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$GL_2(O_K)$-INVARIANT LATTICES IN THE SPACE OF BINARY CUBIC FORMS WITH COEFFICIENTS IN THE NUMBER FIELD K

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Abstract. In 2008, Ohno, Taniguchi and Wakatsuki obtained a classification of all $GL_2(\mathbb{Z})$-invariant lattices in the space of binary cubic forms with coefficients in $\mathbb{Q}$. In this paper, we aim to generalize their result by replacing the rational field with an arbitrary algebraic number field, K. We conclude the paper by connecting the lattices described in our main result to a zeta function developed by Datskovsky and Wright, which yields a functional equation for certain Dirichlet series attached to the lattices.

1. Introduction

We begin our work with a discussion of the space of binary cubic forms over K, where K is a field. This is necessary to describe the action of GL_2 on this space and to define the zeta function mentioned above. To simplify our exposition, we introduce some notation.

Notation 1. Throughout this paper, V denotes the four dimensional affine space. Also, we will let G denote the general linear group, GL_2, and B will represent the subgroup of G consisting of lower triangular matrices. Thus $V_K = K^4$, $G_K = GL_2(K)$, and $B_K = \{A \in G_K : A \text{ is lower triangular}\}$.

The space of binary cubic forms over K is the set

$$\{x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3 : x_i \in K\}.$$

We identify the cubic form $x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3$ with the point $x = (x_1, x_2, x_3, x_4) \in V_K$, and we will denote the form as either x or $F_x(u, v)$. The group G_K acts on the space of binary cubic forms by linear change of variables, modified to allow for scalar multiplication.

Definition 1. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_K$, and let $x \in V_K$. We define the action of g on x by

$$g \cdot x = F_{g\cdot x}(u, v) = \det(g)^{-1} F_x\left((u\, v)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right).$$

The twist by $\det(g)^{-1}$ ensures that if $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, then $g \cdot x = ax$.

For a form x, we let $P(x)$ denote the discriminant of the polynomial $F_x(u, 1)$: for $x = (x_1, x_2, x_3, x_4)$, we have

$$P(x) = x_2^2x_3^2 + 18x_1x_2x_3x_4 - 4x_2^3x_4 - 4x_1x_3^3 - 27x_1^2x_4^2.$$

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Observe that for $g \in G_K$, $P(g \cdot x) = \det(g)^2 P(x)$. We call a form x nonsingular if $P(x) \neq 0$.

We will refer to the roots of the polynomial $F_x(u,1)$ as the roots of the form x. $K(x)$ will be the splitting field of $F_x(u,1)$ over K. This is either a cyclic extension of K of degree 3 or less, or a degree 6 extension with Galois group S_3. In the latter case, we may think of $K(x)$ as a Galois conjugacy class of noncyclic cubic extensions of K. We now recall a proposition from Section 2 of [8], giving the orbits in our action of G_K on V_K.

Proposition 1. *The G_K-orbits in V_K are as follows:*
$S_0 = \{0\}$ *(the zero form),*
$S_{1,K} = \{x \in V_K : x$ *has a triple root*$\}$,
$S_{2,K} = \{x \in V_K : x$ *has a double root, as well as a simple root*$\}$,
$V_K(K') = \{x \in V_K : P(x) \neq 0, K(x) = K'\}$.
In the fourth class of orbits, K' runs over all Galois conjugacy classes of extensions of K with degree at most 3.

Indeed, the works of Datskovsky and Wright [8], [2], [3] rely on the fact that the map $x \to K(x)$ induces a one-to-one correspondence between the orbits of nonsingular binary cubic forms over K and the conjugacy classes of extensions of K of degree not exceeding 3.

The authors of [8] also chose standard representatives for each type of orbit, which we will use in a future calculation. For S_0, $S_{1,K}$, and $S_{2,K}$, these are $(0,0,0,0)$, $(1,0,0,0)$, and $(0,1,0,0)$, respectively. For nonsingular forms x with $K(x) = K$, we choose $(0,1,1,0)$ as our standard representative. For forms such that $[K(x) : K] = 2$, with K nonarchimedean, we consider θ such that $K(x) = K(\theta)$, and pick $(0, 1, \theta + \theta', \theta\theta')$, where θ' is the Galois conjugate of θ over K. When $K = \mathbb{R}$ and $K(x) = \mathbb{C}$, we let $x = \frac{1}{\sqrt{2}}(1,0,1,0)$. The exception here is to make sure that the standard representatives for the orbits over $\mathbb{R}$ both have discriminant 1. When $K(x)$ is a conjugacy class of cubic extensions, we again choose θ which generates a member of this class over K, and pick $(1, \theta+\theta'+\theta'', \theta\theta'+\theta\theta''+\theta'\theta'', \theta\theta'\theta'')$ to be our standard representative, where again, θ' and θ'' are the conjugates of θ.

The stabilizers of our nonsingular forms are also known. The following proposition originally appeared in [8].

Proposition 2. *Let $x \in V_K$, with $P(x) \neq 0$.*
(i) If $[K(x) : K] = 1$, then $|\mathrm{Stab}_{G_K}(x)| = 6$.
(ii) If $[K(x) : K] = 2$, then $|\mathrm{Stab}_{G_K}(x)| = 2$.
(iii) If $[K(x) : K] = 3$ and $K(x)$ is cyclic over K, then $|\mathrm{Stab}_{G_K}(x)| = 3$.
(iv) If $K(x)$ is a conjugacy class of noncyclic extensions of K, then $|\mathrm{Stab}_{G_K}(x)| = 1$.

Now, let K be an algebraic number field. We let O_K be the ring of integers in the number field K. $M(K)$ will stand for the set of places of K, while $M_\infty(K)$ and $M_0(K)$ will refer to the sets of infinite and finite places of K, respectively. For $\nu \in M(K)$, we let K_ν denote the completion of K at ν, and if ν is finite, we let O_ν stand for the ring of integers in K_ν. Moreover, $\mathbb{A}_K$ will stand for the ring of adeles of K.

To define the zeta function, we first need to introduce invariant measures on K_ν and G_{K_ν}. On K_ν, for finite ν, we choose the additive measure dx_ν, normalized so that the measure of O_ν is 1, and the multiplicative measure d^*x_ν, which is

normalized so that the measure of O_ν^* is 1. If instead ν is infinite, dx is the usual Lebesgue measure, and d^*x is $\frac{dx}{x}$. We define our measure on G_{K_ν} in stages. First, we introduce a maximal compact subgroup, U_{K_ν} of G_{K_ν}. When ν is real, we let U_{K_ν} be the group of orthogonal matrices in $GL_2(\mathbb{R})$. If ν is complex, U_{K_ν} will be the unitary group. When ν is finite, we let $U_{K_\nu} = GL_2(O_\nu)$. By the Iwasawa decomposition, we know that an element $g \in G_{K_\nu}$ can be written in the form

$$g = k \begin{pmatrix} t & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

for some $k \in U_{K_\nu}$, $t, u \in K_\nu^*$, $c \in K_\nu$. We define

$$a(t,u) = \begin{pmatrix} t & 0 \\ 0 & u \end{pmatrix}$$

and

$$n(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

We have chosen U_{K_ν} to be compact, so we have an invariant measure, dk_ν, such that U_{K_ν} has measure 1. Observe that $a(t,u)n(c) \in B_{K_\nu}$, and every element of B_{K_ν} is of this form. We specify our measure on B_{K_ν} by the formula

$$\int_{B_{K_\nu}} f(b)db_\nu = \int_{K_\nu^*}\int_{K_\nu^*}\int_{K_\nu} \left|\frac{u}{t}\right|_\nu f(a(t,u)n(c))dc_\nu d^*t_\nu d^*u_\nu,$$

where the absolute value is the standard choice on K_ν. This enables us to give a Haar measure, dg_ν, on G_{K_ν} via

$$\int_{G_{K_\nu}} f(g)dg_\nu = \int_{U_{K_\nu}}\int_{B_{K_\nu}} f(kb)db_\nu dk_\nu.$$

We define an additive measure on $\mathbb{A}_K$ via $\prod_{\nu\in M(K)} dx_\nu$. We similarly define a multiplicative measure on $\mathbb{A}_K$ and a measure on $GL_2(\mathbb{A}_K)$ as products of local measures.

With our measures defined, we may go on to define the adelic zeta function. Let Φ be a function on V_K admitting a product $\Phi = \prod_{\nu\in M(K)} \Phi_\nu$ such that Φ_ν is rapidly decreasing if ν is infinite or locally constant with compact support if ν is finite, and all but finitely many of the Φ_ν are characteristic functions of O_ν^4. We will refer to such Φ as Schwartz-Bruhat functions. Let $s \in \mathbb{C}$, and let V_K' denote the set of nonsingular forms in V_K. The adelic zeta function is then defined as

$$Z(s,\Phi) = \int_{G_{\mathbb{A}_K}/G_K} |\det(g)|^s \sum_{x\in V_K'} \Phi(g\cdot x)dg,$$

where we use $|\det(g)|$ to denote the adelic absolute value of $|\det(g)|$. This definition is valid for $\mathrm{Re}(s) > 2$, where the integral converges absolutely and locally uniformly.

Having defined the function, we proceed to describe some of its basic properties. As in [8], it admits a meromorphic continuation to the entire complex plane, except for simple poles at $s = 2, \frac{5}{3}, \frac{1}{3}$, and 0. One has a functional equation for the zeta function, obtained in [8] by means of the Poisson Summation Formula. We briefly recall the Fourier transform on V_K used in [8]. Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ be elements of V_K. We introduce the bilinear form

$$[x,y] = x_1y_4 - \frac{1}{3}x_2y_3 + \frac{1}{3}x_3y_2 - x_4y_1$$

and let $\langle\ \rangle$ be a nontrivial additive character on K. We let dx be the Haar measure on V_K self-dual with respect to the character $\langle [x,y]\rangle$ on $V^2_{\mathbb{A}_K}$. If Φ is a Schwartz-Bruhat function on V_K, we define its Fourier transform by

$$\hat{\Phi}(y) = \int_{V_{\mathbb{A}_K}} \Phi(x)\langle [x,y]\rangle\, dx.$$

The functional equation for the adelic zeta function associated to the space of binary cubic forms is then $Z(2-s,\hat{\Phi}) = Z(s,\Phi)$ (see [8]).

2. Overview of invariant lattices

In this section, we are going to describe the $GL_2(O_K)$-invariant lattices in V_K. We begin with a definition, following Weil [7].

Definition 2. Let K be an algebraic number field, and let O_K be the ring of (algebraic) integers of K. Let V be a finite dimensional vector space over K. A lattice of V is a finitely generated O_K-module in V which contains a basis of V over K.

If K_ν is the completion of K at a finite prime ν, we define lattices in K_ν-vector spaces in the same way. The ring of integers O_ν of K_ν is a local ring, which makes working with it much simpler. Moreover, we have the following lemma, whose proof is given by Weil (Chapter 5 of [7]).

Lemma 1. *Let $M_0(K)$ be the set of finite primes of K, and let $L \subseteq K^4$ be a lattice of K^4. For $\nu \in M_0(K)$, denote by L_ν the closure of L in K^4_ν. Then $L = \bigcap_{\nu\in M_0(K)} (L_\nu \cap K^4)$.*

In light of the preceding lemma, we are able to carry out most of our work in the local fields and reconstruct our global classification from these results. Observe that for K_ν, our lattices are free O_ν-modules of rank 4. To explain our classification, we need the notion of primitivity.

Definition 3. We say a K-lattice, L, is primitive if it is contained in O^4_K, and for every prime ideal, $\mathfrak{p}$, of O_K, we have $\mathfrak{p}^{-1}L \nsubseteq O^4_K$.

Again, we make an analogous definition for local fields. Note that in the local case, we may replace a prime ideal by any of its uniformizers.

We say that two lattices, L and L', are equivalent if there is a fractional ideal, $\mathfrak{p}$, of the integer ring of the base field such that $\mathfrak{p}L = L'$. Every lattice is equivalent to a primitive lattice. Our goal is to classify the primitive, invariant lattices in K^4.

Notation 2. Throughout, E_i, for $i = 1,2,3,4$, will denote the standard basis vectors for K^4. We will write $u(\alpha)$ in place of $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$, and ω for $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Finally, we let $\Psi(x) = (u(1)\cdot x) - x$, for $x \in K^4$. It is easy to check that $\Psi(x) = (x_2 + x_3 + x_4, 2x_3 + 3x_4, 3x_4, 0)$. Also, $\omega \cdot x = (x_4, -x_3, x_2, -x_1)$.

We end our overview by presenting the original result of [4]; we will adhere to the notation of [4] for the lattices we introduce.

Theorem 1. *The $SL_2(\mathbb{Z})$-invariant primitive lattices in $\mathbb{Q}^4$ are as follows:*
$L_1 = \mathbb{Z}^4$,
$L_2 = \{(a,b,c,d) \in \mathbb{Z}^4 : 3|b,c\}$,
$L_3 = \{(a,b,c,d) \in \mathbb{Z}^4 : 2|b+c\}$,
$L_4 = \{(a,b,c,d) \in \mathbb{Z}^4 : 2|b+c,a,d; 3|b,c\}$,
$L_5 = \{(a,b,c,d) \in \mathbb{Z}^4 : 2|b+c,a,d\}$,
$L_6 = \{(a,b,c,d) \in \mathbb{Z}^4 : 2|b+c; 3|b,c\}$,
$L_7 = \{(a,b,c,d) \in \mathbb{Z}^4 : 2|a+b+c, b+c+d\}$,
$L_8 = \{(a,b,c,d) \in \mathbb{Z}^4 : 2|a+b+d, a+c+d; 3|b,c\}$,
$L_9 = \{(a,b,c,d) \in \mathbb{Z}^4 : 2|a+b+d, b+c+d\}$,
$L_{10} = \{(a,b,c,d) \in \mathbb{Z}^4 : 2|a+b+c, b+c+d; 3|b,c\}$.

3. Results and proofs

Let L be a primitive, $GL_2(O_K)$-invariant lattice in V_K.

Lemma 2. *If $\nu \nmid 2,3$, then $L_\nu = L_{1\nu} = O_\nu^4$.*

Proof. Let $x = (x_1, x_2, x_3, x_4)$ be primitive for ν. First, suppose that either x_1 or x_4 is a unit of O_ν. By applying ω if necessary, we may assume, without loss of generality, that $x_4 \in O_\nu^*$, the group of units of O_ν. Let $y_1 = x_4^{-1}u(-3^{-1}x_4^{-1}x_3)\cdot x$. Then the third and fourth coordinates of y_1 are 0 and 1, respectively, so we see that $6^{-1}\Psi(\Psi(y_1)) = (1,1,0,0)$. Now, $E_2 = u(-1)\cdot(1,1,0,0)$ and $E_1 = \Psi(E_2)$, so $E_1, E_2 \in (L)_\nu$, and, by applying ω, one sees easily that $E_3, E_4 \in (L)_\nu$ as well. Hence $(L)_\nu = O_\nu^4$. Next, suppose $x_1, x_4 \notin O_\nu^*$. By primitivity, either x_2 or x_3 is a unit, and again, we may assume $x_3 \in O_\nu^*$ by means of ω. Now, consider $u(1)\cdot x + u(-1)\cdot x - 2x$. This element has $2x_3$ as its first coordinate, and $2x_3 \in O_\nu^*$. Thus, we have reduced the problem to the previous case. This proves our lemma. □

Lemma 3. *If $\nu \mid 3$, then $(L)_\nu = O_\nu \oplus \pi_\nu^m O_\nu \oplus \pi_\nu^m O_\nu \oplus O_\nu$ for some $m \in \{0,1,2,\ldots,ord_\nu(3)\}$.*

Proof. Let x be primitive for ν. We first assume x_2 or x_3 to be a unit. As in the preceeding lemma, we may simply assume that x_3 is a unit. Set $y_1 = (2x_3 + 3x_4)^{-1}\Psi(x) = (x_1^{'}, 1, x_3^{'}, 0)$, and also $y_2 = (2x_3 + 6x_4)^{-1}\Psi(\Psi(x)) = (1, x_2^{'}, 0, 0)$. Because, $x_2^{'} = 6x_4(2x_3+6x_4)^{-1}$ and $x_3^{'} = 3x_4(2x_3+3x_4)^{-1}$, we have $x_2^{'}, x_3^{'} \in 3O_\nu$. Let $y_3 = y_1 - x_1^{'}y_2 = (0, 1 - x_2^{'}x_1^{'}, x_3^{'}, 0)$. Now, $1 - x_2^{'}x_1^{'} \in O_\nu^*$, and we have $y_4 = (1 - x_2^{'}x_1^{'})^{-1}(\omega\cdot y_3) = (0, x_2^{''}, 1, 0)$, where we note that $x_2^{''} \notin O_\nu^*$, since it is divisible by $x_3^{'}$. So let $y_5 = u(-2^{-1}x_2^{''})\cdot y_4 = (x_1^{''}, 0, 1, 0)$ and $y_6 = \Psi(y_5) = (1,2,0,0)$. Then $E_1 = 2^{-1}\Psi(y_6)$ and $E_2 = 2^{-1}(y_6 - E_1)$, so $E_1, E_2, \in (L)_\nu$, and as before, $(L)_\nu = O_\nu^4$.

Let us now suppose that $x_2, x_3, \notin O_\nu^*$. By primitivity, and possibly using ω, we may assume, without loss of generality, that $x_4 \in O_\nu^*$. Let $y_7 = \Psi(x) = (x_2 + x_3 + x_4, 2x_3 + 3x_4, 3x_4, 0)$. Then, we see that $x_2 + x_3 + x_4 \in O_\nu^*$, $3x_4 \in 3O_\nu^*$, and $2x_3 + 3x_4 \in \pi_\nu O_\nu$. For brevity, we write $u = x_2 + x_3 + x_4$, $a = 2x_3 + 3x_4$, and $b = 3x_4$. Then $y_7 = (u, a, b, 0)$. We have $\frac{1}{2}(y_7 + \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\cdot y_7) = (0, a, 0, 0)$. So $y_8 = (u, 0, b, 0) \in (L)_\nu$. $x_4^{-1}\Psi(y_8) = 3E_1 + 6E_2$ and $2^{-1}\Psi(3E_1 + 6E_2) = 3E_1$. Also, $3E_2 = 2^{-1}((3E_1 + 6E_2) - 3E1)$ and $E_1 = u^{-1}(y_8 - x_4 3E_3)$, so $E_1, 3E_2 \in (L)_\nu$. We already saw that $(0, a, 0, 0) \in (L)_\nu$, and it follows that $\pi_\nu^{ord_\nu(a)}E_2 \in (L)_\nu$. Let m be the smallest integer such that $\pi_\nu^m E_2 \in (L)_\nu$, and note that $m \in 0, 1, 2, \ldots, ord_\nu(3)$. Then by applying ω, we see that m is also the smallest integer such that $\pi_\nu^m E_3 \in L_\nu$.

Suppose $cE_2 + dE_3$ lies in $(L)_\nu$ and that either c or d has order at ν less than m. (We may disregard the E_1 and E_4 coordinates since we know $E_1, E_4 \in (L)_\nu$.) If $x' = cE_2 + dE_3 \in (L)_\nu$, then $\frac{1}{2}(x' \pm \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)x') \in (L)_\nu$, and we see that both $(0, c, 0, 0)$ and $(0, 0, d, 0)$ lie in $(L)_\nu$, violating the minimality of m. So we conclude that $(L)_\nu = O_\nu \oplus \pi_\nu^m O_\nu \oplus \pi_\nu^m O_\nu \oplus O_\nu$. □

Lemma 4. *If L_ν contains an element of the form $(\alpha, 1, 1, 0)$, then it contains $2E_i$, for $i = 1, 2, 3, 4$.*

Proof. Let $f = (\alpha, 1, 1, 0) \in L_\nu$. Then $\Psi(x) = 2E_1 + 2E_2$. Now, $\Psi(2E_1 + 2E_2) = 2E_1$, and the lemma follows easily. □

Lemma 5. *If $\nu \mid 2$ and $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] > 1$, then $L_\nu = L_{1\nu}$.*

Proof. As before, we choose $x = (x_1, x_2, x_3, x_4)$ to be primitive for ν. First, we suppose that either $x_1 \in O_\nu^*$ or $x_4 \in O_\nu^*$. Applying ω if necessary, we may simply assume that $x_4 \in O_\nu^*$. We set $y_1 = u(-\frac{1}{3}x_4^{-1}x_3) \cdot x$, and note that the third and fourth entries of y_1 are 0 and x_4, respectively. We next set $y_2 = (3x_4)^{-1}\Psi(y_1)$, which becomes $(x_1^{'}, 1, 1, 0)$. By Lemma 4, $2E_i \in L_\nu$, for $i = 1, 2, 3, 4$. Because the residue field extension is nontrivial, there exists $u \in O_\nu^*$ such that $1 - u \in O_\nu^*$. Then also $1 - u^2 \in O_\nu^*$, for $1 - u^2 = (1 - u)(1 + u)$ and $1 + u = (1 - u) + 2u$. Let $y_3 = (u)\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) \cdot y_2 = (x_1^{'}, u, u^2, 0)$, and then let $y_4 = y_2 - y_3 = (0, 1 - u, 1 - u^2, 0)$. Observe that $(1 - u) + (1 - u^2) = (1 - u)(1 + (1 + u)) = (1 - u)(2 + u)$, which is a unit. So we have shown that L_ν contains an element $(0, a, b, 0)$ such that a, b, and $a + b$ are all units. $\Psi((0, a, b, 0)) = (a + b, 2b, 0, 0)$, and since we already know $2E_2 \in L_\nu$, we see that $(a + b, 0, 0, 0) \in L_\nu$. It follows easily that $E_1, E_4 \in L_\nu$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) \cdot (0, a, b, 0) = (0, a, ub, 0)$, and $(0, a, b, 0) - (0, a, ub, 0) = (0, 0, (1 - u)b, 0)$. But we have chosen u so that $1 - u$ is a unit, so we see $E_2, E_3 \in L_\nu$. Hence $L_\nu = L_{1\nu}$.

Next, assume x_1 and x_4 lie in $\pi_\nu O_\nu$. By primitivity, we have that $x_2 \in O_\nu^*$ or $x_3 \in O_\nu^*$. If $x_2 + x_3 \notin O_\nu^*$, then both x_2 and x_3 are units. There exists $v \in O_\nu^*$ such that $1 + v \in O_\nu^*$, and applying $\left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right)$ to x gives $(v^{-1}x_1, x_2, vx_3, v^2x_4)$. Now, $(x_2 + x_3) + (x_2 + vx_3) = 2x_2 + (1 + v)x_3 \in O_\nu^*$, for we have assumed $x_2 + x_3 \in \pi_\nu O_\nu$. This reasoning lets us assume, without loss of generality, that $x = (x_1, x_2, x_3, x_4)$ is such that $x_2 + x_3 \in O_\nu^*$. But then we can apply Ψ to x and get a form with a unit in its first coordinate, thus reducing the proof to the first case of this lemma. □

Lemma 6. *If $\nu \mid 2$ and $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] = 1$, then $L_{5\nu} \subseteq L_\nu$ or $L_{9\nu} \subseteq L_\nu$. Here, $L_{5\nu} = \{(a, b, c, d) \in O_\nu^4 : \pi_\nu \mid a, d, b + c\}$ and $L_{9\nu} = \{(a, b, c, d) \in O_\nu^4 : \pi_\nu \mid a + b + d, a + c + d\}$.*

Proof. If ν is unramified over 2, the argument in [4] applies verbatim. So assume ν is ramified. Let $x \in L$ be primitive for ν. As usual, suppose that either $x_1 \in O_\nu^*$ or $x_4 \in O_\nu^*$. As in the preceeding lemmas, this reduces to the assumption that $x_4 \in O_\nu^*$. This time, we choose the same y_1 as in the previous lemma, we find $y_2 = (x_1^{'}, 1, 1, 0) \in L_\nu$, and we see that $2E_i \in L_\nu$ for $i = 1, 2, 3, 4$. Pick $u = 1 + \pi_\nu \in O_\nu^*$, so that $1 - u$ has order 1 at π_ν. Following Lemma 5, let $y_3 = u\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) \cdot y_2 = (x_1^{'}, u, u^2, 0)$. Now, $y_2 - y_3 = (0, 1 - u, 1 - u^2, 0)$. Also, $1 + u = 1 + 1 + \pi_\nu = 2 + \pi_\nu$, hence $1 + u$ has order 1 at ν. Since $1 - u$ also has order 1, it follows that $1 - u^2 \in \pi_\nu^2 O_\nu^*$. Applying matrices of the forms $\left(\begin{smallmatrix} v_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & v_2 \end{smallmatrix}\right)$, with $(v_i \in O_\nu^*)$, to $y_2 - y_3$, we see that $(0, \pi_\nu, \pi_\nu^2, 0) = \pi_\nu E_2 + \pi_\nu^2 E_3 \in L_\nu$. Now, $\Psi((0, \pi_\nu, \pi_\nu^2, 0)) = (\pi_\nu + \pi_\nu^2, 2\pi_\nu^2, 0, 0)$. Since $2\pi_\nu^2 E_2 \in L_\nu$, $(\pi_\nu + \pi_\nu^2, 0, 0, 0)$ and, in turn, $\pi_\nu E_1, \pi_\nu E_4$ lie in L_ν. Then also

$(0, \pi_\nu, \pi_\nu^2, -\frac{1}{3}\pi_\nu^2) \in L_\nu$. Next, $u(1) \cdot (0, \pi_\nu, \pi_\nu^2, -\frac{1}{3}\pi_\nu^2) = (z, \pi_\nu + \pi_\nu^2, 0, -\frac{1}{3}\pi_\nu^2)$. Here, $z = \pi_\nu + \pi_\nu^2 - \frac{1}{3}\pi_\nu^2 \in \pi_\nu O_\nu$. Since $\pi_\nu E_1$ and $\pi_\nu E_4$ are in L_ν, we can show that $(0, \pi_\nu + \pi_\nu^2, 0, 0) \in L_\nu$, and it follows easily that $\pi_\nu E_2$ and $\pi_\nu E_3$ are in L_ν. We have thus shown that $\pi_\nu E_i \in L_\nu$ for $i = 1, 2, 3, 4$.

Now, consider the case where x_1 and x_4 are nonunits, so that x_2 or x_3 is a unit. If $x_2 + x_3 \in O_\nu^*$, we can apply Ψ to x and use the above case to see that $\pi_\nu E_i \in L_\nu$ for $i = 1, 2, 3, 4$. Otherwise, we have that x_2 and x_3 are both units. Choose $v \in O_\nu^*$ so that $x_2 + vx_3 = 0$, and replace x by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right) \cdot x$. The new $x = (x_1, x_2, x_3, x_4)$ satisfies $x_2 + x_3 = 0$. We proceed as follows: let $w'(z) = -\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot z$; that is, w' reverses the coordinates,

$$\Psi(x) = (x_2 + x_3 + x_4, 2x_3 + 3x_4, 3x_4, 0),$$

and since $x_2 + x_3 = 0$,

$$w'(\Psi(x)) = (0, 3x_4, 2x_3 + 3x_4, x_4).$$

Subtracting this result from x, we get $(z', a, b, 0)$, where $a, b \in O_\nu^*$. Using the matrices of the forms $\left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right)$ (for units u, v), we may insist that $a = b = 1$. Lemma 4 now shows that $2E_i \in L_\nu$ for $i = 1, 2, 3, 4$, and from here the argument from the previous case (with y_2 replaced by $(z', 1, 1, 0)$) can be used to show that $\pi_\nu E_i \in L_\nu$ for $i = 1, 2, 3, 4$.

So regardless of which coordinates of x are initially taken to be units, we have seen that L_ν has an element of the form $(z, 1, 1, 0)$. Now, observe that $L_{5\nu} = O_\nu(\pi_\nu E_1) \oplus O_\nu(\pi_\nu E_4) \oplus O_\nu(E_2 + E_3) \oplus O_\nu(\pi_\nu E_2)$ and

$$L_{9\nu} = O_\nu(E_1 + E_2 + E_3) \oplus O_\nu(E_2 + E_3 + E_4) \oplus O_\nu(\pi_\nu E_2) \oplus O_\nu(\pi_\nu E_3).$$

In addition to $(z, 1, 1, 0)$, L_ν also contains $\pi_\nu E_i$, for $i = 1, 2, 3, 4$. If $z \notin O_\nu^*$, then $(z, 1, 1, 0) - (\pi_\nu^{-1} z)\pi_\nu E_1 = E_2 + E_3 \in L_\nu$, so $L_{5\nu} \subseteq L_\nu$. If $z \in O_\nu^*$, then we can write $z = 1 + z'$, where $z' \in \pi_\nu O_\nu$, since $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] = 1$. Then $(z, 1, 1, 0) - (\pi_\nu^{-1} z')\pi_\nu E_1 = E_1 + E_2 + E_3 \in L_\nu$, and by applying w', we can conclude that $E_2 + E_3 + E_4 \in L_\nu$. Hence $L_{9\nu} \subseteq L_\nu$. This completes the proof of the lemma. □

Notation 3. Up to this point, we have defined L_ν, $L_{5\nu}$, and $L_{9\nu}$. Now, we introduce $L_{3\nu} = \{(a, b, c, d) \in O_\nu^4 : \pi_\nu \mid b + c\}$ and $L_{7\nu} = \{(a, b, c, d) \in O_\nu^4 : \pi_\nu \mid a + b + c, b + c + d\}$.

In view of Lemma 6, we can prove the following, more precise lemma.

Lemma 7. *If $\nu \mid 2$ and $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] = 1$, then $L_\nu \in \{L_{1\nu}, L_{3\nu}, L_{5\nu}, L_{7\nu}, L_{9\nu}\}$.*

Proof. By Lemma 6, either $L_{5\nu} \subseteq L_\nu \subseteq L_{1\nu}$ or $L_{9\nu} \subseteq L_\nu \subseteq L_{1\nu}$.

Case I. Assume that $L_{5\nu} \subseteq L_\nu \subseteq L_{1\nu}$. If $L_\nu = L_{5\nu}$, there is nothing to prove, so assume that L_ν properly contains $L_{5\nu}$. We will write $L_{1\nu} = O_\nu E_1 \oplus O_\nu E_4 \oplus O_\nu(E_2 + E_3) \oplus O_\nu E_2$ and $L_{5\nu} = O_\nu \pi_\nu E_1 \oplus O_\nu \pi_\nu E_4 \oplus O_\nu(E_2 + E_3) \oplus O_\nu \pi_\nu E_2$. Then $\{aE_1 + bE_4 + cE_2 : a, b \in \{0, 1\}\}$ is a set of coset representatives for $L_{1\nu}/L_{5\nu}$. The fact that $L_{5\nu} \subsetneq L_\nu \subseteq L_{1\nu}$ implies that one of our nonzero coset representatives lies in L_ν. Suppose the representative that lies in L_ν is E_1, E_4, or $E_1 + E_4$. L_ν also contains $L_{5\nu}$, so we can show $L_{3\nu} \subseteq L_\nu$ for $L_{3\nu} = O_\nu E_1 \oplus O_\nu E_4 \oplus O_\nu(E_2 + E_3) \oplus O_\nu \pi_\nu E_2$. Indeed, the E_1 and E_4 cases are obvious. If instead

we have $E_1+E_4 \in L_\nu$, note that $\Psi(E_1+E_4) = E_1+3(E_2+E_3)$, which reduces this case to that of E_1. Hence $L_{3\nu} \subseteq L_\nu$, as desired. But $L_{1\nu}/L_{3\nu} \cong \mathbb{Z}/2\mathbb{Z}$, and so no lattices lie (properly) between $L_{1\nu}$ and $L_{3\nu}$. Hence either $L_\nu = L_{3\nu}$ or $L_\nu = L_{1\nu}$. Next, suppose that the coset representative is E_2, E_1+E_2, or E_2+E_4. Recall that $2E_2 \in L_{5\nu} \subseteq L_\nu$. Now, $E_1 = \Psi(E_2) = \Psi(E_1+E_2) = \Psi(\omega(E_2+E_4)) - 2E_2$, and we can easily show that both E_1 and E_2 are in L_ν. In this case, $L_\nu = L_{1\nu}$. To finish this case, suppose the coset representative that lies in L_ν is $E_1+E_2+E_4$. Write $L_{7\nu} = O_\nu(E_1+E_2+E_4) \oplus O_\nu(E_1+E_3+E_4) \oplus O_\nu\pi_\nu E_1 \oplus O_\nu\pi_\nu E_4$. Now, $E_2+E_3 \in L_{5\nu}$, and clearly, $(E_1+E_2+E_4)-(E_2+E_3)+2E_3 = (E_1+E_3+E_4)$, so $L_{7\nu} \subseteq L_\nu$. Moreover, $L_{1\nu}/L_{7\nu}$ has $\{0, E_1, E_4, E_1+E_4\}$ as a set of coset representatives. L_ν must contain one of these, and if it contains any of the nonzero members of this set, it contains them all (see the argument above). We have seen that this forces $L_\nu = L_{1\nu}$. Otherwise, $L_\nu = L_{7\nu}$.

Case II. Assume that $L_{9\nu} \subseteq L_\nu \subseteq L_{1\nu}$. We have that $L_{9\nu} = O_\nu(E_1+E_2+E_3) \oplus O_\nu(E_2+E_3+E_4) \oplus O_\nu\pi_\nu E_1 \oplus O_\nu\pi_\nu E_2$ and $L_{1\nu} = O_\nu(E_1+E_2+E_3) \oplus O_\nu(E_2+E_3+E_4) \oplus O_\nu E_2 \oplus O_\nu E_3$. A set of coset representatives for $L_{1\nu}/L_{9\nu}$ is given by $\{aE_2+bE_3 : a,b \in \{0,1\}\}$. Again, L_ν contains at least one coset representative, and if the only representative in L_ν is 0, then $L_\nu = L_{9\nu}$. If L_ν contains E_2 or E_3, it clearly contains both, and $\Psi(E_2) = E_1$ implies that also $E_1 \in L_\nu$. Hence $L_\nu = L_{1\nu}$. If instead the coset representative contained in L_ν is E_2+E_3, then L_ν contains $(E_1+E_2+E_3)-(E_2+E_3) = E_1$. It follows that $E_4 \in L_\nu$, and thus $L_{3\nu} \subseteq L_\nu$. We have seen that this implies that either $L_\nu = L_{3\nu}$ or $L_\nu = L_{1\nu}$. This completes the proof. □

We are now in a position to state and prove our main theorem.

Theorem 2. *Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ be the prime ideal divisors of $3O_K$, and let $\mathfrak{p}_{r+1}$, $\mathfrak{p}_{r+2}, \ldots, \mathfrak{p}_t$ be the divisors of $2O_K$ with residue field $\mathbb{Z}/2\mathbb{Z}$. Then the primitive, $GL_2(O_K)$-invariant lattices of K^4 are of the form $\{(a,b,c,d) \in O_K^4 : b,c \in \mathfrak{p}_1^{m_1}\cdots\mathfrak{p}_r^{m_r}; (b+c) \in \mathfrak{p}_{r+1}^{n_{r+1,1}}\cdots\mathfrak{p}_t^{n_{t,1}}; a,d,(b+c) \in \mathfrak{p}_{r+1}^{n_{r+1,2}}\cdots\mathfrak{p}_t^{n_{t,2}}; (a+b+c),\ (b+c+d) \in \mathfrak{p}_{r+1}^{n_{r+1,3}}\cdots\mathfrak{p}_t^{n_{t,3}}; (a+b+d),(a+c+d) \in \mathfrak{p}_{r+1}^{n_{r+1,4}}\cdots\mathfrak{p}_t^{n_{t,4}}\}$, where $0 \le m_i \le ord_{\mathfrak{p}_i}(3)$, $0 \le n_{i,j} \le 1$, and $n_{i,j} = 1$ implies $n_{i,k} = 0$ for any $k > j$.*

Proof. Combine Lemma 2 through Lemma 7, and use the fact that a lattice $L \subseteq K^4$ is given by $\bigcap_{\nu \in M_0(K)} (L_\nu \bigcap K^4)$, where L_ν is the closure in K_ν^4 of L. □

4. Dirichlet series and the functional equation

In this section, we connect our invariant lattices to the adelic zeta function for the space of binary cubic forms (see [2] and [8]). Before proceeding, we need to introduce some terminology pertaining to the notion of a splitting type. Let $S \subseteq M(K)$. For each $\nu \in S$, let A_ν be the set of Galois conjugacy classes of extensions of the local field K_ν of degree not exceeding 3. Let $A_S = \prod_{\nu \in S} A_\nu$, so the elements α of A_S are of the form $\alpha = (K_{\alpha\nu_1}, K_{\alpha\nu_2}, \ldots)$, where each $K_{\alpha\nu}$ is an extension of K_ν. Let L be an extension of K of degree at most 3. Then $K_\nu \bigotimes_K L$ is a direct sum of extensions of K_ν of total degree $[L:K]$. The structure of the summands uniquely determines how the prime ν of K factors in L. For each ν, let $F_{L\nu}$ denote the summand of $K_\nu \bigotimes_K L$ with highest degree over K_ν. Let $\alpha \in A_S$. We say that L has S-splitting type α if for each $\nu \in S$, $K_{\alpha\nu} \cong F_{L\nu}$. We will also

call α the S-splitting signature of L, and write $L \sim \alpha$. For $x \in V_K^{'}$, we write $x \sim \alpha$ if the splitting field $K(x)$ has S-splitting signature α. If α_ν is a Galois conjugacy class as described above, we let $Z_{\alpha_\nu}(s, \Phi_\nu) = \int_{G_\nu} |\det(g_\nu)|_\nu^s \Phi(g_\nu \cdot X_\nu) dg_\nu$, where X_ν is the standard orbital representative for the conjugacy class of extensions α_ν. If $S \subseteq M(K)$ and $\alpha \in A_S$, we define $Z_\alpha(s, \Phi) = \prod_{\nu \in S} Z_{\alpha_\nu}(s, \Phi_\nu)$.

We begin by fixing a number field, K, and let $\mathbb{A}$ denote the ring of adeles of K. We abbreviate $GL_2(\mathbb{A})$ simply as G, and write G_K for $GL_2(K)$, viewed as a subgroup of G. We introduce

$$G(\infty) = \prod_{\nu \in M_\infty(K)} GL_2(K_\nu) \times \prod_{\nu \in M_0(K)} GL_2(O_\nu).$$

Observe that the number of double cosets of $G(\infty)\backslash G/G_K$ is just h_K, the class number of K. We have the decomposition

$$G = \bigcup_{t \in \mathbb{A}^*/K^*\mathbb{A}^*(\infty)} G(\infty) \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right) G_K.$$

Observe also that $G(\infty)G_K = \{g \in G : \det(g) \in K^*\mathbb{A}^*(\infty)\}$, where

$$\mathbb{A}^*(\infty) = \prod_{\nu \in M_\infty(K)} K_\nu^* \times \prod_{\nu \in M_0(K)} O_\nu^*.$$

Now we fix a (primitive) invariant lattice $L \subseteq V_K$. As in our introduction, we write $|a|$ for the adelic absolute value of $a \in \mathbb{A}_K$. We choose a Schwartz-Bruhat function, $\Phi = \prod_\nu \Phi_\nu$, whose finite components are the characteristic functions for the respective closures of L. We note that for $\nu \in M_0(K)$, this closure is a free, $GL_2(O_\nu)$-invariant O_ν-module of rank 4. For notational convenience, we also write $\Phi = \prod_{\nu \in M_\infty(K)} \Phi_\nu \times 1_{L_0}$, where $L_0 = \prod_{\nu \in M_0(K)} L_\nu$, and 1_{L_0} is the characteristic function for L_0. Let $H = \mathbb{A}^*/K^*\mathbb{A}^*(\infty)$, and let $\hat{H}$ be the dual of H. We now consider the sum

$$\frac{1}{h_K} \sum_{\chi \in \hat{H}} Z(\chi\omega_s, \Phi).$$

Proposition 3.

$$\frac{1}{h_K} \sum_{\chi \in \hat{H}} Z(\chi\omega_s, \Phi) = \int_{G(\infty)/GL_2(O_K)} |\det(g)|^s \sum_{x \in L'} \Phi(g \cdot x) dg,$$

where $L^{'} = \{x \in L : P(x) \neq 0\}$.

Proof. First, note that

$$\frac{1}{h_K} \sum_{\chi \in \hat{H}} Z(\chi\omega_s, \Phi) = \int_{G/G_K} \frac{1}{h_K} \sum_{\chi \in \hat{H}} \chi(\det(g)) |\det(g)|^s \sum_{x \in V_K^{'}} \Phi(g \cdot x) dg.$$

This reduces to

$$\int_{G(\infty)G_K/G_K} |\det(g)|^s \sum_{x \in V_K^{'}} \Phi(g \cdot x) dg,$$

since $\sum_{\chi\in\hat{H}}\chi(\det(g))=0$ whenever $g\notin G(\infty)G_K$. By an isomorphism theorem, this integral is just

$$\int_{G(\infty)/G_K\bigcap G(\infty)}|\det(g)|^s\sum_{x\in V_K'}\Phi(g\cdot x)dg.$$

Note that $G_K\cap G(\infty)=GL_2(O_K)$. Now if $g\in G(\infty)$, then $g=(g_\nu)_\nu$, where $g_\nu\in GL_2(O_\nu)$ for all finite places ν. We have that $g_\nu\cdot x\in L_\nu$ iff $x\in g_\nu^{-1}L_\nu$, where L_ν is the local component of L_0. Now, since L_ν is $GL_2(O_\nu)$-invariant, we see that $g_\nu\cdot x\in L_\nu$ iff $x\in L_\nu$. If $x\in L_\nu$ for all finite ν, then we have $x\in K\bigcap\prod_{\nu\in M_0(K)}L_\nu=L$, by Lemma 1 of Section 2. Hence

$$\int_{G(\infty)/G_K\bigcap G(\infty)}|\det(g)|^s\sum_{x\in V_K'}\Phi(g\cdot x)dg=\int_{G(\infty)/GL_2(O_K)}|\det(g)|^s\sum_{x\in L'}\Phi(g\cdot x)dg,$$

where L' is as described in the proposition. □

The integral on the right hand side of Proposition 3 may be rewritten as a sum, namely

$$\sum_{x\in GL_2(O_K)\backslash L'}\frac{1}{|\mathrm{Stab}_{G_K}(x)|}\int_{G(\infty)}|\det(g)|^s\Phi(g\cdot x)dg.$$

The integrals under the sum are of the form

$$\int_{G(\infty)}|\det(g)|^s\Phi(g\cdot x)dg$$
$$=\prod_{\nu\in M_\infty(K)}\int_{GL_2(K_\nu)}|\det(g_\nu)|_\nu^s\Phi_\nu(g_\nu\cdot x)dg_\nu\times\prod_{\nu\in M_0(K)}\int_{GL_2(O_\nu)}1_{L_\nu}(g_\nu\cdot x)dg_\nu.$$

All of the finite local factors evaluate to 1. We denote the product of the infinite local factors by $Z_{x_\infty}(s,\Phi_\infty)$. Our work and conventions thus show:

Proposition 4.

$$\frac{1}{h_K}\sum_{\chi\in\hat{H}}Z(\chi\omega_s,\Phi)=\sum_{x\in GL_2(O_K)\backslash L'}\frac{1}{|\mathrm{Stab}_{G_K}(x)|}Z_{x_\infty}(s,\Phi_\infty),$$

where $Z_{x_\infty}(s,\Phi_\infty)=\prod_{\nu\in M_\infty(K)}\int_{GL_2(K_\nu)}|\det(g_\nu)|_\nu^s\Phi_\nu(g_\nu\cdot x)dg_\nu.$

The next proposition will help us relate these integrals to discriminants of cubic forms.

Proposition 5.

$$Z_{x_\infty}(s,\Phi_\infty)=Z_\alpha(s,\Phi_\infty)|N_{K/\mathbb{Q}}(P(x))|^{\frac{-s}{2}},$$

where α *is the* $M_\infty(K)$*-splitting type of* x.

Proof. In each $Z_{x_\infty}(s,\Phi_\infty)$ replace x by its corresponding standard orbital representative, which we will call X. Then

$$Z_{x_\infty}(s,\Phi_\infty)=Z_{X_\infty}(s,\Phi_\infty)|N_{K/\mathbb{Q}}(P(x))|^{\frac{-s}{2}}.$$

This immediately proves our proposition, since if x has $M_\infty(K)$-splitting type $\alpha = (\alpha_\nu)_{\nu\in M_\infty(K)}$ and X_∞ is the product of the standard orbital representatives for the conjugacy classes of extensions α_ν, then $Z_{X_\infty}(s,\Phi_\infty) = Z_\alpha(s,\Phi_\infty)$.

To see how the norm of the discriminant arises, let $g = (g_\nu)_\nu$ be the matrix such that $g\cdot x = X$. In each local factor of $Z_{x_\infty}(s,\Phi_\infty)$, which is of the form $\int_{GL_2(K_\nu)} |\det(h_\nu)|_\nu^s \Phi_\nu(h_\nu\cdot x)dh_\nu$, we make a change of variables, replacing h_ν by $h_\nu g_\nu$. Then the local factor becomes $|\det(g_\nu)|_\nu^s Z_{X_\nu}(s,\Phi_\nu)$ and the product of the local factors becomes $\prod_{\nu\in M_\infty(K)} |\det(g_\nu)|_\nu^s Z_X(s,\Phi_\infty)$. Now, observe that $\det(g_\nu)^2 P(x_\nu) = P(g_\nu\cdot x_\nu) = P(X_\nu)$. Thus $\prod_{\nu\in M_\infty(K)} |\det(g_\nu)|_\nu^2 = \dfrac{\prod_{\nu\in M_\infty(K)} |P(X_\nu)|_\nu}{|N_{K/\mathbb{Q}}(P(x))|}$.

But at each infinite place ν, $|P(X_\nu)|_\nu = 1$, by choice of orbital representative. $\square$

Now, let

$$E_L(s,\Phi) = \frac{1}{h_K}\sum_{\chi\in\hat{H}} Z(\chi\omega_s,\Phi).$$

Let α denote an $M_\infty(K)$-splitting type. We define

$$\xi_\alpha(s,L) = \sum_{\substack{x\in GL_2(O_K)\backslash L' \\ x\sim\alpha}} \frac{1}{|\mathrm{Stab}_{G_K}(x)|}|N_{K/\mathbb{Q}}(P(x))|^{-\frac{s}{2}}.$$

Taken together, Propositions 4 and 5 prove the following:

Proposition 6.

$$E_L(s,\Phi) = \sum_\alpha Z_\alpha(s,\Phi_\infty)\xi_\alpha(s,L).$$

In the sum on the right, α runs through all possible $M_\infty(K)$-splitting types.

We conclude this paper by deriving a functional equation for the series $\xi_\alpha(s,L)$ that relates $\xi_\alpha(2-s,L)$ to $\xi_\alpha(s,\hat{L})$, the series associated with the dual lattice $\hat{L}$ of L. Using the functional equation for the adelic zeta function, we have

$$E_L(s,\Phi) = \frac{1}{h_K}\sum_{\chi\in\hat{H}} Z(\chi\omega_s,\Phi) = \frac{1}{h_K}\sum_{\chi\in\hat{H}} Z(\chi\omega_{2-s},\hat{\Phi}) = E_{\hat{L}}(2-s,\hat{\Phi}).$$

Combining this equation with Proposition 6 shows that

$$\sum_\alpha Z_\alpha(2-s,\Phi)\xi_\alpha(2-s,L) = \sum_\beta Z_\beta(s,\hat{\Phi})\xi_\beta(s,\hat{L}). \tag{1}$$

In [2], Datskovsky and Wright gave a functional equation for $Z_\beta(s,\hat{\Phi})$. Indeed, we have

$$Z_\beta(s,\hat{\Phi}) = \sum_\alpha C_{\beta\alpha}(s)Z_\alpha(2-s,\Phi). \tag{2}$$

Here, as in [2], $C_{\beta\alpha}(s)$ is a product of local factors, running over $M_\infty(K)$. If we write $\gamma(s) = \Gamma(s-\frac{1}{6})\Gamma(s)^2\Gamma(s+\frac{1}{6})$ and $\rho(s) = \frac{1}{2}\pi^{-2s}3^{3s-3}$, then we can say that,

for a real prime ν of K, the local factor $C_{\beta_\nu\alpha_\nu}(s)$ of $C_{\beta\alpha}(s)$ is

$$C_{\beta_\nu\alpha_\nu}(s) = \begin{cases} \rho(s)\gamma(\frac{s}{2})\sin(\frac{\pi s}{2}) & \text{if only } \beta_\nu \text{ is split,} \\ \rho(s)\gamma(\frac{s}{2})\sin(\frac{3\pi s}{2}) & \text{if only } \alpha_\nu \text{ is split,} \\ \rho(s)\gamma(\frac{s}{2})\sin(\pi s) & \text{otherwise.} \end{cases}$$

When ν is instead complex, we let $r(s) = \pi^{-2s}3^{3s}\gamma(\frac{s}{2})$, and find that

$$C_{\beta_\nu\alpha_\nu}(s) = \frac{r(s)}{r(2-s)}.$$

With this definition of $C_{\beta\alpha}(s)$ in place, we may now give our functional equation for $\xi_\alpha(s, L)$.

Theorem 3.

$$\xi_\alpha(2-s, L) = \sum_\beta C_{\beta\alpha}(s)\xi_\beta(s, \hat{L}).$$

Proof. We can use equation (2) to rewrite

$$\sum_\beta Z_\beta(s, \hat{\Phi})\xi_\beta(s, \hat{L})$$

as

$$\sum_\beta Z_\beta(s, \hat{\Phi})\xi_\beta(s, \hat{L}) = \sum_\beta \left(\sum_\alpha C_{\beta\alpha}(s)Z_\alpha(2-s, \Phi)\right)\xi_\beta(s, \hat{L}).$$

Then

$$\begin{aligned} \sum_\alpha Z_\alpha(2-s, \Phi)\xi_\alpha(2-s, L) &= \sum_\beta \left(\sum_\alpha C_{\beta\alpha}(s)Z_\alpha(2-s, \Phi)\right)\xi_\beta(s, \hat{L}) \\ &= \sum_\alpha \left(\sum_\beta C_{\beta\alpha}(s)\xi_\beta(s, \hat{L})\right) Z_\alpha(2-s, \Phi). \end{aligned}$$

Given α and β, we may choose Φ so that $Z_\alpha(2-s, \Phi) = \delta_{\alpha\beta}$. Upon doing so, we find that

$$\xi_\alpha(2-s, L) = \sum_\beta C_{\beta\alpha}(s)\xi_\beta(s, \hat{L}),$$

as desired. □

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