LOG-CONCAVITY OF THE DUISTERMAAT-HECKMAN MEASURE FOR SEMIFREE HAMILTONIAN $S^1$-ACTIONS

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(Communicated by Lei Ni)

This paper is dedicated to my father

Abstract. The Ginzburg-Knutson conjecture states that for any Hamiltonian Lie group $G$-action, the corresponding Duistermaat-Heckman measure is log-concave. It turns out that the conjecture is not true in general, but every well-known counterexample has non-isolated fixed points. In this paper, we prove that if the Hamiltonian circle action on a compact symplectic manifold $(M,\omega)$ is semifree and all fixed points are isolated, then the Duistermaat-Heckman measure is log-concave. With the same assumption, we also prove that $\omega$ and every reduced symplectic form satisfy the hard Lefschetz property.

1. Introduction

In statistical mechanics, consider the relation $S(E) = k \log W(E)$, which is called Boltzmann’s principle, where $W(E)$ is the number of states with given values of macroscopic parameters $E$ (like energy, temperature, · · · ), $k$ is the Boltzmann’s constant, and $S$ is the entropy of the system which measures the degree of disorder in the system. For the additive values $E$, it is well-known that the entropy is always a concave function. (See [18] for more details.)

Now, consider a Hamiltonian $G$-manifold $(M,\omega)$ with the moment map $\mu : M \to g^*$. The Liouville measure $m_L$ is defined by

$$m_L(U) := \int_U \omega^n \frac{n!}{n!}$$

for any open set $U \subset M$. The push-forward measure $m_{DH} := \mu_* m_L$ is called the Duistermaat-Heckman measure. Then $m_{DH}$ can be regarded as a measure on $g^*$ such that for any Borel subset $B \subset g^*$, $m_{DH}(B) = \int_{\mu^{-1}(B)} \omega^n$ tells us how many states of our system have momenta in $B$. By the Duistermaat-Heckman theorem [5], $m_{DH}$ can be expressed in terms of the density function $DH(\xi)$ with respect to the Lebesgue measure on $g^*$. Hence if we consider Boltzmann’s principle in our Hamiltonian system with an identification $W = DH$, it is natural to ask whether the Duistermaat-Heckman measure $m_{DH}$ is log-concave. As noted in [17], [11], and [14], V. Ginzburg and A. Knutson conjectured that for any closed Hamiltonian $T$-manifold, the corresponding Duistermaat-Heckman measure is log-concave.
The log-concavity problem of the Duistermaat-Heckman measure is proved by A. Okounkov [16] when $M$ is a co-adjoint orbit of the classical Lie groups of type $A_n$, $B_n$, or $C_n$ with the maximal torus action. Around the same time, W. Graham [7] proved that the log-concavity property holds for any holomorphic Hamiltonian circle action on any Kähler manifold. But the counterexample was found by Y. Karshon [11]. By using Lerman’s symplectic cutting method, she constructed a closed 6-dimensional semifree Hamiltonian $S^1$-manifold with two fixed components such that the Duistermaat-Heckman measure is not log-concave. Later, Y. Lin [14] generalized the construction of 6-dimensional Hamiltonian $S^1$-manifolds which do not satisfy the log-concavity of the Duistermaat-Heckman measure. But all counterexamples of Karshon and Lin are the cases when each fixed component is of codimension two (i.e. non-isolated). So, the log-concavity problem is still open for the case when $(M, \omega)$ is a Hamiltonian $S^1$-manifold whose fixed components are of codimension greater than two. In this paper, we will show that

**Theorem 1.1.** Let $(M, \omega)$ be a closed symplectic manifold with a semifree Hamiltonian $S^1$-action whose fixed point set $M^{S^1}$ consists of isolated points. Then the Duistermaat-Heckman measure is log-concave.

The conditions “semifree” and “isolated fixed points” enable us to use the Tolman-Weitsman basis [19] of the equivariant cohomology $H^{S^1}_*(M)$. As you will see in Section 2 any semifree Hamiltonian $S^1$-manifold with only isolated fixed points has a lot of remarkable properties. In fact, the cohomology ring and the equivariant cohomology ring of $M$ are the same as those of $S^2 \times \cdots \times S^2$ with a diagonal semifree circle action. In particular, $(M, \omega)$ is equivariantly symplectomorphic to the product space of $S^2$ copies $(S^2 \times \cdots \times S^2, \sigma)$ with some $S^1$-invariant Kähler structure $\sigma$ when dim $M \leq 6$. (See [12] and [6].) Therefore, we may ask whether $(M, \omega)$ satisfies the properties which the diagonal circle action on $(S^2 \times \cdots \times S^2, \sigma)$ satisfies. In this paper, we prove the following.

**Theorem 1.2.** Let $(M, \omega)$ be a closed semifree Hamiltonian $S^1$-manifold whose fixed points are all isolated, and let $\mu$ be the moment map. Then $\omega$ satisfies the hard Lefschetz property. Moreover, the reduced symplectic form $\omega_t$ satisfies the hard Lefschetz property for every regular value $t$.

In Section 2 we briefly review Tolman and Weitsman’s work [19] which is very powerful to analyze the equivariant cohomology of the Hamiltonian $S^1$-manifold with isolated fixed points as we mentioned above. Especially we use the Tolman-Weitsman basis of the equivariant cohomology $H^{S^1}_*(M)$ which is constructed by using the equivariant version of Morse theory [20]. In Section 3 we express the Duistermaat-Heckman function explicitly in terms of the integration of some cohomology class on the reduced space. Then we compute the integration by using the Kirwan-Jeffrey residue formula [10]. Consequently, we will show that the log-concavity of the Duistermaat-Heckman measure is completely determined by the set of pairs $\{(\mu(F), m_F) | F \in M^{S^1}\}$, where $\mu(F)$ is the image of the moment map of $F$ and $m_F$ is the product of all weights of the $S^1$-representation on $T_F M$ (Proposition 3.9). In Section 4 we will prove Theorem 1.1 and we will prove Theorem 1.2 in Section 5.
2. Tolman-Weitsman basis
of the equivariant cohomology $H^*_S(M; \mathbb{Z})$

In this section we briefly review Tolman and Weitsman’s results in [19]. Throughout this section, we assume that $(M^{2n}, \omega)$ is a closed semifree Hamiltonian $S^1$-manifold whose fixed points are isolated. Note that for each fixed point $p \in M^{S^1}$, the index of $p$ is the Morse index of the moment map at $p$, which is the same as twice the number of negative weights of the tangential $S^1$-representation at $p$.

**Proposition 2.1** ([19]). Let $N_k$ be the number of fixed points of index $2k$. Then $N_k = \binom{2k}{k}$. Hence $N_k$ is the same as that of the standard diagonal circle action on $(S^2 \times \cdots \times S^2, \omega_1 \oplus \cdots \oplus \omega_n)$, where $\omega_1$ is the Fubini-Study form on $S^2$ of $i$-th factor.

**Theorem 2.2** ([19]). Let $2^{[n]}$ be the power set of $\{1, \cdots, n\}$. Then there exists a bijection $\phi : M^{S^1} \to 2^{[n]}$ satisfying the following:

1. For each index-2k fixed point $x \in M^{S^1}$, $|\phi(x)| = k$.
2. Let $u$ be the generator of $H^*(BS^1, \mathbb{Z})$. For each index-2k fixed point $x \in M^{S^1}$, there exists a unique cohomology class $\alpha_x \in H^2_{S^1}(M; \mathbb{Z})$ such that for any $x' \in M^{S^1}$,
   - $\alpha_{x|x'} = u^k$ if $\phi(x) \subset \phi(x')$.
   - $\alpha_{x|x'} = 0$ otherwise.

Here, $\alpha_{x|x'}$ means the image $\pi_{x'}(i^*(\alpha))$ where $i^* : H^*_S(M; \mathbb{Z}) \to H^*_S(M^{S^1}; \mathbb{Z})$ is a homomorphism induced by an inclusion $i : M^{S^1} \hookrightarrow M$, and $\pi : H^*_S(M^{S^1}; \mathbb{Z}) \to H^*_S(x'; \mathbb{Z})$ is a natural projection. Moreover $\{\alpha_x | x \in M^{S^1}\}$ forms a basis of $H^*_S(M; \mathbb{Z})$.

If we apply Theorem 2.2 to $(S^2 \times \cdots \times S^2, \omega_1 \oplus \cdots \oplus \omega_n)$ with the diagonal semifree Hamiltonian circle action, we get a bijection $\psi : (S^2 \times \cdots \times S^2)^{S^1} \to 2^{[n]}$ and there is a basis $\{\beta_y | y \in (S^2 \times \cdots \times S^2)^{S^1}\}$ of $H^*_S(S^2 \times \cdots \times S^2; \mathbb{Z})$ that satisfies the conditions in Theorem 2.2. Hence we have an identification map

$$
\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}
$$

and $\psi^{-1} \circ \phi$ preserves the indices of the fixed points.

Note that $\psi^{-1} \circ \phi$ gives an identification between $H^*_S(M; \mathbb{Z})$ and $H^*_S(S^2 \times \cdots \times S^2; \mathbb{Z})$ as follows. Let $a_i = \alpha_{\phi^{-1}(i)} \in H^2_{S^1}(M; \mathbb{Z})$ and $b_i = \beta_{\psi^{-1}(i)} \in H^2_{S^1}(S^2 \times \cdots \times S^2; \mathbb{Z})$. The following lemma is proved by Tolman and Weitsman in [19], but we give a complete proof here to use their idea in the rest of this paper.

**Lemma 2.3** ([19]). For each $x \in M^{S^1}$, we have $\alpha_x = \prod_{j \in \phi(x)} a_j$. Similarly, we have $\beta_y = \prod_{j \in \psi(y)} b_j$ for each $y \in (S^2 \times \cdots \times S^2)^{S^1}$.

**Proof.** For an inclusion $i : M^{S^1} \rightarrow M$, we have a natural ring homomorphism $i^* : H^*_S(M) \rightarrow H^*_S(M^{S^1}) \cong H^*(M^{S^1}) \otimes H^*(BS^1)$. Kirwan’s injectivity theorem [13] implies that $i^*$ is an injective ring homomorphism. Hence it is enough to show that $\alpha_x | z = \prod_{j \in \phi(x)} a_j | z$ for all $x, z \in M^{S^1}$. For any $x, z \in M^{S^1}$ with $\text{Ind}(x) = 2k$,

- $\alpha_x | z = u^k$ if $\phi(x) \subset \phi(z)$.
- $\alpha_x | z = 0$ otherwise.
On the other hand, \( \langle \prod_{j \in \phi(x)} a_j \rangle_z = \prod_{j \in \phi(x)} a_j |_z \). Since \( a_j |_z = u \) if and only if \( j \in \phi(z) \), we have

- \( \langle \prod_{j \in \phi(x)} a_j \rangle_z = u^k \) if \( \phi(x) \subset \phi(z) \).
- \( \langle \prod_{j \in \phi(x)} a_j \rangle_z = 0 \) otherwise.

Therefore, we have \( \alpha_x = \prod_{j \in \phi(x)} a_j \). The proof of the second statement is similar. \( \square \)

Hence the \( H^*(BS^1) \)-module isomorphism \( f : H^*_{S^1}(M; \mathbb{Z}) \to H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{Z}) \) which sends \( \alpha_x \) to \( \beta_{\psi^{-1} \phi(x)} \) for each \( x \in M_{S^1} \) is in fact a ring isomorphism by Lemma 2.3. To sum up, we have the following corollary.

**Corollary 2.4** ([19]). There is a ring isomorphism

\[
 f : H^*_{S^1}(M; \mathbb{Z}) \to H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{Z})
\]

which sends \( \alpha_x \) to \( \beta_{\psi^{-1} \phi(x)} \). Moreover, for any \( \alpha \in H^*_{S^1}(M; \mathbb{Z}) \) and any fixed point \( x \in M_{S^1} \), we have \( \alpha_x = f(\alpha) |_{\psi^{-1} \phi(x)} \).

3. The Duistermaat-Heckman function and the residue formula

Let \( (M, \omega) \) be a 2n-dimensional closed Hamiltonian \( S^1 \)-manifold with the moment map \( \mu : M \to \mathbb{R} \). We may assume that 0 is a regular value of \( \mu \) such that \( \mu^{-1}(0) \) is non-empty. Choose two consecutive critical values \( c_1 \) and \( c_2 \) of \( \mu \) so that the open interval \( (c_1, c_2) \) consists of regular values of \( \mu \) and contains 0. By the Duistermaat-Heckman’s theorem [5], \( [\omega] = [\omega_0] - e \) where \( e \) is the Euler class of \( S^1 \)-fibration \( \mu^{-1}(0) \to M_0 \), where \( M_0 \) is the symplectic reduction at 0 with the induced symplectic form \( \omega_0 \). Hence we have

\[
 \text{DH}(t) = \int_{M_0} \frac{1}{(n-1)!} ([\omega_0] - et)^{n-1}
\]

on \( (c_1, c_2) \subset \text{Im} \mu \).

Note that a continuous function on an open interval \( g : (a, b) \to \mathbb{R} \) is concave if \( g(tc + (1 - t)d) \geq tg(c) + (1 - t)g(d) \) for any \( c, d \in (a, b) \) and for any \( t \in (0, 1) \). We remark the basic property of a concave function as follows.

**Remark 3.1.** Let \( g \) be a continuous, piecewise smooth function on a connected interval \( I \subset \mathbb{R} \). Then \( g \) is concave on \( I \) if and only if the derivative of \( g \) is decreasing, i.e. \( g''(t) \leq 0 \) for every smooth point \( t \in I \) and \( g'_+(c) - g'_-(c) < 0 \) for every singular point \( c \in I \), where \( g'_+(c) = \lim_{t \to c, t > c} g'(t) \) and \( g'_-(c) = \lim_{t \to c, t < c} g'(t) \).

Note that Duistermaat and Heckman proved that DH is a polynomial on a connected regular open interval \( U \subset \mu(M) \). The following formula due to Guillemin, Lerman, and Sternberg describes the behavior of DH near the critical value of \( \mu \). In particular, it implies that DH is \( k \)-times differentiable at a critical value \( c \in \mu(M) \) if and only if \( \mu^{-1}(c) \) does not contain a fixed component whose codimension is less than \( 4 + 2k \).

**Theorem 3.2** ([8]). Assume that \( c \) is a critical value which corresponds to the fixed components \( C_i \)'s. Then the jump of \( \text{DH}(t) \) at \( c \) is given by

\[
 \text{DH}^+ - \text{DH}^- = \sum_i \text{vol}(C_i) (d_i - 1)! \prod_j w_j (t - c)^{d_i - 1} + O((t - c)^{d_i})
\]
where the sum is over the components $C_i$ of $M^{S^1} \cap \mu^{-1}(c)$, $d_i$ is half the real codimension of $C_i$ in $M$, and the $w_j$’s are the weights of the $S^1$-representation on the normal bundle of $C_i$.

If $c$ is a critical value which is not an extremum, then the codimension of the fixed point set in $\mu^{-1}(c)$ is at least 4. Therefore Theorem 3.2 implies that $DH$ is continuous at non-extremal critical values and $DH'(t)$ jumps at $c$ when $d$ equals 2. In the case when $d = 2$, the two non-zero weights must have opposite signs, so the jump in the derivative is negative, i.e. $DH'(t)$ decreases when it passes through the critical value with $d = 2$. Since $DH$ is continuous, the jump in $\frac{d}{dt}\ln DH(t)$ is negative at $c$. Combining with Remark 3.1 we have the following corollary.

Corollary 3.3. Let $(M, \omega)$ be a closed Hamiltonian $S^1$-manifold with the moment map $\mu : M \to \mathbb{R}$. Then the corresponding Duistermaat-Heckman function $DH$ is log-concave on $\mu(M)$ if $(\log DH(t))'' \leq 0$ for every regular value $t \in \mu(M)$.

Note that $(\log DH(t))'' = \frac{DH(t) \cdot DH''(t) - DH(t)^2}{DH(t)^2}$. Therefore $(\log DH(t))'' \leq 0$ is equivalent to $DH(t) \cdot DH''(t) - DH(t)^2 \leq 0$. Equation (3.1) implies that

$$ DH(t) \cdot DH''(t) = (n-1)(n-2) \int_{M_0} e^{2[\omega_t]} - 3 \cdot \int_{M_0} [\omega_t]^{n-1} $$

and

$$ DH'(t)^2 = (n-1)^2 \left( \int_{M_0} [\omega_t]^{n-2} \right)^2. $$

To compute the integrals appearing in equations (3.2) and (3.3), we need the following procedures. For an inclusion $i : \mu^{-1}(0) \hookrightarrow M$, we have a ring homomorphism $\kappa : H^*_S(M; \mathbb{R}) \to H^*_S(\mu^{-1}(0); \mathbb{R}) \cong H^*(M_0; \mathbb{R})$ which is called the Kirwan map. Due to the Kirwan surjectivity $\mathbb{R}$, $\kappa$ is a ring surjection. Now, consider a 2-form $\bar{\omega} := \omega - d(\mu\theta)$ on $M \times ES^1$ where $\theta$ is the pull-back of the connection form on $ES^1$ along the projection $M \times ES^1 \to ES^1$. We denote $x = \pi^*u \in H^*_S(\; (M; \mathbb{Z})$ where $\pi : M \times S^1 \to BS^1$ and $u$ is a generator of $H^*(BS^1; \mathbb{Z})$ such that the Euler class of the Hopf bundle $ES^1 \to BS^1$ is $-u$. Some part of the following two lemmas is given in [1], but we give the complete proofs here.

Lemma 3.4. $\bar{\omega}$ is $S^1$-invariant and closed, and $i_X \bar{\omega} = 0$ so that $\bar{\omega}$ represents a cohomology class in $H^*_S(M; \mathbb{R})$. Moreover, for any fixed component $F \in M^{S^1}$, we have $\kappa([\bar{\omega}]) = [\omega_0]$ and $[\bar{\omega}]|_F = [\omega]|_F + \mu(F)u$. In particular, if $F$ is isolated, then $[\bar{\omega}]|_F = \mu(F)u$.

Proof. For the first statement, it is enough to show that $i_X \bar{\omega}$ and $L_X \bar{\omega}$ vanish. Note that $i_X \bar{\omega} = i_X \omega + i_X d(\mu\theta) = -d\mu + di_X (\mu\theta) - L_X (\mu\theta)$ by Cartan’s formula. Since $i_X (\mu\theta) = \mu$ and $\mu$ is invariant under the circle action, we have $i_X \bar{\omega} = -d\mu + d\mu = 0$. Moreover, it is obvious that $\bar{\omega}$ is closed by definition. Hence $L_X \bar{\omega} = 0$ by Cartan’s formula again.

To prove the second statement, consider the following diagram:

$$
\begin{align*}
\mu^{-1}(0) \times ES^1 & \hookrightarrow M \times ES^1 \\
\mu^{-1}(0) \times S^1 \to ES^1 & \hookrightarrow M \times S^1 \times ES^1 \\
\mu^{-1}(0)/S^1 & \cong M_{\text{red}}
\end{align*}
$$
Since $d\mu$ is zero on the tangent bundle $\mu^{-1}(0) \times ES^1$, the pull-back of $\tilde{\omega} = \omega - d\mu \wedge \theta - \mu d\theta$ to $\mu^{-1}(0) \times ES^1$ is the restriction $\omega|_{\mu^{-1}(0) \times ES^1}$ and the push-forward of $\omega|_{\mu^{-1}(0) \times ES^1}$ to $\mu^{-1}(0)/S^1$ is just a reduced symplectic form at the level 0. Hence $\kappa(\tilde{\omega}) = [\omega_0]$.

To show the last statement, consider $[\tilde{\omega}]|_F = [\omega - d\mu \wedge \theta - \mu d\theta]|_F$. Since the restriction $d\mu|_{F \times ES^1}$ vanishes, we have $[\tilde{\omega}]|_F = [\omega]|_F - \mu(F) \cdot [d\theta]|_{ES^1} = [\omega]|_F + \mu(F)u$. In particular, if $F$ is isolated, then we have $[\tilde{\omega}]|_F = [\mu(F)u]$.

Lemma 3.5. Consider a 2-form $d\theta$ on $M \times ES^1$, where $\theta$ is the pull-back of the connection form on $ES^1$ along the projection $M \times ES^1 \to ES^1$. Then we can push $d\theta$ forward to $M \times S^1 ES^1$ so that $d\theta$ represents a cohomology class in $H^*_S(M; \mathbb{R})$. Moreover, $[d\theta] = -x$ and $\kappa([d\theta]) = -\kappa(x) = e$, where $e$ is the Euler class of the $S^1$-fibration $\mu^{-1}(0) \to M_{red}$.

Proof. Note that $i_X d\theta = L_X \theta - di_X \theta = 0$. Hence we can push $d\theta$ forward to $M \times S^1 ES^1$. For any fixed point $p \in M^{S^1}$, the restriction $[d\theta]|_p$ is the Euler class of $p \times ES^1 \to BS^1$. Hence $[d\theta] = -u \cdot 1 = -x$. Here, the multiplication “.” comes from the $H^*(BS^1)$-module structure on $H^*_S(M)$. By the diagram in the proof of Lemma 3.3, $\kappa([d\theta])$ is just the Euler class of the $S^1$-fibration $\mu^{-1}(0) \to M_{red}$. Therefore $\kappa([d\theta]) = -\kappa(x) = e$.

Combining equations (3.2), (3.3), Lemma 3.4 and Lemma 3.5 we have the following corollary.

Corollary 3.6. $DH(0) \cdot DH''(0) - DH'(0)^2 \leq 0$ if and only if

$$(n - 2) \int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3}) \cdot \int_{M_0} \kappa([\tilde{\omega}]^{n-1}) - (n - 1) \left( \int_{M_0} \kappa([d\theta][\tilde{\omega}]^{n-2}) \right)^2 \leq 0.$$

To compute the above integrals $\int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3})$, $\int_{M_0} \kappa([\tilde{\omega}]^{n-1})$, and $\int_{M_0} \kappa([d\theta][\tilde{\omega}]^{n-2})$, we need the residue formula due to Jeffrey and Kirwan. (See [10] and [9].)

Theorem 3.7 ([10]). Let $\nu \in H^*_S(M; \mathbb{R})$. Then

$$\int_{M_0} \kappa(\nu) = \sum_{F \in M^{S^1}, \mu(F) > 0} \text{Res}_{u=0} \left( \frac{\nu|_F}{e_F} \right).$$

Here, $e_F$ is the equivariant Euler class of the normal bundle to $F$ so that we can regard $\nu|_F$ and $e_F$ as polynomials with one variable $u$. $\text{Res}_{u=0}(f)$ means a residue of $f$ where $f$ is a rational function with one variable $u$.

Now, let’s compute $\int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3})$. By Theorem 3.7

$$\int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3}) = \sum_{F \in M^{S^1}, \mu(F) > 0} \text{Res}_{u=0} \left( \frac{[d\theta]^2 [\tilde{\omega}]^{n-3}|_F}{e_F} \right).$$

Since $[\tilde{\omega}]|_z = \mu(z)u$ and $[d\theta]|_z = -u$ by Lemma 3.3 and 3.5 we have

$$\int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3}) = \sum_{F \in M^{S^1}, \mu(F) > 0} \text{Res}_{u=0} \left( \frac{[d\theta]^2 [\tilde{\omega}]^{n-3}}{e_F} \right).$$

Since $[\tilde{\omega}]|_z = \mu(z)u$ and $[d\theta]|_z = -u$ by Lemma 3.3 and 3.5 we have

$$\int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3}) = \sum_{F \in M^{S^1}, \mu(F) > 0} \text{Res}_{u=0} \left( \frac{[d\theta]^2 [\tilde{\omega}]^{n-3}}{e_F} \right).$$

$$= \sum_{F \in M^{S^1}, \mu(F) > 0} \frac{1}{m_F} \mu(F)^{n-3},$$
where \( m_F \) is the product of all weights of tangential \( S^1 \)-representation at \( F \). Similarly, if \( \xi \in \mathbb{R} \) is a regular value of \( \mu \), then we let \( \tilde{\mu} = \mu - \xi \) be the new moment map. By the same argument, we have the following lemma.

**Lemma 3.8.** For a regular value \( \xi \) of the moment map \( \mu \), we have the following:

\[
\begin{align*}
\int_{M_0} \kappa([d\theta]^2 \omega^{n-3}) &= \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-3}, \\
\int_{M_0} \kappa([d\theta]\omega)^{n-2} &= \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-2}, \\
\int_{M_0} \kappa(\omega)^{n-1} &= \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-1}.
\end{align*}
\]

Combining Corollary 3.6 and Lemma 3.8 we have the following proposition.

**Proposition 3.9.** Let \( (M,\omega) \) be a closed Hamiltonian \( S^1 \)-manifold with the moment map \( \mu : M \to \mathbb{R} \). Assume that \( M_{S^1} \) consists of isolated fixed points. Then a density function of the Duistermaat-Heckman measure with respect to \( \mu \) is log-concave if and only if

\[
\sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-3} - \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-1} - \left( \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-2} \right)^2 \leq 0
\]

for every regular value \( \xi \in \mu(M) \), where \( m_F \) is the product of all weights of the \( S^1 \)-representation on \( T_FM \). In particular, the log-concavity of the Duistermaat-Heckman measure is completely determined by the set \( \{(\mu(F), m_F)| F \in M_{S^1}\} \).

**Corollary 3.10.** Let \( (M^{2n},\omega) \) and \( (N^{2n},\sigma) \) be two closed Hamiltonian \( S^1 \)-manifolds with the moment maps \( \mu_1 \) and \( \mu_2 \), respectively. Assume there exists a bijection \( \phi : M_{S^1} \to N_{S^1} \) which satisfies

1. for each \( F \in M_{S^1}, m_F = m_{\phi(F)} \), and
2. for each \( F \in M_{S^1}, \mu_1(F) = \mu_2(\phi(F)) \),

where \( m_F \) is the product of all weights of the tangential \( S^1 \)-representation at \( F \). If \( N \) satisfies the log-concavity of the Duistermaat-Heckman measure with respect to \( \mu_2 \), then so does \( M \) with respect to \( \mu_1 \).

**Remark 3.11.** The integration formulae (1) and (3) in Lemma 3.8 are proved by Wu by using the stationary phase method. See Theorem 5.2 in [21] for the details.

### 4. Proof of Theorem 1.1

As noted in the introduction, if a Hamiltonian \( S^1 \)-action on the Kähler manifold is holomorphic, then the corresponding Duistermaat-Heckman function is log-concave by [7]. Let \( (M^{2n},\omega) \) be a closed semifree Hamiltonian \( S^1 \)-manifold with the moment map \( \mu \). Assume that all fixed points are isolated. Let \( \text{DH} \) be the corresponding Duistermaat-Heckman function with respect to \( \mu \). We will show that there
is a Kähler form $\omega_1 \oplus \cdots \oplus \omega_n$ on $S^2 \times \cdots \times S^2$ with the standard diagonal holomorphic semifree circle action such that a bijection $\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$ given in Section 2 satisfies the conditions in Corollary 3.10 which implies the log-concavity of DH. Now we start with the lemma below.

Lemma 4.1. Let $(M^{2n}, \omega)$ be a closed semifree Hamiltonian circle action with the moment map $\mu$. Assume that all fixed points are isolated. Then $\{\mu(F), m_F|F \in M^{S^1}\}$ is completely determined by $\mu(p_0^1), \mu(p_1^1), \cdots, \mu(p_n^1)$, where the $p_j^i$'s are the fixed points of index $2k$ for $j = 1, \cdots, \binom{n}{k}$.

Proof. Consider an equivariant symplectic 2-form $\tilde{\omega}$ on $M \times_{S^1} ES^1$ which is given in Section 3. Since the set $\{x, a_1, \cdots, a_n\}$ is a basis of $H^2_{S^1}(M; \mathbb{Z})$, we may assume

$$[\tilde{\omega}] = m_0 x + m_1 a_1 + \cdots + m_n a_n$$

for some real numbers $m_i$. (See Section 2 for the definition of $x, a_1, \cdots, a_n$.) Therefore, for any fixed point $p_i^j \in M^{S^1}$, we have

$$[\tilde{\omega}]|_{p_i^j} = (m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_i^j}.$$

When $i = 0$ and $j = 1$, Lemma 3.4 implies that

$$[\tilde{\omega}]|_{p_0^1} = m_0 u.$$

Since every $a_i$ vanishes on $p_0^1$, the right hand side is

$$(m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_0^1} = m_0 u.$$

Hence $m_0 = \mu(p_0^1)$. Similarly,

$$[\tilde{\omega}]|_{p_i^1} = \mu(p_1^i)u$$

and

$$(m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_i^j} = m_0 u + m_i u.$$

Hence we have $m_i = \mu(p_i) - m_0 = \mu(p_i^j) - \mu(p_0^1)$ for each $i = 1, \cdots, n$. Therefore $\{\mu(p_0^1), \mu(p_1^1), \cdots, \mu(p_n^1)\}$ determines the coefficients $m_i$ of $[\tilde{\omega}]$.

For $p_k^j$ with $k > 1$, the relation $[\tilde{\omega}]|_{p_k^j} = (m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_k^j}$ implies

- $[\tilde{\omega}]|_{p_k^j} = \mu(p_k^j)u$ and
- $(m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_k^j} = m_0 u + \sum_{i \in \phi(p_k^j)} m_i u$.

Therefore, for fixed $k$, the set $\{(\mu(p_k^j), m_{p_k^j})|j = 1, \cdots, \binom{n}{k}\}$ is just

$${((m_0 + m_{i_1} + \cdots + m_{i_k}), (-1)^k)_{\{i_1, \ldots, i_k\}}|\{i_1, \ldots, i_k\} \subset \{1, 2, \cdots, n\},}$$

and this set does not depend on the ordering of $p_k^j$. Hence $\{\mu(F), m_F|F \in M^{S^1}\} = \bigcup_{k=0}^n \{((\mu(p_k^j), m_{p_k^j})|j = 1, \cdots, \binom{n}{k}\}$ is completely determined by $\mu(p_0^1), \mu(p_1^1), \cdots, \mu(p_n^1)$. \hfill $\square$

Now we are ready to prove Theorem 1.1.

Theorem 4.2 (Theorem 1.1). Let $(M, \omega)$ be a closed symplectic manifold with a semifree Hamiltonian $S^1$-action whose fixed point set $M^{S^1}$ consists of isolated points. Then the Duistermaat-Heckman measure is log-concave.
Proof. Let \( \mu : M \to \mathbb{R} \) be a moment map and let \( M^{S^1} = \{ p^i_k | k = 0, \ldots, n, j = 1, \ldots, \binom{n}{j} \} \) be the fixed point set, where \( p^i_k \) is a fixed point of index \( 2k \) labeled by \( j = 1, \ldots, \binom{n}{j} \). Note that \( \mu(p^i_k) \) is the minimum value of \( \mu \). Let \( \phi : M^{S^1} \to 2^{[n]} \) be the identification map between the fixed point set \( M^{S^1} \) and the power set \( 2^{[n]} \) defined in Theorem 2.2. Then we may assume (by re-labeling, if necessary) that \( \phi(p^i_k) = \{ j \} \) for \( j = 1, \ldots, n \). We will show that there exists a semifree holomorphic Hamiltonian \( S^1 \)-manifold \((S^2 \times \cdots \times S^2, \sigma)\) with the moment map \( \mu' \) such that \( \psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1} \) preserves their indices, weights, and the values of the moment map, where \( \psi : (S^2 \times \cdots \times S^2)^{S^1} \to 2^{[n]} \) is the identification map described in Theorem 2.2.

Let \( \omega_i \) be the Fubini-Study form on \( S^2 \) such that the symplectic volume is \( \mu(p^i_1) - \mu(p^i_0) \). Let \( S \) be the south pole and \( N \) be the north pole of \( S^2 \) so that \( S \) is the minimum (\( N \) is the maximum, respectively) of the moment map on \((S^2, \omega_i)\) with the standard semifree circle action on \( S^2 \). Then \((S^2 \times \cdots \times S^2, \omega_i \oplus \cdots \oplus \omega_n)\) is a symplectic manifold with the diagonal semifree Hamiltonian circle action. Let \( \mu' : S^2 \times \cdots \times S^2 \to \mathbb{R} \) be the moment map whose minimum is \( \mu(p^i_1) \). Let \( \psi : (S^2 \times \cdots \times S^2)^{S^1} \to 2^{[n]} \) be the identification map between the fixed point set \((S^2 \times \cdots \times S^2)^{S^1}\) and the power set \( 2^{[n]} \) such that

\[
\psi^{-1}(\{i\}) := q^i_1 = (S, \ldots, S, N, S, \ldots, S)
\]

for all \( i = 1, \ldots, n \), where \( q^i_1 = (S, \ldots, S, N, S, \ldots, S) \) is a fixed point on \((S^2 \times \cdots \times S^2)\) of index 2 such that the \( i \)-th coordinate is \( N \) and the other coordinates are \( S \). Then we can easily see that \( \psi^{-1} \circ \phi(p^i_k) = q^i_1 \) and \( \mu(p^i_1) = \mu'(q^i_1) = \mu(p^i_1) - \mu(p^i_0) \) for all \( j = 1, \ldots, n \). By Lemma 3.1 we have

\[
\{(\mu(F), m_F) | F \in M^{S^1} \} = \{(\mu'(F), m_F) | F \in (S^2 \times \cdots \times S^2)^{S^1} \},
\]

and \( \psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1} \) satisfies the condition in Corollary 3.10. Therefore the Duistermaat-Heckman measure is log-concave on \( \mu(M) \).

\[\square\]

Remark 4.3 (Summary). Let \( (M, \omega) \) be a \( 2n \)-dimensional compact semifree Hamiltonian \( S^1 \)-manifold with isolated fixed points. Let \( \mu : M \to \mathbb{R} \) be a moment map. Let \( \phi : M^{S^1} \to 2^{[n]} \) be the identification described in Theorem 2.2. In the proof of Theorem 4.4 we proved that there exists a Kähler form \( \omega_1 \oplus \cdots \oplus \omega_n \) on \( S^2 \times \cdots \times S^2 \) with the diagonal semifree Hamiltonian action with the moment map \( \mu' : S^2 \times \cdots \times S^2 \to \mathbb{R} \) satisfying the following:

- There is an identification \( \psi : (S^2 \times \cdots \times S^2)^{S^1} \to 2^{[n]} \) such that \( \psi^{-1}(\{i\}) = q^i_1 = (S, \ldots, S, N, S, \ldots, S) \), where \( q^i_1 = (S, \ldots, S, N, S, \ldots, S) \) is a fixed point on \((S^2 \times \cdots \times S^2)\) of index 2 such that the \( i \)-th coordinate is \( N \) and the other coordinates are \( S \).
- The composition map \( \psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1} \) preserves their indices, weights, and the values of the moment map.
- By Corollary 2.3 \( \psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1} \) induces an isomorphism \( f : H^*_{S^1}(M; \mathbb{Z}) \to H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{Z}) \). Moreover, by the proof of Lemma 4.1 \( f \) sends the equivariant symplectic class \( [\omega] \) in \( H^2_{S^1}(M; \mathbb{R}) \) to the one in \( H^2_{S^1}(S^2 \times \cdots \times S^2; \mathbb{R}) \).

We will refer to Remark 4.3 in the proof of Theorem 4.4 in Section 5.
5. THE HARD LEFSCHETZ PROPERTY OF THE REDUCED SYMPLECTIC FORMS

For a Kähler manifold \((N, \sigma)\) with a holomorphic circle action preserving \(\sigma\), let \(t \in \mathbb{R}\) be any regular value of the moment map \(H : N \to \mathbb{R}\). Since the reduced space \(N_t := H^{-1}(t)/S^1\) with the reduced symplectic form \(\sigma_t\) is again Kähler, \(\sigma_t\) satisfies the hard Lefschetz property for every regular value \(t \in \mathbb{R}\). In this section, we show that the same thing happens when \((M, \omega)\) is a closed semifree Hamiltonian \(S^1\)-manifold whose fixed points are all isolated. The following theorem is due to Tolman and Weitsman.

**Theorem 5.1** ([20]). Let \((M, \omega)\) be a closed Hamiltonian \(S^1\)-manifold with a moment map \(\mu : M \to \mathbb{R}\). Assume that all fixed points are isolated and 0 is a regular value. Let \(M^{S^1}\) be the set of fixed points. Define \(K^+_M := \{\alpha \in H^{S^1}_\ast(M; \mathbb{Z}) \mid \alpha|_{F^+} = 0\}\) where \(F^+ := M^{S^1} \cap \mu^{-1}(0, \infty)\) and \(K^-_M := \{\alpha \in H^{S^1}_\ast(M; \mathbb{Z}) \mid \alpha|_{F^-} = 0\}\) where \(F^- := M^{S^1} \cap \mu^{-1}(-\infty, 0)\). Then there is a short exact sequence

\[0 \to K \to H^{S^1}_\ast(M; \mathbb{Z}) \xrightarrow{\kappa} (M_{\text{red}}; \mathbb{Z}) \to 0\]

where \(\kappa\) is the Kirwan map.

Now we are ready to prove Theorem 1.2.

**Theorem 5.2** (Theorem 1.2). Let \((M, \omega)\) be a closed semifree Hamiltonian \(S^1\)-manifold whose fixed points are all isolated, and let \(\mu\) be the moment map. Then \(\omega\) satisfies the hard Lefschetz property. Moreover, the reduced symplectic form \(\omega_t\) satisfies the hard Lefschetz property for every regular value \(t\).

**Proof.** Let \(\mu : M \to \mathbb{R}\) be a moment map such that 0 is a regular value of \(\mu\). For \(M_{\text{red}} \cong \mu^{-1}(0)/S^1\) with the reduced symplectic form \(\omega_0\), let \(\kappa_M : H^{S^1}_\ast(M; \mathbb{R}) \to H^{S^1}_\ast(M_{\text{red}}; \mathbb{R})\) be the Kirwan map for \((M, \omega)\). Let \(\kappa\) be the one for \((S^2 \times \cdots \times S^2, \sigma)\), where \(\sigma := \omega_1 \oplus \cdots \oplus \omega_n\) is chosen in the proof of Theorem 1.1 in Section 4. (See also Remark 4.3.) As in Remark 4.3, we proved that there exists a semifree holomorphic Hamiltonian \(S^1\)-manifold \((S^2 \times \cdots \times S^2, \sigma)\) with the moment map \(\mu'\) such that \(\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}\) preserves their indices, weights, and the values of the moment map. Also, the induced ring isomorphism \(f : H^{S^1}_\ast(M; \mathbb{Z}) \to H^{S^1}_\ast(M_{\text{red}}; \mathbb{Z})\) given in Corollary 2.4 satisfies \(\alpha|_x = f(\alpha)|_{\psi^{-1} \circ \phi(x)}\) for any \(\alpha \in H^{S^1}_\ast(M; \mathbb{Z})\) and any fixed point \(x \in M^{S^1}\). Hence \(\psi^{-1} \circ \phi\) identifies \(K^+_M\) with \(K^{S^2 \times \cdots \times S^2}_+\) and \(K^-_M\) with \(K^{S^2 \times \cdots \times S^2}_-\). Hence if \(\alpha \in K^+_M\), then \(f(\alpha) \in K^{S^2 \times \cdots \times S^2}_+\). Similarly for any \(\alpha \in K^-_M\), we have \(f(\alpha) \in K^{S^2 \times \cdots \times S^2}_-\). Therefore \(f(\alpha)\) is in \(\ker \kappa\) if and only if \(\alpha \in \ker \kappa_M\) by Theorem 5.1.

Now, let \(\tilde{\omega}\) be the equivariant symplectic form with respect to the moment map \(\mu\). (See Section 4) Note that \(\kappa(f([\tilde{\omega}]))\) is the cohomology class of the reduced symplectic form of \(S^2 \times \cdots \times S^2\) at \(\mu^{-1}(0)/S^1\). Since the Kähler quotient of the holomorphic action is again Kähler, \(\kappa(f([\tilde{\omega}]))\) satisfies the hard Lefschetz property. Now, assume that \(\omega_0\) does not satisfy the hard Lefschetz property. Then there exists a positive integer \(k(< n)\) and some non-zero \(\alpha \in H^k(M_{\text{red}}; \mathbb{R})\) such that \(\alpha \cdot [\omega_0]^{n-k} = 0\) in \(H^{2n-k}(M_{\text{red}})\). By the Kirwan surjectivity theorem 1.3, we can find \(\tilde{\alpha} \in H^{S^1}_\ast(M; \mathbb{R})\) with \(\kappa(\tilde{\alpha}) = \alpha\). Then \(\tilde{\alpha} \cdot [\tilde{\omega}]^{n-k}\) is in \(\ker \kappa_M\) and hence the image of \(f(\tilde{\alpha} \cdot [\tilde{\omega}]^{n-k})\) is in \(\ker \kappa\). It implies that \(f(\tilde{\alpha}) = 0\) by the hard Lefschetz
condition for \( f(\tilde{\omega}) \). Since \([\tilde{\alpha}] \in \ker \kappa_M\) if and only if \( f([\tilde{\alpha}]) \in \ker \kappa\) and \([\tilde{\alpha}]\) is not in \( \ker \kappa_M\), it is a contradiction.

It remains to show that \((M, \omega)\) satisfies the hard Lefschetz property. Recall that \(\psi^{-1} \circ \phi : M^{S^1} \rightarrow (S^2 \times \cdots \times S^2)^{S^1}\) induces an isomorphism

\[
f : H^*_S(M; \mathbb{Z}) \rightarrow H^*_S(S^2 \times \cdots \times S^2; \mathbb{Z}),
\]

which sends the equivariant symplectic class \([\tilde{\omega}]\) to \([\tilde{\sigma}]\) as we have seen in Section 4 (See Remark 4.3). Here, \(\tilde{\sigma}\) is an equivariant symplectic form induced by \(\sigma - d(\mu' \theta)\) in \(H^*_S(S^2 \times \cdots \times S^2; \mathbb{R})\). Since \(f\) is an \(H^*(BS^1; \mathbb{R})\)-algebra isomorphism, it induces a ring isomorphism

\[
f_u : H^*_S(M; \mathbb{R}) \rightarrow H^*_S(S^2 \times \cdots \times S^2; \mathbb{R}).
\]

Moreover, the quotient map \(\pi_M : H^*_S(M; \mathbb{R}) \rightarrow H^*_S(M; \mathbb{R})/H^*_S(M; \mathbb{Z}) \cong H^*(M; \mathbb{R})\) (\(\pi_{S^2 \times \cdots \times S^2}\), resp.) is a ring homomorphism which comes from an inclusion \(M \hookrightarrow M \times S^1 E^S\) as a fiber. Therefore \(\pi_M([\tilde{\omega}]) = [\omega]\) and \(\pi_{S^2 \times \cdots \times S^2}([\tilde{\sigma}]) = [\sigma]\). Therefore the isomorphism \(f_u\) maps \([\omega]\) to \([\sigma]\). Since \(\sigma\) is a Kähler form, it satisfies the hard Lefschetz property. Hence so does \(\omega\).

\[\square\]

References


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