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MAXIMIZATION OF THE SECOND CONFORMAL EIGENVALUE OF SPHERES

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Abstract. In this paper we establish an upper bound on the second eigenvalue of n-dimensional spheres in the conformal class of the round sphere. This upper bound holds in all dimensions and is asymptotically sharp as the dimension increases.

Given (M,g) a smooth compact Riemannian manifold (without boundary), the spectrum of the Laplacian $\Delta_g = -div_g\,(\nabla)$ is a discrete sequence of eigenvalues

$$0 = \lambda_0\,(M,g) < \lambda_1\,(M,g) \leq \lambda_2\,(M,g) \leq \cdots \leq \lambda_k\,(M,g) \leq \cdots$$

which goes to $+\infty$ as $k \to +\infty$. The eigenfunctions associated to the simple eigenvalue $\lambda_0 = 0$ are the constant functions. A natural, and often addressed, question is how to get estimates on the eigenvalues thanks to some geometric assumptions. In this paper, we discuss the maximization of eigenvalues for metrics in a given conformal class with fixed volume. We focus on the case of the standard sphere.

We let $\mathbf{S}^n$ be the unit sphere of $\mathbf{R}^{n+1}$ for $n \geq 2$. If g is a metric on $\mathbf{S}^n$, we are interested in the scale invariant quantity

$$\Lambda_{n,k}(g) = \lambda_k(\mathbf{S}^n, g)\, Vol_g(\mathbf{S}^n)^{\frac{2}{n}}.$$

In dimension 2, we can maximize $\Lambda_{2,k}$ on regular metrics. An inequality has been proved for $k = 1$ by Hersch [6]:

$$\Lambda_{2,1}(g) \leq 8\pi,$$

with equality iff g is the round metric. He followed the proof of the maximization by Szegö [9] of the first non-zero Neumann eigenvalue for planar domains, attained by discs. Nadirashvili found an optimal maximization for $k = 2$. He proved in [8] that

$$\Lambda_{2,2}(g) < 16\pi,$$

where the supremum is attained in the degenerate case of the union of two identical spheres. His idea was used later in [5] to show that among simply connected planar domains, the second non-zero Neumann eigenvalue is maximal in the degenerate case of two discs of the same area.

If we look for an analogous inequality in dimension $n \geq 3$, we have to restrict our attention to some classes of metrics since $\Lambda_{n,k}$ is not bounded on the set of regular metrics (see [2]). It is natural, as suggested in [4] and [3], to consider the set of

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metrics in some conformal class. Indeed, in any given conformal class, $\Lambda_{n,k}(g)$ admits some upper bound (see [7]). Thus we define the conformal spectrum of $(\mathbf{S}^n, [g_0])$, where $[g_0]$ is the class of metrics conformal to the round metric g_0, by

$$\lambda_k^c(\mathbf{S}^n, [g_0]) = \sup_{g\in[g_0]} \Lambda_{n,k}(g).$$

The theorem of Hersch was generalized in this framework in [4]. We have that

$$\lambda_1^c(\mathbf{S}^n, [g_0]) = n\sigma_n^{\frac{2}{n}},$$

where σ_n is the volume of the unit n-dimensional sphere. We know almost nothing about $\lambda_k^c(\mathbf{S}^n, [g_0])$ for $k \geq 2$. A lower bound was obtained by a method of conformal surgery in [3]. For all k, we have that

$$\lambda_k^c(\mathbf{S}^n, [g_0]) \geq n(k\sigma_n)^{\frac{2}{n}}.$$

Nadirashvili, Girouard and Polterovich conjectured in [5] that this inequality is an equality in all dimensions for $k = 2$, where the supremum is attained for the union of two identical spheres:

Conjecture ([5])**.** *For any metric $g \in [g_0]$,*

$$\lambda_2\left(\mathbf{S}^n, g\right) Vol_g\left(\mathbf{S}^n\right)^{\frac{2}{n}} < n\left(2\sigma_n\right)^{\frac{2}{n}}.$$

Towards this conjecture, the following theorem gives an "asymptotically sharp" upper bound:

Theorem. *Let $n \geq 2$ and $g \in [g_0]$ be a metric on S^n conformal to the round metric. Then*

$$\lambda_2\left(\mathbf{S}^n, g\right) Vol_g\left(\mathbf{S}^n\right)^{\frac{2}{n}} < K_n n(2\sigma_n)^{\frac{2}{n}},$$

where K_n is a constant independant of $g \in [g_0]$ given by

$$K_n = \frac{n+1}{n}\left(\frac{\Gamma(n)\Gamma(\frac{n+1}{2})}{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2})}\right)^{\frac{2}{n}}.$$

Note that $K_2 = 1$, that $1 < K_n \leq 1.04$ for all $n \geq 3$ and that $\lim_{n\to\infty} K_n = 1$. The Theorem is sharp in dimension 2 and was in fact already proved by Nadirashvili in [8]. In [5], Girouard, Nadirashvili and Polterovich established this inequality in odd dimensions.

In this paper we prove this theorem in all dimensions, unifying the previous proofs in dimension $n = 2$ and in odd dimensions and along the way extending it. The starting point of the proof is a construction, described in section 1 below, initiated by Nadirashvili [8] and used by Girouard, Nadirashvili and Polterovich [5] in odd dimension. However, our use of this construction differs from that of these two papers: we use the min-max characterization of the second eigenvalue up to the end of the proof (see section 3), capitalizing on a new topological fact proved in section 2.

1. Construction of test functions

In this section, we describe the construction of Nadirashvili [8] (see also [5]), which is at the basis of our theorem as well as of the previous results. Let g be a metric on $\mathbf{S}^n$ conformal to g_0 of volume 1. We denote by dv_g the measure associated to g. In this paper we shall use the min-max characterization of the second eigenvalue of the Laplacian, which tells us in particular that

$$\lambda_2(\mathbf{S}^n, g) \leq \sup_{u \in E\setminus\{0\}} \frac{\int_{\mathbf{S}^n} |\nabla_g u|_g^2 \, dv_g}{\int_{\mathbf{S}^n} u^2 dv_g} \tag{1}$$

for all 2-dimensional subspaces E of functions in $H^1(\mathbf{S}^n)$ with mean value 0. The aim is to find a suitable space E of test functions such that (1) gives the estimate of the Theorem.

On $(\mathbf{S}^n, g_0)$, the eigenspace associated to $\lambda_1(\mathbf{S}^n, g_0)$ has dimension $n+1$: it is the set of linear forms of $\mathbf{R}^{n+1}$ written $X_s = (s, .)$ for $s \in \mathbf{R}^{n+1}$. We will build E with these functions, and as Hersch did for $\lambda_1(\mathbf{S}^n, g)$, we proceed to a renormalization of measures in order to keep the orthogonality to constants. For $\xi \in \mathbf{B}^{n+1}$, we let $d_\xi : \overline{\mathbf{B}^{n+1}} \to \overline{\mathbf{B}^{n+1}}$ be defined by

$$d_\xi(x) = \frac{(1-|\xi|^2)x + (1 + 2\xi.x + |x|^2)\xi}{1 + 2\xi.x + |x|^2 |\xi|^2},$$

which is a conformal transformation when restricted to the unit sphere.

We say that d_ξ renormalizes a finite measure $d\nu$ on the n-sphere if

$$\forall s \in \mathbf{S}^n, \qquad \int_{\mathbf{S}^n} X_s \circ d_\xi d\nu = 0.$$

The Hersch lemma says that for all finite measures $d\nu$, such a ξ exists. Moreover, it is unique and depends continuously on $d\nu$ (the set of finite measures is considered as the topological dual of the continuous bounded functions) as proved in [5], Proposition 4.1.5. We call ξ the renormalization point of $d\nu$.

We also define families of measures parametrized by the set of caps of $\mathbf{S}^n$, denoted by $\mathcal{C}$:

$$a_{0,p} = \{x \in \mathbf{S}^n; x.p > 0\}, \qquad a_{r,p} = d_{rp}(a_{0,p}), \qquad (r,p) \in (-1,1) \times \mathbf{S}^n.$$

We denote by $d\mu_a$ the "lift" of the measure dv_g by the cap $a \in \mathcal{C}$:

$$d\mu_a = \begin{cases} dv_g + (\tau_a)^* \, dv_g & \text{on } a, \\ 0 & \text{on } a^*, \end{cases}$$

where $a^* = \mathbf{S}^n \setminus \bar{a}$ and τ_a is the conformal reflection with respect to the boundary circle of a. That is,

$$\tau_{a_{r,p}} = d_{rp} \circ R_p \circ d_{-rp},$$

where

$$R_p(x) = x - 2(p,x)p$$

is the reflection of $\mathbf{R}^{n+1}$ with respect to the hyperplane orthogonal to p. Let $\xi(a)$ be the renormalization point of $d\mu_a$. We set $d\nu_a = (d_{\xi(a)})_* d\mu_a$. Thanks to this family of measures, we can define a new family of test functions orthogonal to the constants:

$$u_a^s = \begin{cases} X_s \circ d_{\xi(a)} & \text{on } a, \\ X_s \circ d_{\xi(a)} \circ \tau_a & \text{on } a^*. \end{cases}$$

By a Hölder inequality, the numerator of the Rayleigh quotient is less than a conformal invariant:

$$\begin{aligned}\int_{\mathbf{S}^n} |\nabla_g u_a^s|_g^2\, dv_g &< \left(\int_{\mathbf{S}^n} |\nabla_g u_a^s|_g^n\, dv_g\right)^{\frac{2}{n}} \\ &= \left(2\int_{d_{\xi(a)}(a)} |\nabla_g X_s|_g^n\, dv_g\right)^{\frac{2}{n}} \\ &< \left(2\int_{\mathbf{S}^n} |\nabla_{g_0} X_s|_{g_0}^n\, dv_{g_0}\right)^{\frac{2}{n}}.\end{aligned} \tag{2}$$

Let us define the multiplicity of a finite measure:

Definition. The multiplicity of a finite measure $d\nu$ on $\mathbf{S}^n$ is the dimension of the eigenspace W associated to the maximal eigenvalue of the quadratic form:

$$Q(s) = \int_{\mathbf{S}^n} X_s^2 d\nu.$$

We say that $d\nu$ is multiple if its multiplicity is greater than or equal to 2. Otherwise, we say that $d\nu$ is simple.

As was noticed in [5], we know that if dv_g is multiple, then we can choose $E = \{X_s; s \in W\}$ in (1) to get that $\lambda_2(\mathbf{S}^n, g) \leq n(2\sigma_n)^{\frac{2}{n}}$. We also know that if there is a cap $a \in \mathcal{C}$ such that $d\nu_a$ is multiple, then $\lambda_2(\mathbf{S}^n, g) < K_n n(2\sigma_n)^{\frac{2}{n}}$ using the space of test functions $E = \{u_a^s; s \in W\}$ in (1). In this case, the Theorem would be proved. In [5], it was proved that there necessarily exists such a multiple measure in odd dimensions (see below).

Let us now assume that all measures dv_g and $d\nu_a$, for $a \in \mathcal{C}$, are simple. Up to a renormalization and a rotation, we may assume that

$$\forall t \in \mathbf{S}^n, \int_{\mathbf{S}^n} X_t dv_g = 0$$

and that

$$\forall t \in \mathbf{S}^n \setminus [e_1], \int_{\mathbf{S}^n} X_t^2 dv_g < \int_{\mathbf{S}^n} X_{e_1}^2 dv_g.$$

We denote by $[s(a)]$ the unique direction of maximization of the quadratic form associated to $d\nu_a$. With the parametrization $(r,p) \in (-1,1) \times \mathbf{S}^n$ of $\mathcal{C}$, the maps $\xi : \mathcal{C} \to \mathbf{B}^{n+1}$ and $[s] : \mathcal{C} \to \mathbf{R}P^n$ are continuous. Moreover, one may prove that if $r \to -1$, that is, $a \to \mathbf{S}^n$, we have:

$$\lim_{a \to \mathbf{S}^n} \xi(a) = 0, \qquad \lim_{a \to \mathbf{S}^n} [s(a)] = [e_1]. \tag{3}$$

2. Properties of the lift of the maximal direction

Let us study the maps ξ and $[s]$ in light of the links between a cap $a \in \mathcal{C}$ and its symmetrical cap $a^* = \mathbf{S}^n \setminus \bar{a}$. With the parameter $(r,p) \in (-1,1) \times \mathbf{S}^n$, notice that $a^*_{r,p} = a_{-r,-p}$.

Claim 1. For $a \in \mathcal{C}$, we write $\xi^* = \xi(a^*)$, $[s^*] = [s(a^*)]$. Then

$$-\xi^* = \tau_a(-\xi) \qquad \text{and} \qquad [s^*] = R_a[s],$$

where $R_a = d_{\xi^*(a)} \circ \tau_a \circ d_{-\xi(a)}$ is an orthogonal map.

Proof. We set $\eta = -\tau_a(-\xi)$. Let $t \in \mathbf{S}^n$; then

$$\int_{\mathbf{S}^n} X_t \circ d_\eta \, d\mu_{a^*} = \int_{\mathbf{S}^n} X_t \circ d_\eta \circ \tau_a \, d\mu_a.$$

One can check that $d\mu_{a^*} = (\tau_a)^* d\mu_a$. The map $R_a = d_\eta \circ \tau_a \circ d_{-\xi(a)}$ is orthogonal because it is a Möbius transformation of the unit ball preserving the origin ([1], Theorem 3.4.1). Thus we have that

$$\int_{\mathbf{S}^n} X_t \circ d_\eta \, d\mu_{a^*} = \int_{\mathbf{S}^n} X_t \circ R_a \circ d_\xi \, d\mu_a = \int_{\mathbf{S}^n} X_{R_a^{-1}(t)} \circ d_\xi \, d\mu_a = 0.$$

This is true for all $t \in \mathbf{S}^n$, and uniqueness of the renormalization point ensures that $\xi^* = \eta$.

The same argument with the function $(X_t \circ d_{\xi^*})^2$ leads to

$$\forall t \in \mathbf{S}^n, \int_{\mathbf{S}^n} (X_t \circ d_{\xi^*})^2 \, d\mu_{a^*} = \int_{\mathbf{S}^n} \left(X_{R_a^{-1}(t)} \circ d_\xi\right)^2 d\mu_a,$$

and once again we can conclude by uniqueness of the maximal direction that $[s^*] = R_a[s]$. □

Remark. Thanks to Claim 1, we can prove the Theorem in odd dimensions. Indeed, when $r \to 1$, that is, $a \to \{p\}$, we use (3) in order to obtain:

$$\lim_{a \to \{p\}} R_a = R_p.$$

Then, $[s(a)] = R_a^{-1}[s^*(a)] \to R_p[e_1]$ when $a \to \{p\}$ by (3). Therefore, following [5] in odd dimensions, the map $[s] : [-1, 1] \times \mathbf{S}^n \to \mathbf{R}P^n$ defines a homotopy between the constant map $[e_1]$ of degree 0 and $\phi(p) = R_p[e_1]$ of degree 4. Thus, there is a contradiction and there exists a multiple measure among dv_g and $d\nu_a$ for $a \in \mathcal{C}$.

We do not prove that the assumption that all measures are simple leads to a contradiction. Indeed, it is not clear that in even dimensions such a configuration cannot happen. Instead, we look for suitable test functions as in Nadirashvili's proof in dimension 2 [8]. However, inspired by the method of [5], we use a topological argument to get symmetric properties of the lifts of the maximal directions.

The continuous map $[s] : [-1, 1) \times \mathbf{S}^n \to \mathbf{R}P^n$ has exactly two continuous lifts because the set $[-1, 1) \times \mathbf{S}^n$ is simply connected. We denote by s the continuous lift such that $s(-1, .) = -e_1$; the other continuous lift is $-s$. Thanks to Claim 1,

$$s(-r, -p) = \epsilon(r, p) R_{a_{r,p}} s(r, p),$$

where $\epsilon : [-1, 1) \times \mathbf{S}^n \to \{\pm 1\}$ is a continuous map. Since $s \neq 0$ and $[-1, 1) \times \mathbf{S}^n$ is connected, ϵ is a constant map.

Claim 2. We have that $\epsilon = -1$. In other words,

$$s(a^*) = -R_a s(a)$$

for all caps a.

Proof. We assume by contradiction that $\epsilon = 1$. We set $f(p) = s(0, p)$ for $p \in \mathbf{S}^n$. This function f is continuous on the sphere and satisfies

$$\forall p \in \mathbf{S}^n, f(-p) = R_p f(p). \tag{4}$$

Indeed, $R_{a_{0,p}} = R_p$ because $\tau_{a_{0,p}} = R_p$. Using Claim 3 below, we know that such a map f cannot have degree 0. However, the map $s : [-1,0] \times \mathbf{S}^n \to \mathbf{S}^n$ defines a homotopy between $s_0 = f$ and $s_{-1} = -e_1$ of degree zero. Thus, there is a contradiction. □

We use the following topology result:

Claim 3. Let $f : \mathbf{S}^n \to \mathbf{S}^n$ be a continuous map which satisfies (4). Then, if n is odd, $\deg(f) = 1$, and if n is even, $\deg(f) \in 2\mathbf{Z} + 1$.

Proof. We first prove the claim for smooth functions which have a property of transversality (Step 1) and we show that this case is generic (Step 2).

Step 1. Let $f : \mathbf{S}^n \to \mathbf{S}^n$ be a smooth function which satisfies (4). Let us assume that for all fix points $x \in \mathbf{S}^n$ of f, $T_x f - I : T_x\mathbf{S}^n \to T_x\mathbf{S}^n$ is an isomorphism. Then, if n is odd, $\deg(f) = 1$, and if n is even, $\deg(f) \in 2\mathbf{Z} + 1$.

Proof of Step 1. Let F be defined by

$$\begin{array}{rccl} F: & \mathbf{S}^n \times [-1,1] & \longrightarrow & \mathbf{R}^{n+1}, \\ & (x,t) & \longmapsto & \frac{1}{2}\left(f(x) - x + t(f(x)+x)\right). \end{array}$$

We notice that if F never vanishes, $\frac{F}{|F|}$ defines a homotopy between f and σ, the antipodal map and $\deg(f) = \deg(\sigma) = (-1)^{n+1}$.

Now, $F(x,t) = 0$ if and only if $t = 0$ and x is a fix point of f, and then

$$\forall (v,t) \in T_x\mathbf{S}^n \times \mathbf{R}, DF(x,0)(v,t) = \frac{1}{2}(T_x f - I)v + xt.$$

Thus, $DF(x,0)$ is an isomorphism, and 0 is a regular value. We write $(x_1,0), \cdots, (x_r,0)$ as the regular points of $F^{-1}(0)$. Let's approximate F by its differential in the neighborhood of its zeros. Let $\alpha > 0$, and set for $1 \le i \le r$, $\phi_i : B_{x_i}(\alpha) \to B_0(\alpha) \subset T_{x_i}\mathbf{S}^n$ the exponential chart at x_i. We obtain for $(x,t) \in B_{x_i}(\alpha) \times (-\alpha,\alpha)$:

$$F(x,t) = DF(x_i,0)(\phi_i(x),t) + R_i(\phi_i(x),t),$$

where $\frac{R_i(v,t)}{|(v,t)|} \to 0$ when $(v,t) \to 0$. We write for $x \in \mathbf{S}^n$ that

$$F_t(x) = F(x,t), \quad L_t(x) = \begin{cases} DF(x_i,0)(\phi_i(x),t) & \text{if } (x,t) \in B_{x_i}(\alpha) \times (-\alpha,\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

We define a cut-off function $0 \le \psi \le 1$ such that $\psi = 1$ on $K_1 = \bigcup_{i=1}^r \overline{B_{x_i}(\frac{\alpha}{2})}$ and $\psi = 0$ on $K_2 = \mathbf{S}^n \setminus \bigcup_{i=1}^r B_{x_i}(\alpha)$. We set for $s \in [0,1]$,

$$G_s^t = \frac{s\psi L_t + (1-s\psi)F_t}{|s\psi L_t + (1-s\psi)F_t|}.$$

One may choose $\alpha > 0$ small enough so that G_s^t is well defined for all $t \in (-\alpha,\alpha) \setminus \{0\}$. Then, for $0 < t < \alpha$, G_1^t is homotopic to $G_0^t = \frac{F_t}{|F_t|}$, and so to f, and G_1^{-t} is homotopic to σ. We now write, for $t \in (-\alpha,\alpha)$, $g_t = G_1^t$.

Let us look at the behaviour of $g_t = \frac{L_t}{|L_t|}$ in the balls $\overline{B_{x_i}(\frac{\alpha}{2})}$ when $t \to 0$. We recall that

$$L_t(x) = \frac{1}{2}(T_{x_i} f - I)\phi_i(x) + x_i t.$$

Therefore, the image $I_{x_i}^t = g_t(\overline{B_{x_i}(\frac{\alpha}{2})})$ blows up to the half-sphere $D_{x_i} = \{x \in \mathbf{S}^n; (x,x_i) > 0\}$ when $t \to 0$.

Thanks to (4), x is a fix point of f if and only if $-x$ is a fix point also. Moreover, by differentiating (4) at a fix point x, we obtain $T_{-x}f - I = -(T_x f - I)$.

Let's renumber the fixed points $x_1, \cdots, x_k, -x_1, \cdots, -x_k$ (with $r = 2k$), so that $x_1, \cdots, x_k$ are in a same half-sphere $D_p = \{(x,p) > 0\}$. We choose $\epsilon < \alpha$ small enough so that $\bigcap_{i=1}^k I_{x_i}^\epsilon$ has a non-empty interior I. Then, for $z \in I$, there is a unique point in $g_t^{-1}(z) \cap B_{x_i}(\frac{\alpha}{2})$ for all $0 < t < \epsilon$. Since $g_\epsilon(x) = g_{-\epsilon}(-x)$, if $z \in I$, then $z \in I_{-x_i}^{-\epsilon}$ and $z \notin I_{-x_i}^\epsilon \cup I_{x_i}^{-\epsilon}$.

For $1 \leq i \leq k$, let $\{a_i\} = B_{x_i}(\frac{\alpha}{2}) \cap g_\epsilon^{-1}(z)$. Then by the definition of degree and homotopy,

$$\deg(f) - \deg(\sigma) = \deg(g_\epsilon) - \deg(g_{-\epsilon}) = \sum_{i=1}^k \mathrm{ind}_{a_i}(g_\epsilon) - \mathrm{ind}_{-a_i}(g_{-\epsilon}) = \sum_{i=1}^k (1-(-1)^{n+1})\nu_i,$$

where $\nu_i = \mathrm{ind}_{a_i}(g_\epsilon) \in \pm 1$. In odd dimensions, $\deg(f) = \deg(\sigma) = 1$, and in even dimensions, $\deg(f) \in 2\mathbf{Z} + 1$. This ends the proof of Step 1.

Step 2. Let $f : \mathbf{S}^n \to \mathbf{S}^n$ be a continuous map which satisfies (4). Then there exists a map, homotopic to f, which satisfies the assumptions of Step 1.

Proof of Step 2. Denote by $(e_0, e_1, \cdots, e_n)$ the canonical basis of $\mathbf{R}^{n+1}$ and $B_k^\alpha \subset D_{e_k} = \{(x, e_k) > 0\}$ the ball centered at e_k such that $d(B_k^\alpha, D_{-e_k}) = \alpha > 0$. Choose α small enough so that

$$\bigcup_{i=0}^n B_i^{2\alpha} \cup (-B_i^{2\alpha}) = \mathbf{S}^n.$$

Let $\epsilon > 0$. We build by induction maps $g_k : \mathbf{S}^n \to \mathbf{S}^n$ such that $g_0 = f$ and, for $0 \leq k \leq n$,

- $g_{k+1} = g_k$ on $\mathbf{S}^n \setminus (B_k^\alpha \cup (-B_k^\alpha))$,
- g_{k+1} is smooth on $\bigcup_{i=0}^k B_i^{2\alpha} \cup (-B_i^{2\alpha})$,
- $\|g_{k+1} - g_k\|_{\mathcal{C}^0} < \epsilon$,
- g_{k+1} satisfies (4).

By density of smooth maps $\mathbf{S}^n \to \mathbf{R}^{n+1}$, choose h_k such that $\|h_k - g_k\|_{\mathcal{C}^0} < \epsilon$. Let $0 \leq \phi \leq 1$ be a smooth cut-off function such that $\phi = 1$ on $B_i^{2\alpha}$ and $\phi = 0$ on $\mathbf{S}^n \setminus B_i^\alpha$. We let g_{k+1} be defined, provided ϵ is small enough, by

$$g_{k+1}(x) = \frac{\phi h_k + (1-\phi) g_k}{|\phi h_k + (1-\phi) g_k|} \text{ and } g_{k+1}(-x) = R_x \circ g_{k+1}(x)$$

for $x \in \overline{D_{e_k}}$. Therefore $g = g_{n+1}$ is smooth, satisfies (4) and $\|g - f\|_{\mathcal{C}^0} < C\epsilon$. If ϵ is small enough, g is homotopic to f.

Let's now tackle the transversality condition. We write g in the following way:

$$g(x) = X(x) + \lambda(x) x,$$

where X is a tangent vector field of the sphere and $|X|^2 + \lambda^2 = 1$. Then, g satisfies (4) if and only if X and λ are even maps. By differentiating these equalities at a fixed point x (with $\lambda(x) = 1$ and $X(x) = 0$), one may find $T_x g - I = T_x X$. Then, $T_x g - I$ is an isomorphism for all fixed points x if and only if X is transverse to the zero vector field. Then, one may build by induction, with Sard's theorem in

n-dimensional charts on D_{e_k}, smooth tangent vector fields X_k such that $X_0 = X$ and for $0 \leq k \leq n$:

- $X_{k+1} = X_k$ on $\mathbf{S}^n \setminus (B_k^\alpha \cup (-B_k^\alpha))$,
- X_{k+1} is transverse to 0 on $\bigcup_{i=0}^k B_i^{2\alpha} \cup (-B_i^{2\alpha})$,
- $\|X_{k+1} - X_k\|_{\mathcal{C}^0} < \epsilon$,
- X_{k+1} is an even map.

Set

$$\bar{f}(x) = \frac{X_{n+1}(x) + \lambda(x)x}{|X_{n+1}(x)|^2 + \lambda(x)^2}.$$

If ϵ is small enough, then $\bar{f}$ is well defined, satisfies the assumptions of Step 1 and is homotopic to f. This ends the proof of Step 2.

These two steps clearly end the proof of the claim. □

3. Choice of test functions

Thanks to Claim 2, one may easily deduce that

$$\forall a \in \mathcal{C}, u_{a^*} = -u_a, \tag{5}$$

where we have set, for this section, $u_a = u_a^{s(a)}$. Let $r \in (-1,1)$. We look at the space E generated by

$$\phi = X_{e_1} \text{ and } \psi_r = u_{a_{r,e_1}}.$$

One may deduce from the continuity of ξ and s, (3) and (5) that

Claim 4. The map $r \in (-1,1) \mapsto \psi_r \in \left(L^2(\mathbf{S}^n, g), \|.\|_{L^2}\right)$ is continuous and

$$\lim_{r \to -1} \psi_r = -\phi, \qquad \lim_{r \to 1} \psi_r = \phi.$$

For $(x,y) \in \mathbf{R}^2 \setminus \{0\}$, we set $f_r = x\phi + y\psi_r \in E$. Conformal invariance gives that

$$\begin{aligned}\frac{\int_{\mathbf{S}^n} |\nabla_g f_r|_g^2 \, dv_g}{\int_{\mathbf{S}^2} f_r^2 dv_g} &= \frac{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n \, dv_g\right)^{\frac{2}{n}}}{\frac{1}{n+1}} \frac{\sigma x^2 + \tau_r y^2 + 2\alpha_r xy}{Ix^2 + J_r y^2 + 2\beta_r xy} \\ &:= (n+1)\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n \, dv_g\right)^{\frac{2}{n}} q(x,y),\end{aligned}$$

where we set for $r \in (-1,1)$

$$\sigma = \frac{\int_{\mathbf{S}^n} |\nabla_g \phi|_g^2 \, dv_g}{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n \, dv_g\right)^{\frac{2}{n}}} < 1, \qquad \tau_r = \frac{\int_{\mathbf{S}^n} |\nabla_g \psi_r|_g^2 \, dv_g}{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n \, dv_g\right)^{\frac{2}{n}}} < 2^{\frac{2}{n}},$$

$$\alpha_r = \frac{\int_{\mathbf{S}^n} g(\nabla_g \psi_r, \nabla_g \phi) dv_g}{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n \, dv_g\right)^{\frac{2}{n}}}, \qquad \beta_r = (n+1)\int_{\mathbf{S}^n} \phi \psi_r dv_g,$$

$$I = (n+1)\int_{\mathbf{S}^n} \phi^2 dv_g > 1, \qquad J_r = (n+1)\int_{\mathbf{S}^n} \psi_r^2 dv_g > 1.$$

By (2), $\tau_r < 2^{\frac{2}{n}}$, and by maximality of ϕ and ψ_r, $I > 1$ and $J_r > 1$.

The value $(n+1)2^{\frac{2}{n}}\left(\int_{\mathbf{S}^n}|\nabla_g\phi|_g^n\,dv_g\right)^{\frac{2}{n}}$, which also appears in (2), is independent of the metric $g\in[g_0]$ thanks to conformal invariance. The quotient K_n given in the theorem compares this value with the constant of the conjecture $n(2\sigma_n)^{\frac{2}{n}}$:

$$K_n := \frac{(n+1)2^{\frac{2}{n}}\left(\int_{\mathbf{S}^n}|\nabla_{g_0}\phi|_{g_0}^n\,dv_{g_0}\right)^{\frac{2}{n}}}{n(2\sigma_n)^{\frac{2}{n}}}$$

$$= \frac{n+1}{n}\left(\frac{1}{\sigma_n}\int_{\mathbf{S}^n}(1-X_{e_1}^2)dv_{g_0}\right)^{\frac{2}{n}} \tag{6}$$

$$= \frac{n+1}{n}\left(\frac{\sigma_{n-1}}{\sigma_n}\int_0^\pi(\sin\theta)^{2n-1}d\theta\right)^{\frac{2}{n}}.$$

The computation of the explicit value of K_n is classical (see for instance [5]). Thus, in order to get the estimate of the Theorem and using the min-max principle (1), we look for $r\in(-1,1)$ such that for all $(x,y)\in\mathbf{R}^2\setminus\{0\}$,

$$q(x,y)<2^{\frac{2}{n}}.$$

Since $I>1$ and $J_r>1$, we look for $r\in(-1,1)$ such that

$$(\sigma-2^{\frac{2}{n}})x^2+2(\alpha_r-2^{\frac{2}{n}}\beta_r)yx+(\tau_r-2^{\frac{2}{n}})y^2<0.$$

Moreover, since $\sigma<1$ and $\tau_r-2^{\frac{2}{n}}<0$, it is sufficient to find $r\in(-1,1)$ such that

$$\alpha_r-2^{\frac{2}{n}}\beta_r=0.$$

By Claim 4, we know that

$$\alpha_r=\frac{-\int_{\mathbf{S}^n}\psi_r\,(\Delta_g\phi)\,dv_g}{\left(\int_{\mathbf{S}^n}|\nabla_g\phi|_g^n\,dv_g\right)^{\frac{2}{n}}}\underset{r\to1}{\longrightarrow}\frac{-\int_{\mathbf{S}^n}\phi\,(\Delta_g\phi)\,dv_g}{\left(\int_{\mathbf{S}^n}|\nabla_g\phi|_g^n\,dv_g\right)^{\frac{2}{n}}}=\sigma$$

and that

$$\beta_r=(n+1)\int_{\mathbf{S}^n}\phi\psi_r dv_g\underset{r\to1}{\longrightarrow}(n+1)\int_{\mathbf{S}^n}\phi^2dv_g=I.$$

Thus, when $r\to1$ and, in an analogous way, when $r\to-1$ (see Claim 4),

$$\alpha_r-2^{\frac{2}{n}}\beta_r\underset{r\to1}{\longrightarrow}\sigma-2^{\frac{2}{n}}I<0$$

and

$$\alpha_r-2^{\frac{2}{n}}\beta_r\underset{r\to-1}{\longrightarrow}2^{\frac{2}{n}}I-\sigma>0.$$

By continuity (Claim 4), there exists $r\in(-1,1)$ such that $\alpha_r-2^{\frac{2}{n}}\beta_r=0$. As has already been said, this completes the proof of the Theorem.

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