GL\(_2(\mathbb{O}_K)\)-INVARIANT LATTICES IN THE SPACE OF BINARY CUBIC FORMS WITH COEFFICIENTS IN THE NUMBER FIELD \(K\)

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Abstract. In 2008, Ohno, Taniguchi and Wakatsuki obtained a classification of all \(GL_2(\mathbb{Z})\)-invariant lattices in the space of binary cubic forms with coefficients in \(\mathbb{Q}\). In this paper, we aim to generalize their result by replacing the rational field with an arbitrary algebraic number field, \(K\). We conclude the paper by connecting the lattices described in our main result to a zeta function developed by Datskovsky and Wright, which yields a functional equation for certain Dirichlet series attached to the lattices.

1. Introduction

We begin our work with a discussion of the space of binary cubic forms over \(K\), where \(K\) is a field. This is necessary to describe the action of \(GL_2\) on this space and to define the zeta function mentioned above. To simplify our exposition, we introduce some notation.

**Notation 1.** Throughout this paper, \(V\) denotes the four dimensional affine space. Also, we let \(G\) denote the general linear group, \(GL_2\), and \(B\) will represent the subgroup of \(G\) consisting of lower triangular matrices. Thus \(V_K = \mathbb{K}^4\), \(G_K = GL_2(K)\), and \(B_K = \{A \in G_K : A\) is lower triangular\}.

The space of binary cubic forms over \(K\) is the set
\[
\{x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3 : x_i \in K\}.
\]
We identify the cubic form \(x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3\) with the point \(x = (x_1, x_2, x_3, x_4) \in V_K\), and we will denote the form as either \(x\) or \(F_x(u,v)\). The group \(G_K\) acts on the space of binary cubic forms by linear change of variables, modified to allow for scalar multiplication.

**Definition 1.** Let \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_K\), and let \(x \in V_K\). We define the action of \(g\) on \(x\) by
\[
g \cdot x = F_g \cdot x = F_{g \cdot x}(u,v) = \det(g)^{-1}F_x((u,v)(\begin{pmatrix} a & b \\ c & d \end{pmatrix})).
\]
The twist by \(\det(g)^{-1}\) ensures that if \(g = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\), then \(g \cdot x = ax\).

For a form \(x\), we let \(P(x)\) denote the discriminant of the polynomial \(F_x(u,1)\): for \(x = (x_1, x_2, x_3, x_4)\), we have
\[
P(x) = x_2^2x_3^2 + 18x_1x_2x_3x_4 - 4x_2^3x_4 - 4x_1x_3^3 - 27x_1^2x_4^2.
\]
Observe that for \( g \in G_K \), \( P(g \cdot x) = \det(g)^2 P(x) \). We call a form \( x \) nonsingular if \( P(x) \neq 0 \).

We will refer to the roots of the polynomial \( F_x(u, 1) \) as the roots of the form \( x \). \( K(x) \) will be the splitting field of \( F_x(u, 1) \) over \( K \). This is either a cyclic extension of \( K \) of degree 3 or less, or a degree 6 extension with Galois group \( S_3 \). In the latter case, we may think of \( K(x) \) as a Galois conjugacy class of noncyclic cubic extensions of \( K \). We now recall a proposition from Section 2 of [8], giving the orbits in our action of \( G_K \) on \( V_K \).

**Proposition 1.** The \( G_K \)-orbits in \( V_K \) are as follows:

- \( S_0 = \{0\} \) (the zero form),
- \( S_{1,K} = \{x \in V_K : x \text{ has a triple root}\} \),
- \( S_{2,K} = \{x \in V_K : x \text{ has a double root, as well as a simple root}\} \),
- \( V_K(K') = \{x \in V_K : P(x) \neq 0, K(x) = K'\} \).

In the fourth class of orbits, \( K' \) runs over all Galois conjugacy classes of extensions of \( K \) with degree at most 3.

Indeed, the works of Datskovsky and Wright [3, 2, 8] rely on the fact that the map \( x \to K(x) \) induces a one-to-one correspondence between the orbits of nonsingular binary cubic forms over \( K \) and the conjugacy classes of extensions of \( K \) of degree not exceeding 3.

The authors of [8] also chose standard representatives for each type of orbit, which we will use in a future calculation. For \( S_0, S_{1,K}, \) and \( S_{2,K} \), these are \((0,0,0,0), (1,0,0,0), \) and \((0,1,0,0)\), respectively. For nonsingular forms \( x \) with \( K(x) = K \), we choose \((0,1,0,0)\) as our standard representative. For forms such that \( K(x) : K = 2 \), with \( K \) nonarchimedean, we consider \( \theta \) such that \( K(x) = K(\theta) \), and pick \((0,1,\theta+\theta',\theta\theta')\), where \( \theta' \) is the Galois conjugate of \( \theta \) over \( K \). When \( K = \mathbb{R} \) and \( K(x) = \mathbb{C} \), we let \( x = \frac{1}{\sqrt{2}}(1,0,1,0) \). The exception here is to make sure that the standard representatives for the orbits over \( \mathbb{R} \) both have discriminant 1. When \( K(x) \) is a conjugacy class of cubic extensions, we again choose \( \theta \) which generates a member of this class over \( K \), and pick \((1,\theta+\theta'+\theta''',\theta\theta'+\theta\theta''+\theta''\theta',\theta\theta\theta'')\) to be our standard representative, where again, \( \theta', \theta'' \) are the conjugates of \( \theta \).

The stabilizers of our nonsingular forms are also known. The following proposition originally appeared in [8].

**Proposition 2.** Let \( x \in V_K \), with \( P(x) \neq 0 \).

1. If \( [K(x) : K] = 1 \), then \(|\text{Stab}_{G_K}(x)| = 6|\).
2. If \( [K(x) : K] = 2 \), then \(|\text{Stab}_{G_K}(x)| = 2|\).
3. If \( [K(x) : K] = 3 \) and \( K(x) \) is cyclic over \( K \), then \(|\text{Stab}_{G_K}(x)| = 3|\).
4. If \( K(x) \) is a conjugacy class of noncyclic extensions of \( K \), then \(|\text{Stab}_{G_K}(x)| = 1|\).

Now, let \( K \) be an algebraic number field. We let \( O_K \) be the ring of integers in the number field \( K \). \( M(K) \) will stand for the set of places of \( K \), while \( M_\infty(K) \) and \( M_0(K) \) will refer to the sets of infinite and finite places of \( K \), respectively. For \( \nu \in M(K) \), we let \( K_\nu \) denote the completion of \( K \) at \( \nu \), and if \( \nu \) is finite, we let \( O_\nu \) stand for the ring of integers in \( K_\nu \). Moreover, \( \mathbb{A}_K \) will stand for the ring of adeles of \( K \).

To define the zeta function, we first need to introduce invariant measures on \( K_\nu \) and \( G_{K_\nu} \). On \( K_\nu \), for finite \( \nu \), we choose the additive measure \( dx_\nu \), normalized so that the measure of \( O_\nu \) is 1, and the multiplicative measure \( d^* x_\nu \), which is
normalized so that the measure of $O^*_\nu$ is 1. If instead $\nu$ is infinite, $dx$ is the usual Lebesgue measure, and $d^*x$ is $\frac{dx}{x}$. We define our measure on $G_{K,\nu}$ in stages. First, we introduce a maximal compact subgroup, $U_{K,\nu}$ of $G_{K,\nu}$. When $\nu$ is real, we let $U_{K,\nu}$ be the group of orthogonal matrices in $GL_2(\mathbb{R})$. If $\nu$ is complex, $U_{K,\nu}$ will be the unitary group. When $\nu$ is finite, we let $U_{K,\nu} = GL_2(O_\nu)$. By the Iwasawa decomposition, we know that an element $g \in G_{K,\nu}$ can be written in the form

$$g = k \left( \frac{t}{u} \right) \frac{1}{c} \left( \frac{1}{c} \right),$$

for some $k \in U_{K,\nu}$, $t, u \in K^*_\nu$, $c \in K_{\nu}$. We define

$$a(t, u) = \left( \frac{1}{c} \right)$$

and

$$n(c) = \left( \frac{1}{c} \right).$$

We have chosen $U_{K,\nu}$ to be compact, so we have an invariant measure, $dk_{\nu}$, such that $U_{K,\nu}$ has measure 1. Observe that $a(t, u)n(c) \in B_{K,\nu}$, and every element of $B_{K,\nu}$ is of this form. We specify our measure on $B_{K,\nu}$ by the formula

$$\int_{B_{K,\nu}} f(b)db_{\nu} = \int_{U_{K,\nu}} \int_{K^*_\nu} \int_{K_{\nu}} \left| \frac{u}{t} \right| f(a(t, u)n(c))dc_{\nu}d^*t_{\nu}d^*u_{\nu},$$

where the absolute value is the standard choice on $K_{\nu}$. This enables us to give a Haar measure, $dg_{\nu}$, on $G_{K,\nu}$ via

$$\int_{G_{K,\nu}} f(g)dg_{\nu} = \int_{U_{K,\nu}} \int_{B_{K,\nu}} f(bk)db_{\nu}dk_{\nu}. $$

We define an additive measure on $\mathbb{A}_K$ via $\prod_{\nu \in M(K)} dx_{\nu}$. We similarly define a multiplicative measure on $\mathbb{A}_K$ and a measure on $GL_2(\mathbb{A}_K)$ as products of local measures.

With our measures defined, we may go on to define the adelic zeta function. Let $\Phi$ be a function on $V_K$ admitting a product $\Phi = \prod_{\nu \in M(K)} \Phi_\nu$ such that $\Phi_\nu$ is rapidly decreasing if $\nu$ is infinite or locally constant with compact support if $\nu$ is finite, and all but finitely many of the $\Phi_\nu$ are characteristic functions of $O^*_\nu$. We will refer to such $\Phi$ as Schwartz-Bruhat functions. Let $s \in \mathbb{C}$, and let $V'_{K}$ denote the set of nonsingular forms in $V_K$. The adelic zeta function is then defined as

$$Z(s, \Phi) = \int_{G_{\mathbb{A}_K}/G_K} \left| \det(g) \right|^s \sum_{x \in V'_{K}} \Phi(g \cdot x)dg,$$

where we use $|\det(g)|$ to denote the adelic absolute value of $|\det(g)|$. This definition is valid for $\text{Re}(s) > 2$, where the integral converges absolutely and locally uniformly.

Having defined the function, we proceed to describe some of its basic properties. As in $\mathbb{S}$, it admits a meromorphic continuation to the entire complex plane, except for simple poles at $s = 2, \frac{5}{3}, \frac{1}{3},$ and 0. One has a functional equation for the zeta function, obtained in $\mathbb{S}$ by means of the Poisson Summation Formula. We briefly recall the Fourier transform on $V_K$ used in $\mathbb{S}$. Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ be elements of $V_K$. We introduce the bilinear form

$$[x, y] = x_1y_4 - \frac{1}{3}x_2y_3 + \frac{1}{3}x_3y_2 - x_4y_1$$
and let $\langle \rangle$ be a nontrivial additive character on $K$. We let $dx$ be the Haar measure on $V_K$ self-dual with respect to the character $\langle [x,y] \rangle$ on $V^2_{k_K}$. If $\Phi$ is a Schwartz-Bruhat function on $V_K$, we define its Fourier transform by

$$\hat{\Phi}(y) = \int_{V_{k_K}} \Phi(x) \langle [x,y] \rangle \, dx.$$ 

The functional equation for the adelic zeta function associated to the space of binary cubic forms is then $Z(2-s, \hat{\Phi}) = Z(s, \Phi)$ (see [8]).

2. Overview of invariant lattices

In this section, we are going to describe the $GL_2(O_K)$-invariant lattices in $V_K$. We begin with a definition, following Weil [7].

**Definition 2.** Let $K$ be an algebraic number field, and let $O_K$ be the ring of (algebraic) integers of $K$. Let $V$ be a finite dimensional vector space over $K$. A lattice of $V$ is a finitely generated $O_K$-module in $V$ which contains a basis of $V$ over $K$.

If $K_\nu$ is the completion of $K$ at a finite prime $\nu$, we define lattices in $K_\nu$-vector spaces in the same way. The ring of integers $O_\nu$ of $K_\nu$ is a local ring, which makes working with it much simpler. Moreover, we have the following lemma, whose proof is given by Weil (Chapter 5 of [7]).

**Lemma 1.** Let $M_0(K)$ be the set of finite primes of $K$, and let $L \subseteq K^4$ be a lattice of $K^4$. For $\nu \in M_0(K)$, denote by $L_\nu$ the closure of $L$ in $K_\nu^4$. Then $L = \bigcap_{\nu \in M_0(K)} (L_\nu \cap K^4)$.

In light of the preceding lemma, we are able to carry out most of our work in the local fields and reconstruct our global classification from these results. Observe that for $K_\nu$, our lattices are free $O_\nu$-modules of rank 4. To explain our classification, we need the notion of primitivity.

**Definition 3.** We say a $K$-lattice, $L$, is primitive if it is contained in $O_K^4$, and for every prime ideal, $p$, of $O_K$, we have $p^{-1}L \not\subseteq O_K^4$.

Again, we make an analogous definition for local fields. Note that in the local case, we may replace a prime ideal by any of its uniformizers.

We say that two lattices, $L$ and $L'$, are equivalent if there is a fractional ideal, $p$, of the integer ring of the base field such that $pL = L'$. Every lattice is equivalent to a primitive lattice. Our goal is to classify the primitive, invariant lattices in $K^4$.

**Notation 2.** Throughout, $E_i$, for $i = 1, 2, 3, 4$, will denote the standard basis vectors for $K^4$. We will write $u(\alpha)$ in place of $(\begin{smallmatrix} \alpha & \alpha \\ 0 & 1 \end{smallmatrix})$, and $\omega$ for $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. Finally, we let $\Psi(x) = (u(1) \cdot x) - x$, for $x \in K^4$. It is easy to check that $\Psi(x) = (x_2 + x_3 + x_4, 2x_3 + 3x_4, 3x_4, 0)$. Also, $\omega \cdot x = (x_4, -x_3, x_2, -x_1)$.

We end our overview by presenting the original result of [4]; we will adhere to the notation of [4] for the lattices we introduce.
Theorem 1. The $SL_2(\mathbb{Z})$-invariant primitive lattices in $\mathbb{Q}^4$ are as follows:

$$L_1 = \mathbb{Z}^4,$$

$$L_2 = \{(a, b, c, d) \in \mathbb{Z}^4 : 3|b, c\},$$

$$L_3 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|b + c\},$$

$$L_4 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|b + c, a, d; 3|b, c\},$$

$$L_5 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|b + c, a, d\},$$

$$L_6 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|b + c, 3|b, c\},$$

$$L_7 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|a + b + c, b + c + d\},$$

$$L_8 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|a + b + d, a + c + d; 3|b, c\},$$

$$L_9 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|a + b + d, b + c + d\},$$

$$L_{10} = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|a + b + c, b + c + d; 3|b, c\}.$$  

3. Results and Proofs

Let $L$ be a primitive, $GL_2(O_K)$-invariant lattice in $V_K$.

Lemma 2. If $\nu \mid 2, 3$, then $L_{\nu} = L_{1\nu} = O_{\nu}^1$.

Proof. Let $x = (x_1, x_2, x_3, x_4)$ be primitive for $\nu$. First, suppose that either $x_1$ or $x_2$ is a unit of $O_{\nu}$. By applying $\omega$ if necessary, we may assume, without loss of generality, that $x_4 \in O_{\nu}^\ast$, the group of units of $O_{\nu}$. Let $y_1 = x_4^{-1}u(-3^{-1}x_3^3 \cdot x)$. Then the third and fourth coordinates of $y_1$ are 0 and 1, respectively, so we see that $6^{-1}\Psi(\Psi(y_1)) = (1, 1, 0, 0)$. Now, $E_2 = u(-1) \cdot (1, 1, 0, 0)$ and $E_1 = \Psi(E_2)$, so $E_1, E_2 \in (L)_{\nu}$, and, by applying $\omega$, one sees easily that $E_3, E_4 \in (L)_{\nu}$ as well. Hence $(L)_{\nu} = O_{\nu}^1$. Next, suppose $x_1, x_4 \notin O_{\nu}^\ast$. By primitivity, either $x_2$ or $x_3$ is a unit, and again, we may assume $x_3 \in O_{\nu}^\ast$ by means of $\omega$. Now, consider $u(1) \cdot x + u(-1) \cdot x - 2x$. This element has $2x_3$ as its first coordinate, and $2x_3 \notin O_{\nu}^\ast$. Thus, we have reduced the problem to the previous case. This proves our lemma. □

Lemma 3. If $\nu \mid 3$, then $(L)_{\nu} = O_{\nu} \oplus \pi_{\nu}^m O_{\nu} \oplus \pi_{\nu}^m O_{\nu} \oplus O_{\nu}$ for some $m \in \{0, 1, 2, \ldots, \text{ord}_{\nu}(3)\}$.

Proof. Let $x$ be primitive for $\nu$. We first assume $x_2$ or $x_3$ to be a unit. As in the preceding lemma, we may simply assume that $x_3$ is a unit. Set $y_1 = (2x_3 + 3x_4)^{-1}\Psi(x) = (x_1, 1, x_3, 0)$, and also $y_2 = (2x_3 + 6x_4)^{-1}\Psi(x) = (1, x_2, 0, 0)$. Because, $x_2 = 6x_4(2x_3 + 3x_4)^{-1}$ and $x_3 = 3x_4(2x_3 + 3x_4)^{-1}$, we have $x_2, x_3 \in 3O_{\nu}$. Let $y_3 = y_1 - x_1y_2 = (0, 1 - x_2x_1', x_3, 0)$. Now, $1 - x_2x_1' \in O_{\nu}^\ast$, and we have $y_4 = (1 - x_2x_1')^{-1}(\nu \cdot y_3) = (0, x_2', 1, 0)$, where we note that $x_2 \notin O_{\nu}^\ast$, since it is divisible by $x_3$. So let $y_5 = u(-2^{-1}x_2') \cdot y_4 = (x_1', 0, 1, 0)$ and $y_6 = \Psi(y_5) = (1, 2, 0, 0)$. Then $E_1 = -2^{-1}\Psi(y_6)$ and $E_2 = 2^{-1}(y_6 - E_1)$, so $E_1, E_2 \in (L)_{\nu}$, and as before, $(L)_{\nu} = O_{\nu}^1$.

Let us now suppose that $x_2, x_3 \notin O_{\nu}^\ast$. By primitivity, and possibly using $\omega$, we may assume, without loss of generality, that $x_4 \in O_{\nu}^\ast$. Let $y_7 = \Psi(x) = (x_2 + x_3 + x_4, 2x_3 + 3x_4, 3x_4, 0)$. Then, we see that $x_2 + x_3 + x_4 \in O_{\nu}^\ast$, $3x_4 \in 3O_{\nu}^\ast$, and $2x_3 + 3x_4 \in \pi_{\nu}O_{\nu}$. For brevity, we write $u = x_2 + x_3 + x_4$, $a = 2x_3 + 3x_4$, and $b = 3x_4$. Then $y_7 = (u, a, b, 0)$. We have $\frac{1}{2}(y_7 + (0, 1, 0)) \cdot y_7 = (0, a, 0, 0)$. So $y_8 = (u, 0, b, 0) \in (L)_{\nu}$. $x_4^{-1}\Psi(y_8) = 3E_1 + 6E_2$ and $2^{-1}\Psi(3E_1 + 6E_2) = 3E_1$. Also, $3E_2 = 2^{-1}(3E_1 + E_2 - 3E_1)$ and $E_1 = u^{-1}(y_8 - x_43E_3)$, so $E_1, E_2 \in (L)_{\nu}$. We already saw that $(0, a, 0, 0) \in (L)_{\nu}$, and it follows that $\pi_{\nu}^{\text{ord}_{\nu}(a)}E_2 \in (L)_{\nu}$. Let $m$ be the smallest integer such that $\pi_{\nu}^m E_3 \in (L)_{\nu}$, and note that $m \in \{0, 1, 2, \ldots, \text{ord}_{\nu}(3)\}$. Then by applying $\omega$, we see that $m$ is also the smallest integer such that $\pi_{\nu}^m E_3 \in L_{\nu}$.
Suppose $cE_2 + dE_3$ lies in $(L)_\nu$ and that either $c$ or $d$ has order at $\nu$ less than $m$. (We may disregard the $E_1$ and $E_4$ coordinates since we know $E_1, E_4 \in (L)_\nu$.) If $x' = cE_2 + dE_3 \in (L)_\nu$, then $\frac{1}{2}(x' + (-1, 0)) \in (L)_\nu$, and we see that both $(0, c, 0, 0)$ and $(0, 0, d, 0)$ lie in $(L)_\nu$, violating the minimality of $m$. So we conclude that $(L)_\nu = O_\nu \oplus \pi_\nu^m O_\nu \oplus \pi_\nu^m O_\nu \oplus O_\nu$.

**Lemma 4.** If $L_\nu$ contains an element of the form $(\alpha, 1, 1, 0)$, then it contains $2E_i$, for $i = 1, 2, 3, 4$.

**Proof.** Let $f = (\alpha, 1, 1, 0) \in L_\nu$. Then $\Psi(x) = 2E_1 + 2E_2$. Now, $\Psi(2E_1 + 2E_2) = 2E_4$, and the lemma follows easily.

**Lemma 5.** If $\nu \mid 2$ and $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] > 1$, then $L_\nu = L_{1\nu}$.

**Proof.** As before, we choose $x = (x_1, x_2, x_3, x_4)$ to be primitive for $\nu$. First, we suppose that either $x_1 \in O_\nu^*$ or $x_4 \in O_\nu^*$. Applying $\omega$ if necessary, we may simply assume that $x_4 \in O_\nu^*$. We set $y_1 = u(-\frac{1}{2}x_4^{-1}x_3) \cdot x$, and note that the third and fourth entries of $y_1$ are 0 and $x_4$, respectively. We next set $y_2 = (3x_4)^{-1} \Psi(y_1)$, which becomes $(x_1', 1, 1, 0)$. By Lemma 4, $2E_i \in L_{1\nu}$, for $i = 1, 2, 3, 4$. Because the residue field extension is nontrivial, there exists $\nu \in O_\nu^*$ such that $1 - u \in O_\nu^*$. Then also $1 - u^2 \in O_\nu^*$, for $1 - u^2 = (1 - u)(1 + u)$ and $1 + u = (1 - u) + 2u$. Let $y_3 = (u(\frac{1}{2} u) \cdot y_2 = (x_1, u, u^2, 0)$, and then let $y_4 = y_2 - y_3 = (0, u, 1 - u, u^2, 0)$. Observe that $(1 - u) + (1 - u^2) = (1 - u)(1 + (1 + u)) = (1 - u)(2 + u)$, which is a unit. So we have shown that $L_{1\nu}$ contains an element $(a, b, 0, 0)$ such that $a, b, a + b$ are all units. Then we already know $2E_i \in L_{1\nu}$, we see that $(a + b, 0, 0, 0) \in L_{1\nu}$. It follows easily that $E_1, E_4 \in L_{1\nu}, \Psi((0, a, b, 0)) = (a + b, 2b, 0, 0)$ and since we already have assumed $x_2 + x_3 \in \pi_\nu O_\nu$. This reasoning lets us assume, without loss of generality, that $x = (x_1, x_2, x_3, x_4)$ is such that $x_2 + x_3 \in O_\nu^*$. But then we can apply $\Psi$ to $x$ and get a form with a unit in its first coordinate, thus reducing the proof to the first case of this lemma.

**Lemma 6.** If $\nu \mid 2$ and $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] = 1$, then $L_{2\nu} \subseteq L_\nu$ or $L_{9\nu} \subseteq L_\nu$.

Here, $L_{2\nu} = \{(a, b, c, d) \in O_\nu^4 : \pi_\nu \mid a, d, b + c\}$ and $L_{9\nu} = \{(a, b, c, d) \in O_\nu^4 : \pi_\nu \mid a + b + d, a + c + d\}$.

**Proof.** If $\nu$ is unramified over 2, the argument in 4 applies verbatim. So assume $\nu$ is ramified. Let $x \in \mathbb{E}$ be primitive for $\nu$. As usual, suppose that either $x_1 \in O_\nu^*$ or $x_4 \in O_\nu^*$. In the preceding lemmas, this reduces to the assumption that $x_4 \in O_\nu^*$. This time, we choose the same $y_1$ as in the previous lemma, we find $y_2 = (x_1, 1, 1, 0) \in L_\nu$, and we see that $2E_i \in L_\nu$ for $i = 1, 2, 3, 4$. Pick $u = 1 + \pi_\nu \in O_\nu^*$, so that $1 - u$ has order 1 at $\pi_\nu$. Following Lemma 5, let $y_3 = u(\frac{1}{2} u) \cdot y_2 = (x_1', u, u^2, 0)$. Now, $y_2 - y_3 = (0, 1 - u, 1 - u^2, 0)$. Also, $1 + u = 1 + \pi_\nu = 2 + \pi_\nu$, hence $1 + u$ has order 1 at $\nu$. Since $1 - u$ also has order 1, it follows that $1 - u^2 \in \pi_\nu^2 O_\nu^*$. Applying matrices of the forms $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$, we see that $2\pi_\nu^2 E_2 \in L_\nu$. Now, $\Psi((0, \pi_\nu, \pi_\nu^2, 0)) = (\pi_\nu + \pi_\nu^2, 2\pi_\nu^2, 0, 0)$. Since $2\pi_\nu^2 E_2 \in L_\nu$, $(\pi_\nu + \pi_\nu^2, 0, 0, 0)$ and, in turn, $\pi_\nu E_1, \pi_\nu E_2, \pi_\nu E_4$ lie in $L_\nu$. Then also...
(0, πν, πν2, −1/2πν2) ∈ Lν. Next, u(1) : (0, πν, πν2, −1/2πν2) = (z, πν + πν2, 0, −1/2πν2). Here, z = πν + πν2 − 1/2πν2 ∈ πνOν. Since πνE1 and πνE4 are in Lν, we can show that (0, πν + πν2, 0) ∈ Lν, and it follows easily that πνE2 and πνE3 are in Lν. We have thus shown that πνE1 ∈ Lν for i = 1, 2, 3, 4.

Now, consider the case where x1 and x4 are nonunits, so that x2 or x3 is a unit. If x2 + x3 ∈ Oν∗, we can apply Ψ to x and use the above case to see that πνE1 ∈ Lν for i = 1, 2, 3, 4. Otherwise, we have that x2 and x3 are both units. Choose v ∈ Oν∗ so that x2 + vx3 = 0, and replace x by (1 0 0 1) · x. The new x = (x1, x2, x3, x4) satisfies x2 + x3 = 0. We proceed as follows: let w′(z) = −(1 0 1 0) · z; that is, w′ reverses the coordinates,

\[ \Psi(x) = (x2 + x3 + 4x3 + 3x4 + 3x4, 0), \]

and since x2 + x3 = 0,

\[ w′(\Psi(x)) = (0, 3x4, 2x3 + 3x4, x4). \]

Subtracting this result from x, we get \((z, a, b, 0)\), where \(a, b \in Oν∗\). Using the matrices of the forms \((0 0 1 0)\) and \((1 0 0 1)\) (for units u, v), we may insist that \(a = b = 1\).

Lemma 4 now shows that \(2Ei \in Lν\) for \(i = 1, 2, 3, 4\), and from here the argument from the previous case (with \(y2\) replaced by \((z, 1, 1, 0)\)) can be used to show that \(\piνE1 \in Lν\) for \(i = 1, 2, 3, 4\).

So regardless of which coordinates of x are initially taken to be units, we have seen that Lν has an element of the form \((z, 1, 1, 0)\). Now, observe that L5ν = \(Oν(πνE1) ⊕ Oν(πνE3) ⊕ Oν(E2 + E3) ⊕ Oν(πνE2)\) and

\[ L9ν = Oν(E1 + E2 + E3) ⊕ Oν(E2 + E3 + E4) ⊕ Oν(πνE2) ⊕ Oν(πνE3). \]

In addition to \((z, 1, 1, 0)\), Lν also contains πνE1, for \(i = 1, 2, 3, 4\). If \(z \notin Oν∗\), then \((z, 1, 1, 0) - (πν−1νz)πνE1 = E2 + E3 \in Lν\), so L5ν ⊆ Lν. If \(z \in Oν∗\), then we can write \(z = 1 + z′\), where \(z′ \in πνOν\), since \([Oν/πνOν : Z2 / 2Z2] = 1\). Then \((z, 1, 1, 0) - (πν−1νz′)πνE1 = E1 + E2 + E3 \in Lν\), and by applying \(w′\), we can conclude that \(E2 + E3 + E4 \in Lν\). Hence \(L9ν \subseteq Lν\). This completes the proof of the lemma.

Notation 3. Up to this point, we have defined Lν, L5ν, and L9ν. Now, we introduce L3ν = \\{(a, b, c, d) ∈ O4ν : πν | b + c\} and L7ν = \\{(a, b, c, d) ∈ O4ν : πν | a + b + c, b + c + d\}.

In view of Lemma 6 we can prove the following, more precise lemma.

Lemma 7. If \(ν \mid 2\) and \([Oν/πνOν : Z2 / 2Z2] = 1\), then Lν ∈ \{L1ν, L3ν, L5ν, L7ν, L9ν\}.

Proof. By Lemma 6 either L5ν ⊆ Lν ⊆ L1ν or L9ν ⊆ Lν ⊆ L1ν.

Case I. Assume that L5ν ⊆ Lν ⊆ L1ν. If Lν = L5ν, there is nothing to prove, so assume that Lν properly contains L5ν. We will write \(L1ν = OνE1 ⊕ OνE4 ⊕ Oν(E2 + E3) ⊕ OνE2\) and \(L5ν = OνπνE1 ⊕ OνπνE4 ⊕ Oν(E2 + E3) ⊕ OνπνE2\). Then \\{aE1 + bE4 + cE2 : a, b, c ∈ \{0, 1\}\} is a set of coset representatives for \(L1ν/L5ν\). The fact that \(L5ν \subseteq Lν \subseteq L1ν\) implies that one of our nonzero coset representatives lies in Lν. Suppose the representative that lies in Lν is E1, E4, or E1 + E4. Lν also contains L5ν, so we can show L3ν ⊆ Lν for L3ν = OνE1 ⊕ OνE4 ⊕ Oν(E2 + E3) ⊕ OνπνE2. Indeed, the E1 and E4 cases are obvious. If instead
we have $E_1 + E_4 \in L_\nu$, note that $\Psi(E_1 + E_4) = E_1 + 3(E_2 + E_3)$, which reduces this case to that of $E_1$. Hence $L_{3\nu} \subseteq L_\nu$, as desired. But $L_{1\nu}/L_{3\nu} \cong \mathbb{Z}/2\mathbb{Z}$, and so no lattices lie (properly) between $L_{1\nu}$ and $L_{3\nu}$. Hence either $L_\nu = L_{3\nu}$ or $L_\nu = L_{1\nu}$.

Next, suppose that the coset representative is $E_2$, $E_1 + E_2$, or $E_2 + E_4$. Recall that

$$2E_2 \in L_{5\nu} \subseteq L_\nu.$$ 

Now, $E_1 = \Psi(E_2) = \Psi(E_1 + E_2) = \Psi(\omega(E_2 + E_4)) = 2E_2$, and we can easily show that both $E_1$ and $E_2$ are in $L_\nu$. In this case, $L_\nu = L_{1\nu}$. To finish this case, suppose the coset representative that lies in $L_\nu$ is $E_1 + E_2 + E_4$.

Write $L_{7\nu} = O_\nu E_1 + E_2 + E_4 \oplus O_\nu E_1 + E_3 + E_4 \oplus O_\nu \pi_1 E_1 + O_\nu \pi_2 E_2$ and $L_{1\nu} = O_\nu E_1 + E_2 + E_3$.

A set of coset representatives for $L_{1\nu}/L_{9\nu}$ is given by

$$\{E_2 + bE_3 : a, b \in \{0, 1\}\}.$$ 

Again, $L_\nu$ contains at least one coset representative, and if the only representative in $L_\nu$ is 0, then $L_\nu = L_{9\nu}$. If $L_\nu$ contains $E_2$ or $E_3$, it clearly contains both, and $\Psi(E_2) = E_1$ implies that also $E_1 \in L_\nu$. Hence $L_\nu = L_{1\nu}$. If instead the coset representative contained in $L_\nu$ is $E_2 + E_3$, then $L_\nu$ contains $(E_1 + E_2 + E_3) - (E_2 + E_3) = E_1$. It follows that $E_4 \in L_\nu$, and thus $L_{3\nu} \subseteq L_\nu$. We have seen that this implies that either $L_\nu = L_{3\nu}$ or $L_\nu = L_{1\nu}$.

Case II. Assume that $L_{9\nu} \subseteq L_\nu \subseteq L_{1\nu}$. We have that $L_{9\nu} = O_\nu E_1 + E_2 + E_3 \oplus O_\nu \pi_1 E_1 + O_\nu \pi_2 E_2$ and $L_{1\nu} = O_\nu E_1 + E_2 + E_3 \oplus O_\nu \pi_2 E_3$. A set of coset representatives for $L_{1\nu}/L_{9\nu}$ is given by

$$\{E_2 + bE_3 : a, b \in \{0, 1\}\}.$$ 

Again, $L_\nu$ contains at least one coset representative, and if the only representative in $L_\nu$ is 0, then $L_\nu = L_{9\nu}$. If $L_\nu$ contains $E_2$ or $E_3$, it clearly contains both, and $\Psi(E_2) = E_1$ implies that also $E_1 \in L_\nu$. Hence $L_\nu = L_{1\nu}$. If instead the coset representative contained in $L_\nu$ is $E_2 + E_3$, then $L_\nu$ contains $(E_1 + E_2 + E_3) - (E_2 + E_3) = E_1$. It follows that $E_4 \in L_\nu$, and thus $L_{3\nu} \subseteq L_\nu$. We have seen that this implies that either $L_\nu = L_{3\nu}$ or $L_\nu = L_{1\nu}$.

This completes the proof.

We are now in a position to state and prove our main theorem.

**Theorem 2.** Let $p_1, p_2, \ldots, p_r$ be the prime ideal divisors of $3O_K$, and let $p_{r+1}, p_{r+2}, \ldots, p_t$ be the divisors of $2O_K$ with residue field $\mathbb{Z}/2\mathbb{Z}$. Then the primitive, $GL_2(O_K)$-invariant lattices of $K^4$ are of the form $\{(a, b, c, d) \in O_K^4 : b, c \in p_{r+1}^{n_1} \cdots p_{r+1}^{n_{r+1}} \oplus (b + c) \in p_{r+2}^{n_{r+2}} \cdots p_t^{n_t} : (a + b + c), (b + c + d) \in p_{r+1}^{n_{r+1+3}} \cdots p_r^{n_r} : (a + b + d), (a + c + d) \in p_{r+1}^{n_{r+1+4}} \cdots p_t^{n_t}\}$, where $0 \leq n_i \leq \text{ord}_{p_i}(3)$, $0 \leq n_{i,j} \leq 1$, and $n_{i,j} = 1$ implies $n_{i,k} = 0$ for any $k > j$.

**Proof.** Combine Lemma 2 through Lemma 7 and use the fact that a lattice $L \subseteq K^4$ is given by

$$\bigcap_{\nu \in M_0(K)} (L_\nu \bigcap K^4),$$

where $L_\nu$ is the closure in $K^4$ of $L$.

4. **Dirichlet series and the functional equation**

In this section, we connect our invariant lattices to the adelic zeta function for the space of binary cubic forms (see [2] and [8]). Before proceeding, we need to introduce some terminology pertaining to the notion of a splitting type. Let $S \subseteq M(K)$. For each $\nu \in S$, let $A_\nu$ be the set of Galois conjugacy classes of extensions of the local field $K_\nu$ of degree not exceeding 3. Let $A_S = \prod_{\nu \in S} A_\nu$, so the elements $\alpha$ of $A_S$ are of the form $\alpha = (K_{\alpha_1}, K_{\alpha_2}, \ldots)$, where each $K_{\alpha_\nu}$ is an extension of $K_\nu$. Let $L$ be an extension of $K$ of degree at most 3. Then $K_\nu \otimes K L$ is a direct sum of extensions of $K_\nu$ of total degree $[L : K]$. The structure of the summands uniquely determines how the prime $\nu$ of $K$ factors in $L$. For each $\nu$, let $F_{L_\nu}$ denote the summand of $K_\nu \otimes K L$ with highest degree over $K_\nu$. Let $\alpha \in A_S$. We say that $L$ has $S$-splitting type $\alpha$ if for each $\nu \in S$, $K_{\alpha_\nu} \cong F_{L_\nu}$. We will also
call \( \alpha \) the \( S \)-splitting signature of \( L \), and write \( L \sim \alpha \). For \( x \in V'_K \), we write \( x \sim \alpha \) if the splitting field \( K(x) \) has \( S \)-splitting signature \( \alpha \). If \( \alpha_\nu \) is a Galois conjugacy class as described above, we let \( Z_{\alpha_\nu}(s, \Phi_\nu) = \int_{G_\nu} |\det(g_\nu)|^s \Phi_\nu(g_\nu \cdot X_\nu) dg_\nu \), where \( X_\nu \) is the standard orbital representative for the conjugacy class of extensions \( \alpha_\nu \). If \( S \subseteq M(K) \) and \( \alpha \in A_S \), we define \( Z_\alpha(s, \Phi) = \prod_{\nu \in S} Z_{\alpha_\nu}(s, \Phi_\nu) \).

We begin by fixing a number field, \( K \), and let \( \mathbb{A} \) denote the ring of adeles of \( K \). We abbreviate \( GL_2(\mathbb{A}) \) simply as \( G \), and write \( G_K \) for \( GL_2(K) \), viewed as a subgroup of \( G \). We introduce

\[
G(\infty) = \prod_{\nu \in M_\infty(K)} GL_2(K_\nu) \times \prod_{\nu \in M_0(K)} GL_2(O_\nu).
\]

Observe that the number of double cosets of \( G(\infty) \backslash G/G_K \) is just \( h_K \), the class number of \( K \). We have the decomposition

\[
G = \bigcup_{t \in \mathbb{A}^*/K^*\mathbb{A}^*(\infty)} G(\infty) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) G_K.
\]

Observe also that \( G(\infty)G_K = \{ g \in G : \det(g) \in K^*\mathbb{A}^*(\infty) \} \), where

\[
\mathbb{A}^*(\infty) = \prod_{\nu \in M_\infty(K)} K^*_\nu \times \prod_{\nu \in M_0(K)} O^*_\nu.
\]

Now we fix a (primitive) invariant lattice \( L \subseteq V_K \). As in our introduction, we write \( |a| \) for the adelic absolute value of \( a \in \mathbb{A}_K \). We choose a Schwartz-Bruhat function, \( \Phi = \prod_\nu \Phi_\nu \), whose finite components are the characteristic functions for the respective closures of \( L \). We note that for \( \nu \in M_0(K) \), this closure is a free, \( GL_2(O_\nu) \)-invariant \( O_\nu \)-module of rank 4. For notational convenience, we also write \( \Phi = \prod_{\nu \in M_\infty(K)} \Phi_\nu \times 1_{L_0} \), where \( L_0 = \prod_{\nu \in M_0(K)} L_\nu \), and \( 1_{L_0} \) is the characteristic function for \( L_0 \). Let \( H = \mathbb{A}^*/K^*\mathbb{A}^*(\infty) \), and let \( \hat{H} \) be the dual of \( H \). We now consider the sum

\[
\frac{1}{h_K} \sum_{\chi \in \hat{H}} Z(\chi \omega_s, \Phi).
\]

### Proposition 3.

\[
\frac{1}{h_K} \sum_{\chi \in \hat{H}} Z(\chi \omega_s, \Phi) = \int_{G(\infty)/GL_2(O_K)} |\det(g)|^s \sum_{x \in L'} \Phi(g \cdot x) dg,
\]

where \( L' = \{ x \in L : P(x) \neq 0 \} \).

**Proof.** First, note that

\[
\frac{1}{h_K} \sum_{\chi \in \hat{H}} Z(\chi \omega_s, \Phi) = \int_{G/G_K} \frac{1}{h_K} \sum_{\chi \in \hat{H}} \chi(\det(g)) |\det(g)|^s \sum_{x \in V'_K} \Phi(g \cdot x) dg.
\]

This reduces to

\[
\int_{G(\infty)G_K/G_K} |\det(g)|^s \sum_{x \in V'_K} \Phi(g \cdot x) dg,
\]
since $\sum_{\chi \in \mathcal{H}} \chi(\det(g)) = 0$ whenever $g \notin G(\infty)G_K$. By an isomorphism theorem, this integral is just

$$\int_{G(\infty)/G_K \cap G(\infty)} |\det(g)|^s \sum_{x \in V'_K} \Phi(g \cdot x) dg.$$ 

Note that $G_K \cap G(\infty) = GL_2(O_K)$. Now if $g \in G(\infty)$, then $g = (g_\nu)_\nu$, where $g_\nu \in GL_2(O_\nu)$ for all finite places $\nu$. We have that $g_\nu \cdot x \in L_\nu$ iff $x \in g_\nu^{-1} L_\nu$, where $L_\nu$ is the local component of $L_0$. Now, since $L_\nu$ is $GL_2(O_\nu)$-invariant, we see that $g_\nu \cdot x \in L_\nu$ iff $x \in L_{\nu}$. If $x \in L_{\nu}$ for all finite $\nu$, then we have $x \in K \cap \prod_{\nu \in M_0(K)} L_\nu = L$, by Lemma 1 of Section 2. Hence

$$\int_{G(\infty)/G_K \cap G(\infty)} |\det(g)|^s \sum_{x \in V'_K} \Phi(g \cdot x) dg = \int_{G(\infty)/GL_2(O_K)} |\det(g)|^s \sum_{x \in L'} \Phi(g \cdot x) dg,$$

where $L'$ is as described in the proposition. \qed

The integral on the right hand side of Proposition 3 may be rewritten as a sum, namely

$$\sum_{x \in GL_2(O_K) \backslash L'} \frac{1}{|\text{Stab}_{G_K}(x)|} \int_{G(\infty)} |\det(g)|^s \Phi(g \cdot x) dg.$$

The integrals under the sum are of the form

$$\int_{G(\infty)} |\det(g)|^s \Phi(g \cdot x) dg = \prod_{\nu \in M_\infty(K)} \int_{GL_2(K_\nu)} |\det(g_\nu)|^s \Phi_\nu(g_\nu \cdot x) dg_\nu \times \prod_{\nu \in M_0(K)} \int_{GL_2(O_\nu)} 1_{L_\nu}(g_\nu \cdot x) dg_\nu.$$ 

All of the finite local factors evaluate to 1. We denote the product of the infinite local factors by $Z_{x_\infty}(s, \Phi_\infty)$. Our work and conventions thus show:

**Proposition 4.**

$$\frac{1}{h_K} \sum_{\chi \in \mathcal{H}} Z(\chi \omega_s, \Phi) = \sum_{x \in GL_2(O_K) \backslash L'} \frac{1}{|\text{Stab}_{G_K}(x)|} Z_{x_\infty}(s, \Phi_\infty),$$

where $Z_{x_\infty}(s, \Phi_\infty) = \prod_{\nu \in M_\infty(K)} \int_{GL_2(K_\nu)} |\det(g_\nu)|^s \Phi_\nu(g_\nu \cdot x) dg_\nu$.

The next proposition will help us relate these integrals to discriminants of cubic forms.

**Proposition 5.**

$$Z_{x_\infty}(s, \Phi_\infty) = Z_\alpha(s, \Phi_\infty)|N_{K/\mathbb{Q}}(P(x))|^{\frac{s}{2}},$$

where $\alpha$ is the $M_\infty(K)$-splitting type of $x$.

**Proof.** In each $Z_{x_\infty}(s, \Phi_\infty)$ replace $x$ by its corresponding standard orbital representative, which we will call $X$. Then

$$Z_{x_\infty}(s, \Phi_\infty) = Z_{X_\infty}(s, \Phi_\infty)|N_{K/\mathbb{Q}}(P(x))|^{\frac{s}{2}}.$$
This immediately proves our proposition, since if \( x \) has \( M_{\infty}(K) \)-splitting type \( \alpha = (\alpha_\nu)_{\nu \in M_{\infty}(K)} \) and \( X_\infty \) is the product of the standard orbital representatives for the conjugacy classes of extensions \( \alpha_\nu \), then \( Z_{X_\infty}(s, \Phi_\infty) = Z_\alpha(s, \Phi_\infty) \).

To see how the norm of the discriminant arises, let \( g = (g_\nu)_\nu \) be the matrix such that \( g \cdot x = X \). In each local factor of \( Z_{X_\infty}(s, \Phi_\infty) \), which is of the form \( \int_{GL_2(K_\nu)} |\det(h_\nu)|^s \Phi_\nu(h_\nu \cdot x) dh_\nu \), we make a change of variables, replacing \( h_\nu \) by \( h_\nu g_\nu \). Then the local factor becomes \( |\det(g_\nu)|^s Z_{X_\nu}(s, \Phi_\nu) \) and the product of the local factors becomes \( \prod_{\nu \in M_{\infty}(K)} |\det(g_\nu)|^s Z_{X}(s, \Phi_\infty) \). Now, observe that \( \det(g_\nu)^2 P(x_\nu) = P(g_\nu \cdot x_\nu) = P(X_\nu) \). Thus, \( \prod_{\nu \in M_{\infty}(K)} |\det(g_\nu)|^s = \prod_{\nu \in M_{\infty}(K)} |P(X_\nu)|_\nu \).

But at each infinite place \( \nu \), \( |P(X_\nu)|_\nu = 1 \), by choice of orbital representative. \( \square \)

Now, let

\[
E_L(s, \Phi) = \frac{1}{h_K} \sum_{x \in H} Z(\chi \omega_s, \Phi).
\]

Let \( \alpha \) denote an \( M_{\infty}(K) \)-splitting type. We define

\[
\xi_{\alpha}(s, L) = \sum_{x \in GL_2(Q_K) \setminus L, x \sim \alpha} \frac{1}{|\text{Stab}_{G_K}(x)|} |N_{K/Q}(P(x))|^{-\frac{s}{2}}.
\]

Taken together, Propositions 4 and 5 prove the following:

**Proposition 6.**

\[
E_L(s, \Phi) = \sum_{\alpha} Z_\alpha(s, \Phi_\infty) \xi_{\alpha}(s, L).
\]

In the sum on the right, \( \alpha \) runs through all possible \( M_{\infty}(K) \)-splitting types.

We conclude this paper by deriving a functional equation for the series \( \xi_{\alpha}(s, L) \) that relates \( \xi_{\alpha}(2 - s, L) \) to \( \xi_{\alpha}(s, \hat{L}) \), the series associated with the dual lattice \( \hat{L} \) of \( L \). Using the functional equation for the adelic zeta function, we have

\[
E_L(s, \Phi) = \frac{1}{h_K} \sum_{x \in H} Z(\chi \omega_s, \Phi) = \frac{1}{h_K} \sum_{x \in H} Z(\chi \omega_{2-s}, \Phi) = E_{\hat{L}}(2 - s, \Phi).
\]

Combining this equation with Proposition 5 shows that

\[
\sum_{\alpha} Z_\alpha(2 - s, \Phi) \xi_{\alpha}(2 - s, L) = \sum_{\beta} Z_\beta(s, \hat{\Phi}) \xi_{\beta}(s, \hat{L}).
\]

In [2], Datskovsky and Wright gave a functional equation for \( Z_\beta(s, \hat{\Phi}) \). Indeed, we have

\[
Z_\beta(s, \hat{\Phi}) = \sum_{\alpha} C_{\beta \alpha}(s) Z_\alpha(2 - s, \Phi).
\]

Here, as in [2], \( C_{\beta \alpha}(s) \) is a product of local factors, running over \( M_{\infty}(K) \). If we write \( \gamma(s) = \Gamma(s - \frac{1}{6}) \Gamma(s + \frac{1}{6}) \) and \( \rho(s) = \frac{1}{2} \pi^{-2s} \gamma^{3s-3} \), then we can say that,
for a real prime \( \nu \) of \( K \), the local factor \( C_{\beta, \alpha, \nu}(s) \) of \( C_\beta(s) \) is

\[
C_{\beta, \alpha, \nu}(s) = \begin{cases} 
\rho(s)\gamma \left( \frac{s}{2} \right) \sin \left( \frac{\pi s}{2} \right) & \text{if only } \beta_\nu \text{ is split}, \\
\rho(s)\gamma \left( \frac{s}{2} \right) \sin \left( \frac{3\pi s}{2} \right) & \text{if only } \alpha_\nu \text{ is split}, \\
\rho(s)\gamma \left( \frac{s}{2} \right) \sin (\pi s) & \text{otherwise}.
\end{cases}
\]

When \( \nu \) is instead complex, we let \( r(s) = \pi^{-2s}3^{3s}\gamma \left( \frac{s}{2} \right) \), and find that

\[
C_{\beta, \alpha, \nu}(s) = \frac{r(s)}{r(2-s)}.
\]

With this definition of \( C_{\beta, \alpha}(s) \) in place, we may now give our functional equation for \( \xi_\alpha(s, L) \).

**Theorem 3.**

\[
\xi_\alpha(2-s, L) = \sum_{\beta} C_{\beta, \alpha}(s) \xi_\beta(s, \hat{L}).
\]

**Proof.** We can use equation (2) to rewrite

\[
\sum_{\beta} Z_\beta(s, \hat{\Phi}) \xi_\beta(s, \hat{L})
\]

as

\[
\sum_{\beta} Z_\beta(s, \hat{\Phi}) \xi_\beta(s, \hat{L}) = \sum_{\beta} \left( \sum_{\alpha} C_{\beta, \alpha}(s) Z_\alpha(2-s, \Phi) \right) \xi_\beta(s, \hat{L}).
\]

Then

\[
\sum_{\alpha} Z_\alpha(2-s, \Phi) \xi_\alpha(2-s, L) = \sum_{\beta} \left( \sum_{\alpha} C_{\beta, \alpha}(s) Z_\alpha(2-s, \Phi) \right) \xi_\beta(s, \hat{L})
\]

\[
= \sum_{\alpha} \left( \sum_{\beta} C_{\beta, \alpha}(s) \xi_\beta(s, \hat{L}) \right) Z_\alpha(2-s, \Phi).
\]

Given \( \alpha \) and \( \beta \), we may choose \( \Phi \) so that \( Z_\alpha(2-s, \Phi) = \delta_{\alpha\beta} \). Upon doing so, we find that

\[
\xi_\alpha(2-s, L) = \sum_{\beta} C_{\beta, \alpha}(s) \xi_\beta(s, \hat{L}),
\]

as desired. \[\square\]

**References**


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