\Delta_p^\tau K = f_1(\tau) \cdot K +_p f_2(\tau) \cdot (-K), \tag{1.8}$$

where

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.9}$$

Note that the above functions $f_1(\tau)$ and $f_2(\tau)$ were first defined in [8], [11]. They are chosen such that $\Delta_p^\tau B = B$ for all $p \geq 1$ and $\tau \in [-1, 1]$.

From (1.9), we have that

$$f_1(\tau) + f_2(\tau) = 1, \tag{1.10}$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau). \tag{1.11}$$

In addition, by (1.5), (1.8) and (1.9) we have that if $\tau = 0$, then $\Delta_p^0 K = \Delta_p K$, and if $\tau = \pm 1$, then $\Delta_p^{+1} K = K, \Delta_p^{-1} K = -K$.

We note that asymmetric operators of this type and related maps compatible with linear transformations appear naturally in the theory of valuations in connection with isoperimetric and analytic inequalities (see e.g. [1,2,7,9,10,12,13,17–19, 21,22]). The main goal of this paper is to give the extremum values of the volume of asymmetric L_p-difference bodies and their polars.

Theorem 1.1. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$V(\Delta_p K) \geq V(\Delta_p^\tau K) \geq V(K). \tag{1.12}$$

If K is not origin-symmetric and $p > 1$ (or K is not central if $p = 1$), there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.

Note that inequality (1.12) strengthens and directly implies inequality (1.6).

In the following we denote by $\Delta_p^{\tau,*}K$ the polar body of the asymmetric L_p-difference body $\Delta_p^\tau K$ of K.

Theorem 1.2. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$V(\Delta_p^* K) \leq V(\Delta_p^{\tau,*} K) \leq V(K^*). \tag{1.13}$$

If K is not origin-symmetric, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.

Since $\Delta_p K \in \mathcal{K}_{os}^n$, using Theorems 1.1 and the Blaschke-Santaló inequality for symmetric convex bodies, we deduce

Theorem 1.3. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, then*

$$V(K)V(\Delta_p^* K) \leq \omega_n^2, \tag{1.14}$$

with equality if and only if K is an ellipsoid centered at the origin.

From Theorem 1.1 and the Blaschke-Santaló inequality for general convex bodies we obtain

Theorem 1.4. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$V(K)V(\Delta_p^{\tau,c} K) \leq \omega_n^2, \tag{1.15}$$

with equality if and only if K is an ellipsoid. Here $\Delta_p^{\tau,c}K = (\Delta_p^\tau K)^c$.

The proofs of Theorems 1.1−1.4 are given in section 4 of this paper. In section 3, we derive properties of asymmetric L_p-difference bodies needed in the proof of the main results.

2. Background material

2.1. **Radial functions and polar bodies.** If K is a compact star-shaped set (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0,+\infty)$, is defined by (see [6])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $E \subseteq \mathbb{R}^n$ is non-empty, the polar set E^* of E is defined by (see [6])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in E\}. \tag{2.1}$$

From definition (2.1), it follows that if $K \in \mathcal{K}_o^n$, then

$$h_{K^*} = \frac{1}{\rho_K} \quad \text{and} \quad \rho_{K^*} = \frac{1}{h_K}. \tag{2.2}$$

For $K \in \mathcal{K}_{os}^n$, the well-known Blaschke-Santaló inequality states (see [6]):

Theorem C. *If $K \in \mathcal{K}_{os}^n$, then*

$$V(K)V(K^*) \leq \omega_n^2, \tag{2.3}$$

with equality if and only if K is an ellipsoid.

For a general convex body $K \in \mathcal{K}_o^n$, the Blaschke-Santaló inequality takes the following form (see [14]):

Theorem D. *If $K \in \mathcal{K}^n$, then*

$$V(K)V(K^c) \leq \omega_n^2, \tag{2.4}$$

with equality if and only if K is an ellipsoid. Here, K^c denotes the polar body of K with respect to the centroid of K, i.e., $K^c = (K - cent(K))^*$.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination $\lambda \star K +_{-p} \mu \star L \in \mathcal{S}_o^n$ of K and L is defined by (see [16])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}. \tag{2.5}$$

Note that for convex bodies, L_p-harmonic radial combinations were first studied by Firey (see [4]).

From (1.3), (2.2) and (2.5), it follows that if $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), then

$$(\lambda \cdot K +_p \mu \cdot L)^* = \lambda \star K^* +_{-p} \mu \star L^*. \tag{2.6}$$

2.2. **L_p-dual mixed volumes.** Using L_p-harmonic radial combinations, Lutwak (see [16]) introduced the notion of an L_p-dual mixed volume. For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\varepsilon > 0$, the L_p-dual mixed volume $\widetilde{V}_{-p}(K, L)$ of K and L is defined by

$$\frac{n}{-p} \widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the L_p-dual mixed volume:

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) dS(u). \tag{2.7}$$

Here integration is with respect to spherical Lebesgue measure.

From definition (2.7), it follows immediately that for each $K \in \mathcal{S}_o^n$ and $p \geq 1$,

$$\widetilde{V}_{-p}(K, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(u) dS(u). \tag{2.8}$$

The L_p-dual Minkowski inequality can be stated as follows (see [16]):

Theorem E. *If $K, L \in \mathcal{S}_o^n$ and $p \geq 1$, then*

$$\widetilde{V}_{-p}(K, L) \geq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}}, \tag{2.9}$$

with equality if and only if K and L are dilates.

From Theorem E, Lutwak (see [16]) deduced the following dual Brunn-Minkowski inequality:

Theorem F. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), then*

$$V(\lambda \star K +_{-p} \mu \star L)^{-\frac{p}{n}} \geq \lambda V(K)^{-\frac{p}{n}} + \mu V(L)^{-\frac{p}{n}}, \tag{2.10}$$

with equality if and only if K and L are dilates.

3. Properties of asymmetric L_p-difference bodies

In this section, we establish several properties of asymmetric L_p-difference bodies needed in section 4.

Theorem 3.1. *For $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1, 1]$, if $\phi \in GL(n)$, then*

$$\Delta_p^\tau \phi K = \phi \Delta_p^\tau K. \tag{3.1}$$

Proof. From (1.2) and (1.7), we have that for all $u \in S^{n-1}$,

$$\begin{aligned} h^p(\Delta_p^\tau \phi K, u) &= f_1(\tau) h^p(\phi K, u) + f_2(\tau) h^p(-\phi K, u) \\ &= f_1(\tau) h^p(K, \phi' u) + f_2(\tau) h^p(-K, \phi' u) \\ &= h^p(\Delta_p^\tau K, \phi' u) = h^p(\phi \Delta_p^\tau K, u). \end{aligned}$$

□

Theorem 3.2. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$ and $\tau \neq 0$, then $\Delta_p^\tau K = \Delta_p^{-\tau} K$ for $p > 1$ (or $\Delta_p^\tau K = \Delta_p^{-\tau} K + x$ with $x \in \mathbb{R}^n$ for $p = 1$) if and only if K is an origin-symmetric convex body (or K is a centrally symmetric convex body for $p = 1$).*

Proof. From (1.8) and (1.11), we get that for all $u \in S^{n-1}$,

$$h^p_{\Delta_p^\tau K}(u) = f_1(\tau) h^p_K(u) + f_2(\tau) h^p_{-K}(u), \tag{3.2}$$

$$h^p_{\Delta_p^{-\tau} K}(u) = f_2(\tau) h^p_K(u) + f_1(\tau) h^p_{-K}(u). \tag{3.3}$$

Hence, from (1.10), (3.2) and (3.3), if K is an origin-symmetric convex body for $p > 1$ (or K is a centrally symmetric convex body for $p = 1$), i.e. $K = -K$ for $p > 1$ (or $K = -K + y$ with $y \in \mathbb{R}^n$ for $p = 1$), then for all $u \in S^{n-1}$, we have $h^p(\Delta_p^\tau K, u) = h^p(\Delta_p^{-\tau} K, u)$ for $p > 1$ (or $h^p(\Delta_p^\tau K, u) = h^p(\Delta_p^{-\tau} K, u) + x \cdot u$ for $p = 1$, where $x = (f_1(\tau) - f_2(\tau))y$). Thus, we have $\Delta_p^\tau K = \Delta_p^{-\tau} K$ for $p > 1$ (or $\Delta_p^\tau K = \Delta_p^{-\tau} K + x$ for $p = 1$).

Conversely, if $\Delta_p^\tau K = \Delta_p^{-\tau} K$ for $p > 1$, then (3.2) and (3.3) yield that

$$[f_1(\tau) - f_2(\tau)] h^p_K(u) = [f_1(\tau) - f_2(\tau)] h^p_{-K}(u) \tag{3.4}$$

for all $u \in S^{n-1}$. Since $f_1(\tau) - f_2(\tau) \neq 0$ when $\tau \neq 0$, it follows from (3.4) that $h^p_K(u) = h^p_{-K}(u)$ for all $u \in S^{n-1}$; i.e. K is an origin-symmetric convex body.

If $\Delta_p^\tau K = \Delta_p^{-\tau} K + x$ with $x \in \mathbb{R}^n$ for $p = 1$, then from (3.2) and (3.3) we have

$$[f_1(\tau) - f_2(\tau)] h_K(u) = [f_1(\tau) - f_2(\tau)] h_{-K}(u) + x \cdot u.$$

This yields $h_K(u) = h_{-K}(u) + y \cdot u$ for all $u \in S^{n-1}$, where $y = x/(f_1(\tau) - f_2(\tau))$ when $\tau \neq 0$. Hence we have $K = -K + y$ with $y \in \mathbb{R}^n$; i.e. K is a centrally symmetric convex body. □

From Theorem 3.2, we deduce

Corollary 3.1. *For $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, if K is not an origin-symmetric convex body (or K is not a centrally symmetric convex body), then $\Delta_p^\tau K = \Delta_p^{-\tau} K$ for $p > 1$ (or $\Delta_p^\tau K = \Delta_p^{-\tau} K + x$ with $x \in \mathbb{R}^n$ for $p = 1$) if and only if $\tau = 0$.*

Theorem 3.3. *If $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$\Delta_p^{-\tau} K = \Delta_p^\tau(-K) = -\Delta_p^\tau K. \tag{3.5}$$

Proof. From (3.3), we have that for any $u \in S^{n-1}$,

$$\begin{aligned} h^p_{\Delta_p^{-\tau} K}(u) &= f_2(\tau) h_K^p(u) + f_1(\tau) h_{-K}^p(u) \\ &= f_1(\tau) h_{-K}^p(u) + f_2(\tau) h_{-(-K)}^p(u) = h^p_{\Delta_p^\tau(-K)}(u). \end{aligned}$$

Moreover, using (1.7), we obtain for any $u \in S^{n-1}$,

$$\begin{aligned} h^p(-\Delta_p^\tau K, u) &= h^p(\Delta_p^\tau K, -u) \\ &= h^p(f_1(\tau) \cdot K +_p f_2(\tau) \cdot (-K), -u) \\ &= f_1(\tau) h^p(K, -u) + f_2(\tau) h^p(-K, -u) \\ &= f_1(\tau) h^p(-K, u) + f_2(\tau) h^p(-(-K), u) \\ &= h^p(f_1(\tau) \cdot (-K) +_p f_2(\tau) \cdot (-(-K)), u) \\ &= h^p(\Delta_p^\tau(-K), u). \end{aligned}$$

This yields the right hand equality of (3.5). □

Theorem 3.4. *If $K \in \mathcal{K}_{os}^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$\Delta_p^\tau K = K. \tag{3.6}$$

Proof. Since $K \in \mathcal{K}_{os}^n$, i.e. $K = -K$, by (3.2) and (1.10) we have

$$h^p_{\Delta_p^\tau K}(u) = f_1(\tau) h_K^p(u) + f_2(\tau) h_K^p(u) = h_K^p(u),$$

for all $u \in S^{n-1}$. This gives (3.6). □

Theorem 3.4 immediately implies

Corollary 3.2. *If $K, L \in \mathcal{K}_{os}^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$\Delta_p^\tau K = \Delta_p^\tau L \quad \Longleftrightarrow \quad K = L.$$

4. Proofs of the main results

Proof of Theorem 1.1. Using (1.8), (1.10) and inequality (1.4), we obtain

$$\begin{aligned} V(\Delta_p^\tau K)^{\frac{p}{n}} &= V(f_1(\tau)\cdot K +_p f_2(\tau)\cdot(-K))^{\frac{p}{n}} \\ &\geq f_1(\tau)V(K)^{\frac{p}{n}} + f_2(\tau)V(-K)^{\frac{p}{n}} = V(K)^{\frac{p}{n}}, \end{aligned}$$

i.e., $V(\Delta_p^\tau K) \geq V(K)$. This yields the right hand inequality of (1.12).

Obviously, equality holds in the right hand inequality of (1.12) when $\tau = \pm 1$. Hence, if $\tau \neq \pm 1$, then by the condition of equality in (1.4) we see that equality holds in the right hand inequality of (1.12) for $p > 1$ if and only if K and $-K$ are dilates (or for $p = 1$ if and only if K and $-K$ are homothetic). This yields $K = -K$ for $p > 1$ (or $K = -K + x$ with $x \in \mathbb{R}^n$ for $p = 1$); i.e. K is an origin-symmetric convex body for $p > 1$ (or K is a centrally symmetric convex body for $p = 1$).

Hence, we know that equality holds in the right hand inequality of (1.12) if and only if K is an origin-symmetric convex body for $p > 1$ (or K is a centrally symmetric convex body for $p = 1$) or $\tau = \pm 1$. This means that if K is not origin-symmetric for $p > 1$ (or K is not centrally symmetric for $p = 1$), then equality holds in the right hand inequality of (1.12) if and only if $\tau = \pm 1$.

Now we prove the left hand inequality of (1.12). From (1.8) and (1.10), it follows that for any $u \in S^{n-1}$,

$$\begin{aligned} &h^p(\Delta_p^\tau K, u) + h^p(\Delta_p^{-\tau} K, u) \\ &= f_1(\tau)h^p(K,u) + f_2(\tau)h^p(-K,u) + f_2(\tau)h^p(K,u) + f_1(\tau)h^p(-K,u) \\ &= h^p(K,u) + h^p(-K,u) = h^p(2\cdot\Delta_p K, u). \end{aligned}$$

Thus,

$$\Delta_p^\tau K +_p \Delta_p^{-\tau} K = 2\cdot\Delta_p K. \tag{4.1}$$

Notice that $\lambda \cdot Q = \lambda^{1/p} Q$. Hence,

$$V(2\cdot\Delta_p K)^{\frac{p}{n}} = 2V(\Delta_p K)^{\frac{p}{n}}.$$

Therefore, by (4.1), (3.1) and inequality (1.4) we have

$$\begin{aligned} 2V(\Delta_p K)^{\frac{p}{n}} &\geq V(\Delta_p^\tau K)^{\frac{p}{n}} + V(\Delta_p^{-\tau} K)^{\frac{p}{n}} \\ &= V(\Delta_p^\tau K)^{\frac{p}{n}} + V(-\Delta_p^\tau K)^{\frac{p}{n}} = 2V(\Delta_p^\tau K)^{\frac{p}{n}}, \end{aligned}$$

i.e., $V(\Delta_p K) \geq V(\Delta_p^\tau K)$. This is just the left hand inequality of (1.12).

From the condition of equality in inequality (1.4), we know that equality holds in the left hand inequality of (1.12) if and only if $\Delta_p^\tau K$ and $\Delta_p^{-\tau} K$ are dilates for $p > 1$ (or $\Delta_p^\tau K$ and $\Delta_p^{-\tau} K$ are homothetic for $p = 1$). Thus, $\Delta_p^\tau K = \Delta_p^{-\tau} K$ for $p > 1$ (or $\Delta_p^\tau K = \Delta_p^{-\tau} K + x$ with $x \in \mathbb{R}^n$ for $p = 1$). Hence, using Corollary 3.1, we see that if K is not origin-symmetric for $p > 1$ (or K is not centrally symmetric for $p = 1$), then equality holds in the left hand inequality of (1.12) if and only if $\tau = 0$. □

Proof of Theorem 1.2. From (1.8) and (2.6), we have

$$\Delta_p^{\tau,*} K = f_1(\tau)\star K^* +_{-p} f_2(\tau)\star(-K)^*. \tag{4.2}$$

Using (2.2), we show that for any $u \in S^{n-1}$,

$$h((-K)^*, u) = \frac{1}{\rho(-K,u)} = \frac{1}{\rho(K,-u)} = h(K^*, -u) = h(-K^*, u).$$

This yields $(-K)^* = -K^*$. Hence, (4.2) can be written as

$$\Delta_p^{\tau,*}K = f_1(\tau) \star K^* +_{-p} f_2(\tau) \star (-K^*). \tag{4.3}$$

This combined with (1.10) and inequality (2.10) yields

$$V(\Delta_p^{\tau,*}K)^{-\frac{p}{n}} \geq f_1(\tau)V(K^*)^{-\frac{p}{n}} + f_2(\tau)V(-K^*)^{-\frac{p}{n}} = V(K^*)^{-\frac{p}{n}},$$

i.e., $V(\Delta_p^{\tau,*}K) \leq V(K^*)$. This gives the right hand inequality of (1.13).

Note that equality holds in the right hand inequality of (1.13) when $\tau = \pm 1$. Hence, if $\tau \neq \pm 1$, then by the condition of equality in (2.10) we see that equality holds in the right hand inequality of (1.13) if and only if K^* and $-K^*$ are dilates. This yields $K = -K$, i.e. $K \in \mathcal{K}_{os}^n$.

Therefore, we know that equality holds in the right hand inequality of (1.13) if and only if $K \in \mathcal{K}_{os}^n$ or $\tau = \pm 1$. This means that if K is not an origin-symmetric convex body, then equality holds in the right hand inequality of (1.13) if and only if $\tau = \pm 1$.

Next, we give the proof of the left hand inequality of (1.13). Here, we use techniques from Haberl and Schuster (see [8]).

Using (2.8), we have that

$$V(\Delta_p^{\tau,*}K) = \frac{1}{n}\int_{S^{n-1}} \rho^n_{\Delta_p^{\tau,*}K}(u)dS(u) = \frac{1}{n}\int_{S^{n-1}} (\rho^{-p}_{\Delta_p^{\tau,*}K}(u))^{-\frac{n}{p}} dS(u).$$

By (4.3),

$$\rho^{-p}_{\Delta_p^{\tau,*}K}(u) = f_1(\tau)\rho^{-p}_{K^*}(u) + f_2(\tau)\rho^{-p}_{-K^*}(u) \tag{4.4}$$

for any $u \in S^{n-1}$. Hence, we can calculate the derivative of the function $V(\Delta_p^{\tau,*}K)$ with respect to τ as follows: For every $\tau \in [-1, 1]$,

$$\begin{aligned}\frac{\partial}{\partial\tau}(V(\Delta_p^{\tau,*}K)) &= -\frac{1}{p}\int_{S^{n-1}} (\rho^{-p}_{\Delta_p^{\tau,*}K}(u))^{-\frac{n+p}{p}} \frac{\partial}{\partial\tau}(\rho^{-p}_{\Delta_p^{\tau,*}K}(u))dS(u)\\ &= \frac{f(\tau)}{p}\int_{S^{n-1}} \rho^{n+p}_{\Delta_p^{\tau,*}K}(u)[\rho^{-p}_{K^*}(u) - \rho^{-p}_{-K^*}(u)]dS(u),\end{aligned}$$

where

$$f(\tau) = -f_1^{'}(\tau) = f_2^{'}(\tau) = -\frac{2(1-\tau^2)^{p-1}}{[(1+\tau)^p + (1-\tau)^p]^2} \leq 0, \quad \tau \in [-1, 1].$$

This combined with (2.7) yields

$$\frac{\partial}{\partial\tau}(V(\Delta_p^{\tau,*}K)) = \frac{n}{p}f(\tau)[\widetilde{V}_{-p}(\Delta_p^{\tau,*}K, K^*) - \widetilde{V}_{-p}(\Delta_p^{\tau,*}K, -K^*)].$$

Note that for $p > 1$,

$$f(\tau) = 0 \quad \Longleftrightarrow \quad \tau = \pm 1.$$

Thus, if

$$\frac{\partial}{\partial\tau}(V(\Delta_p^{\tau,*}K)) = 0,$$

we get $\tau = \pm 1$ for $p > 1$ or

$$\widetilde{V}_{-p}(\Delta_p^{\tau,*}K, K^*) = \widetilde{V}_{-p}(\Delta_p^{\tau,*}K, -K^*).$$

Since

$$V(\Delta_p^{\tau,*}K) \le V(K^*),$$

the function $V(\Delta_p^{\tau,*}K)$ attains its maximum at $\tau = \pm 1$. Thus the points where the minimum of $V(\Delta_p^{\tau,*}K)$ is attained are contained in $(-1,1)$. If $\tau = \bar{\tau}$ is such a point, then

$$\frac{\partial}{\partial \tau}(V(\Delta_p^{\tau,*}K))|_{\tau=\bar{\tau}} = 0$$

or equivalently

$$\widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, K^*) = \widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, -K^*). \tag{4.5}$$

Therefore, using (4.3), (2.8) and (2.7), we obtain

$$\begin{aligned} V(\Delta_p^{\bar{\tau},*}K) &= \widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, \Delta_p^{\bar{\tau},*}K) \\ &= \widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, f_1(\bar{\tau}) \star K^* +_{-p} f_2(\bar{\tau}) \star (-K^*)) \\ &= f_1(\bar{\tau})\widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, K^*) + f_2(\bar{\tau})\widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, -K^*). \end{aligned}$$

This together with (4.5) and inequality (2.9) yields

$$\begin{aligned} V(\Delta_p^{\bar{\tau},*}K) &= f_1(\bar{\tau})\widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, -K^*) + f_2(\bar{\tau})\widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, K^*) \\ &= \widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, f_1(\bar{\tau}) \star (-K^*) +_{-p} f_2(\bar{\tau}) \star (-(-K^*))) \\ &= \widetilde{V}_{-p}(\Delta_p^{\bar{\tau},*}K, \Delta_p^{\bar{\tau},*}(-K)) \\ &\ge V(\Delta_p^{\bar{\tau},*}K)^{\frac{n+p}{n}} V(\Delta_p^{\bar{\tau},*}(-K))^{-\frac{p}{n}}, \end{aligned}$$

i.e.

$$V(\Delta_p^{\bar{\tau},*}K) \le V(\Delta_p^{\bar{\tau},*}(-K)). \tag{4.6}$$

According to the condition of equality in (2.9), we know that equality holds in (4.6) if and only if $\Delta_p^{\bar{\tau},*}K$ and $\Delta_p^{\bar{\tau},*}(-K)$ are dilates.

Similarly,

$$V(\Delta_p^{\bar{\tau},*}(-K)) \le V(\Delta_p^{\bar{\tau},*}K), \tag{4.7}$$

with equality if and only if $\Delta_p^{\bar{\tau},*}K$ and $\Delta_p^{\bar{\tau},*}(-K)$ are dilates.

Thus (4.6) and (4.7) show that

$$V(\Delta_p^{\bar{\tau},*}K) = V(\Delta_p^{\bar{\tau},*}(-K))$$

and $\Delta_p^{\bar{\tau},*}K$ and $\Delta_p^{\bar{\tau},*}(-K)$ are dilates. This yields $\Delta_p^{\bar{\tau},*}K = \Delta_p^{\bar{\tau},*}(-K)$, i.e.

$$\rho^{-p}(\Delta_p^{\bar{\tau},*}K, u) = \rho^{-p}(\Delta_p^{\bar{\tau},*}(-K), u)$$

for any $u \in S^{n-1}$. Hence, by (4.4) we obtain

$$[f_1(\bar{\tau}) - f_2(\bar{\tau})][\rho^{-p}(K^*, u) - \rho^{-p}(-K^*, u)] = 0.$$

If $f_1(\bar{\tau}) - f_2(\bar{\tau}) = 0$, then we have $\bar{\tau} = 0$. This means $V(\Delta_p^{\tau,*}K)$ attains its minimum at $\tau = 0$, i.e.

$$V(\Delta_p^{\tau,*}K) \ge V(\Delta_p^{*}K).$$

So we get the left hand inequality of (1.13). If $\rho^{-p}(K^*, u) - \rho^{-p}(-K^*, u) = 0$ for all $u \in S^{n-1}$, then $K = -K$, i.e. $K \in \mathcal{K}_{os}^n$, which implies

$$V(\Delta_p^{\tau,*}K) = V(\Delta_p^{*}K).$$

Therefore, we know that equality holds in the left hand inequality of (1.13) if and only if $K \in \mathcal{K}_{os}^n$ or $\tau = 0$. This means that if K is not origin-symmetric, then equality holds in the left hand inequality of (1.13) if and only if $\tau = 0$. □

Proof of Theorem 1.3. Since $\Delta_p K \in \mathcal{K}_{os}^n$, we can combine with (1.12) the Blaschke-Santaló inequality (2.3) to obtain

$$V(K)V(\Delta_p^* K) \leq V(\Delta_p K)V(\Delta_p^* K) \leq \omega_n^2.$$

The equality conditions of inequalities (1.12) and (2.3) show that equality holds in (1.14) if and only if $K \in \mathcal{K}_{os}^n$ and $\Delta_p K$ is an ellipsoid. This and Theorem 3.4 indicate that equality holds in (1.14) if and only if K is an ellipsoid centered at the origin. □

Proof of Theorem 1.4. Since $K \in \mathcal{K}_o^n$, using inequality (1.12) and the general Blaschke-Santaló inequality (2.4), we obtain

$$V(K)V(\Delta_p^{\tau,c} K) \leq V(\Delta_p^{\tau} K)V(\Delta_p^{\tau,c} K) \leq \omega_n^2.$$

The equality conditions of inequalities (1.12) and (2.4) show that equality holds in (1.15) if and only if K is an ellipsoid. □

Acknowledgement

The authors are most grateful to the referees for the extraordinary attention they gave to our paper.

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