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ASYMPTOTICS OF ORBITS OF A KOLMOGOROV TYPE PLANAR VECTOR FIELD WITH A FIXED NEWTON POLYGON

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Dedicated to the memory of Professor A. M. Molchanov

Abstract. Using the Newton polygon technique we show that the orbits of a Kolmogorov type planar vector field, consisting of a finite sum of power terms, have power asymptotics while tending to the equilibria on the axes and on the boundary of the Poincaré sphere.

1. Introduction

Let us consider a two-dimensional vector field Ω defined on the Poincaré sphere [1] by a Kolmogorov type system

$$\Omega(V_1,V_2)\equiv\Omega(\nu):\ \begin{aligned}\frac{d\nu_1}{dt}&=V_1(\nu_1,\nu_2)\equiv\nu_1P_1(\nu_1,\nu_2),\\ \frac{d\nu_2}{dt}&=V_2(\nu_1,\nu_2)\equiv\nu_2P_2(\nu_1,\nu_2).\end{aligned}\tag{1.1}$$

Here $(\nu_1,\nu_2)\equiv(x_1,x_2), V_1(\nu_1,\nu_2)\equiv X_1(x_1,x_2), V_2(\nu_1,\nu_2)\equiv X_2(x_1,x_2)$ for finite x, and power functions

$$P_1(x_1,x_2)=\sum_{\mu,\nu}p_{\mu\nu}x_1^\mu x_2^\nu,\quad P_2(x_1,x_2)=\sum_{\mu,\nu}q_{\mu\nu}x_1^\mu x_2^\nu,\quad \mu\ge0,\nu\ge0,p_{\mu\nu},q_{\mu\nu}\in\mathbf{R},\tag{1.2}$$

consist of a finite number of summands with distinct powers $(\mu,\nu)\in\mathbf{R}_+^2$:

$$(\nu_1,\nu_2)\equiv(z_1=1/x_1,z_2=x_2/x_1),\tag{1.3}$$

$$\begin{gathered}V_1(\nu_1,\nu_2)\equiv Z_1(z_1,z_2)=z_1Q_1(z_1,z_2),\quad V_2(\nu_1,\nu_2)\equiv Z_2(z_1,z_2)=z_2Q_2(z_1,z_2),\\ Q_1(z_1,z_2)=-P_1(1/z_1,z_2/z_1),\quad Q_2(z_1,z_2)=P_2(1/z_1,z_2/z_1)-P_1(1/z_1,z_2/z_1),\end{gathered}\tag{1.4}$$

$$(\nu_1,\nu_2)\equiv(u_1=x_1/x_2,u_2=1/x_2),\tag{1.5}$$

$$\begin{gathered}V_1(\nu_1,\nu_2)\equiv U_1(u_1,u_2)=u_1R_1(u_1,u_2),\ V_2(\nu_1,\nu_2)\equiv R_2(u_1,u_2)=u_2R_2(u_1,u_2),\\ R_1(u_1,u_2)=P_1(u_1/u_2,1/u_2)-P_2(u_1/u_2,1/u_2),\ R_2(u_1,u_2)=-P_2(u_1/u_2,1/u_2),\end{gathered}\tag{1.6}$$

respectively, close to equators of the Poincaré sphere $z_1u_2=0$.

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The vector field $\Omega(x)$ has a singular point $(x_1 = 0, x_2 = 0)$. This point is isolated only if P_1, P_2 have no common factors x_1, x_2. In such a case $\Omega(x)$ has a singular point $(x_1 = 0, x_2 = a \neq 0)$ if $P_2(0, a) = 0$, and $(x_1 = a \neq 0, x_2 = 0)$ if $P_1(a, 0) = 0$. It is straightforward to verify that vector fields $\Omega(z)$ and $\Omega(u)$ have the singular points $(z_1 = 0, z_2 = 0)$ and $(u_1 = 0, u_2 = 0)$ respectively. Vector fields $\Omega(x)$, $\Omega(z)$ are equivalent everywhere except $x_1 z_1 = 0$, $\Omega(x)$, $\Omega(u)$ are equivalent everywhere except $x_2 u_2 = 0$ ($z_1 = 0, u_2 = 0$ are the equators of the Poincaré sphere) and $\Omega(z)$, $\Omega(u)$ are equivalent everywhere except $z_1 = 0$, $u_2 = 0$; if $\Omega(z)$ has singular point $(z_1 = 0, z_2 = a \neq 0)$, then $\Omega(u)$ has singular point $(u_1 = 1/a \neq 0, u_2 = 0)$.

Without loss of generality we will suppose below that vector fields are *continuously differentiable* (C^1) in both variables; we consider them to be defined in $\mathbf{R}^2_+$.

The objective of this work is to describe the asymptotic behavior of the orbits of the vector fields $\Omega(x)$, $\Omega(z)$, $\Omega(u)$ near the axes $x_1 x_2 = 0$ and the equators of the Poincaré sphere $z_1 u_2 = 0$.

To solve this problem we apply methods of the Newton diagram (polygon) (see [2]–[7] and the references in [2]). We use a modification of the method, which has been developed in [8], [9], for an analysis of trajectories tending to a non-hyperbolic isolated equilibrium $O(0, 0)$ with $t \to \infty$ and/or $t \to -\infty$ (the so-called *O-orbits*). In this paper we generalize these results analyzing isolated singular points (*isp*) $A(a_1, a_2)$ of vector fields $\Omega(x)$, $\Omega(z)$, $\Omega(u)$, for which $a_1 a_2 = 0$. Note that this kind of problem still finds new applications (see, for example, [10], [11]).

2. Newton polygon. The main statement

2.1. **Definitions.** (See Figure 1.) **(1)** The set $\mathrm{M} = \{(u, \nu), |p_{\mu\nu}| + |q_{\mu\nu}| \neq 0\}$ is called *the support* of system (1.1) and of the vector field $\Omega(x)$, and $(p_{\mu\nu}, q_{\mu\nu})$ is called the *vector coefficient* of the point $(\mu, \nu) \in \mathrm{M}$.

(2) The *Newton polygon* (NP) Γ is the convex hull of M. Γ consists of one vertex $\gamma = \gamma^{(0)}(\mu = 0, \nu = 0)$ or is the union of edges $\gamma = \gamma^{(1)} \in \Gamma$ together with their vertices.

(3) The edge $\gamma = \gamma^{(1)} \in \Gamma$ is called *special* and is denoted as $\gamma^\mu, \gamma^\nu, \gamma^{\mu^0}, \gamma^{\nu^0}$ if both of its vertices belong to either the μ-axis or the ν-axis, or the line $\mu = \mu^0$ or the line $\nu = \nu^0$, respectively; the edge $\gamma^{(1)}$ is called *standard* otherwise. *Index* α *of an edge* γ is the number $\alpha = \alpha(\gamma) = \tan(\phi)$, where ϕ is the angle between the edge and the negative direction of the ordinate axis ν; the special edges $\gamma^\mu, \gamma^{\mu^0}$ and $\gamma^\nu, \gamma^{\nu^0}$ have indexes $\alpha = 0$ and $\alpha = \infty$ respectively.

(4) The vertex $\gamma \equiv \gamma^{(0)}(\mu, \nu)$ is called a *boundary* vertex if $\mu\nu = 0$ and *standard* if not. The *index* β of a vertex γ is $\beta \equiv \beta(\gamma) = q_{\mu\nu}/p_{\mu\nu}$ if $p_{\mu\nu} \neq 0$ and $\beta = \infty$ if $p_{\mu\nu} \neq 0$.

(5) *Inscribe* Γ into the rectangle $O_{00}(\mu = \nu = 0)$, $O_{\infty 0}(\mu^0, 0)$, $O_{0\infty}(0, \nu^0)$, $O_{\infty\infty}(\mu^0, \nu^0)$, where $\mu^0 = \max_\mu(\mu, (\mu, \nu) \in \mathrm{M})$, $\nu^0 = \max_\nu(\nu, (\mu, \nu) \in \mathrm{M})$. We call the *Newton diagrams (ND)* Γ_{00}, $\Gamma_{\infty 0}$, $\Gamma_{0\infty}$, $\Gamma_{\infty\infty}$ the parts of Γ which are visible from the corners $O_{00}, O_{\infty 0}, O_{0\infty}, O_{\infty\infty}$ respectively. Thus $\Gamma = \Gamma_{00} \cup \Gamma_{\infty 0} \cup \Gamma_{0\infty} \cup \Gamma_{\infty\infty} \cup \gamma^\mu \cup \gamma^\nu \cup \gamma^{\mu^0} \cup \gamma^{\nu^0}$. Denote coherent sets $\mathrm{M}_\Gamma = \mathrm{M} \cap \Gamma$, $\mathrm{M}_{\Gamma_{0,0}} = \mathrm{M} \cap \Gamma_{00}$, $\mathrm{M}_{\Gamma_{\infty 0}} = \mathrm{M} \cap \Gamma_{\infty 0}, \mathrm{M}_{\Gamma_{\infty\infty}} = \mathrm{M} \cap \Gamma_{\infty\infty}, \mathrm{M}_{\Gamma_{0\infty}} = \mathrm{M} \cap \Gamma_{0\infty}, \mathrm{M} \cap \gamma$, where γ is an edge or a vertex of Γ. Vector fields $\Omega_\Gamma(x)$, $\Omega_{\Gamma_{00}}(x)$, $\Omega_{\Gamma_{\infty 0}}(x)$, $\Omega_{\Gamma_{\infty\infty}}(x)$, $\Omega_{\Gamma_{0\infty}}(x)$ and $\Omega_\gamma(x)$ are called the *truncations* of $\Omega(x)$ (and system (1.1)) onto the corresponding sets. In particular, $\Omega_\Gamma(x)$ is a vector field *with fixed Newton polygon* Γ; $\Omega_{\Gamma_{00}}(x)$ is

a vector field *with fixed Newton diagram* Γ_{00}. Let $\Omega(x) \equiv \Omega(X_1, X_2)$ be defined by systems (1.1), (1.2), which is presented in the form

$$
\begin{aligned}
&\frac{dx_1}{dt} = X_1(x_1,x_2) \equiv x_1(P_1^\gamma(x_1,x_2) + \phi_1(x_1,x_2)),\\
&\frac{dx_2}{dt} = X_2(x_1,x_2) \equiv x_2(P_2^\gamma(x_1,x_2) + \phi_2(x_1,x_2)),\\
&P_1^\gamma = \sum_{(\mu,\nu)\in \mathrm{M}_\gamma} p_{\mu\nu} x_1^\mu x_2^\nu, \quad P_2^\gamma = \sum_{(\mu,\nu)\in \mathrm{M}_\gamma} q_{\mu\nu} x_1^\mu x_2^\nu;\\
&\phi_1 = \sum_{(\mu,\nu)\in \mathrm{M}-\mathrm{M}_\gamma} p_{\mu\nu} x_1^\mu x_2^\nu, \quad \phi_2 = \sum_{(\mu,\nu)\in \mathrm{M}-\mathrm{M}_\gamma} q_{\mu\nu} x_1^\mu x_2^\nu.
\end{aligned}
\tag{2.1}
$$

Then the truncated vector field is $\Omega_\gamma(x) \equiv \Omega_\gamma(X_1^\gamma, X_2^\gamma)$, where $X_1^\gamma \equiv x_1 P_1^\gamma(x_1,x_2)$, $X_2^\gamma \equiv x_2 P_2^\gamma(x_1,x_2)$.

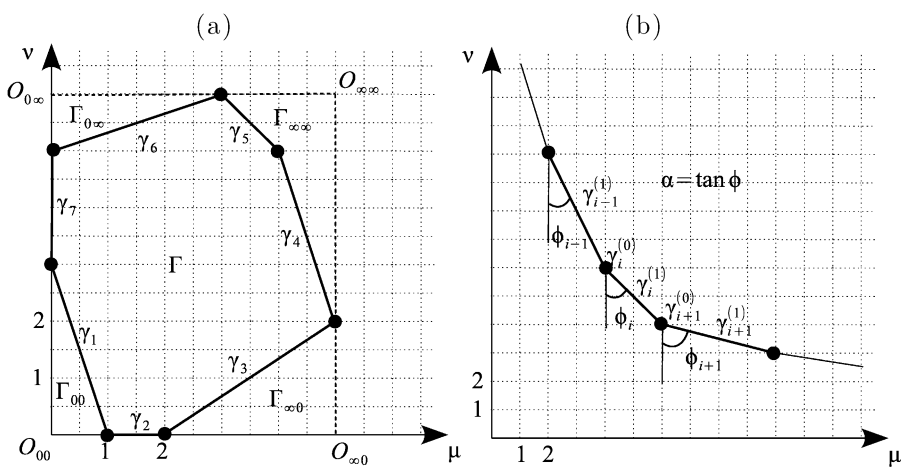

FIGURE 1. (a) Γ is the Newton polygon of the system $x_1' = x_1 P_1(x_1,x_2)$, $x_2' = x_2 P_2(x_1,x_2)$, where $P_1(x_1,x_2) = p_{10}x_1 + p_{20}x_1^2 + p_{52}x_1^5x_2^2 + p_{45}x_1^4x_2^5 + p_{36}x_1^3x_2^6 + p_{05}x_2^5 + p_{03}x_2^3$, $P_2(x_1,x_2) = q_{10}x_1 + q_{20}x_1^2 + q_{52}x_1^5x_2^2 + q_{45}x_1^4x_2^5 + q_{36}x_1^3x_2^6 + q_{05}x_2^5 + q_{03}x_2^3$. Support $\mathrm{M} = \mathrm{M}_\Gamma$ consists of points $(1,0),(2,0),(5,2),(4,5),(3,6),(0,5),(0,3)$; $\Gamma = \Gamma_{00} \cup \Gamma_{\infty 0} \cup \Gamma_{0\infty} \cup \Gamma_{\infty\infty} \cup \gamma^\mu \cup \gamma^\nu$, where $\Gamma_{00} \equiv \gamma_1$ with index $\alpha_1 = 1/3$, $\Gamma_{\infty 0} \equiv \gamma_3$ with index $\alpha_3 = -3/2$, $\Gamma_{0\infty} \equiv \gamma_6$ with index $\alpha_6 = -3$, $\Gamma_{\infty\infty} \equiv \gamma_4 \cup \gamma_5$, $(\alpha_4 = 1/3, \alpha_5 = 1)$ and $\gamma^\mu = \gamma_7$, $(\alpha_7 = 0)$, $\gamma^\nu = \gamma_2$, $(\alpha_2 = \infty)$ are special edges.
(b) A part of Newton diagram Γ_{00} containing consequent edges $\gamma_{i-1}^{(1)}$, $\gamma_i^{(1)}$, $\gamma_{i+1}^{(1)}$ and edge $\gamma_i^{(1)}$ is bounded by vertices $\gamma_i^{(0)}(\mu_i,\nu_i)$, $\gamma_{i+1}^{(0)}(\mu_{i+1},\nu_{i+1})$.

If γ is a vertex (μ,ν), a special edge, say $\gamma=\gamma^\mu$, or a standard edge γ^α with index α, then

$$P_1^\gamma(x_1,x_2)=p_{\mu_\gamma\nu_\gamma}x_1^{\mu_\gamma}x_2^{\nu_\gamma},\quad P_2^\gamma(x_1,x_2)=q_{\mu_\gamma\nu_\gamma}x_1^{\mu_\gamma}x_2^{\nu_\gamma}, \tag{2.2}$$

$$P_1^\gamma(x_1,x_2)=x_1^\mu\sum_\nu p_{\mu\nu}x_2^\nu,\quad P_2^\gamma(x_1,x_2)=x_1^\mu\sum_\nu q_{\mu\nu}x_2^\nu,\ \ (\mu,\nu)\in M_\gamma, \tag{2.3}$$

$$P_1^\gamma(x_1,x_2)=\sum_{\mu+\alpha\nu=\sigma_\gamma}p_{\mu\nu}x_1^\mu x_2^\nu,\quad P_2^\gamma(x_1,x_2)=\sum_{\mu+\alpha\nu=\sigma_\gamma}q_{\mu\nu}x_1^\mu x_2^\nu,\quad \sigma_\gamma\equiv\text{const}, \tag{2.4}$$

respectively. Thus, a truncated vector field $\Omega_\gamma(x)$ is defined by the integrable system (2.1).

Definition (6). The vector field $\Omega_\Gamma(x)(\Omega_{\Gamma_\infty}(x),\ \Omega_{\Gamma_{\infty 0}}(x),\ \Omega_{\Gamma_{\infty\infty}}(x),\ \Omega_{\Gamma_{0\infty}}(x),\ \Omega_\gamma(x))$ is called $\Gamma(\Gamma_{00},\Gamma_{\infty 0},\Gamma_{0\infty},\Gamma_{\infty\infty},\gamma)$-**non-degenerate** with respect to isp $A(a_1,a_2)$, where $a_1a_2=0$ in one of the x,z,u-coordinates, if the following statements are fulfilled: (i) index β of any interior vertex of $\Gamma(\Gamma_{00},\Gamma_{\infty 0},\Gamma_{0\infty},\Gamma_{\infty\infty})$ is not equal to the indexes of the adjacent edges, and index β of any boundary vertex is not equal to $\alpha=0,\infty$; (ii) for any edge γ (having index α) functions $P_1^\alpha(1,\omega)$, $P_2^\alpha(1,\omega)$ have no common non-zero roots; (iii) the function $F^\alpha(\omega)=-\alpha P_1^\alpha(1,\omega)+P_2^\alpha(1,\omega)$ has no multiple non-zero roots.

According to the definition of non-degeneracy we can state

Proposition 2.1. *Among all vector fields $\Omega(\Gamma)$ with the same Newton polygon Γ the set of Γ-non-degenerate vector fields is open and dense.*

2.2. **Asymptotics to equilibria.**

Theorem A. (1) *Let $A(a_1,a_2)$ be a singular point of the Γ-non-degenerate vector field $\Omega(\nu)$ given in the Poincaré sphere by systems* (1.1)–(1.2), (1.1)–(1.4) *or* (1.1)–(1.6), *where $a_1a_2=0$ in coordinates $(\nu_1,\nu_2)=(x_1,x_2)$ or $(\nu_1,\nu_2)=(z_1,z_2)$ or $(\nu_1,\nu_2)=(u_1,u_2)$. Any orbit of $\Omega(\nu)$ which tends to A for $t\to\infty$ or $t\to-\infty$ in the phase coordinates (ν_1,ν_2) has either power or trivial asymptotics; i.e., it can be presented in the form*

$$\nu_2-a_2=k\nu_1^\rho(1+o(1)),\quad \nu_1-a_1=\tilde k\nu_2^{\tilde\rho}(1+o(1)), \tag{2.5}$$

where $(a_2,\rho\ge 0)$, $(a_1,\tilde\rho\ge 0)$, constant $k\ne 0$ if $\rho>0$, $\tilde k\ne 0$ if $\tilde p>0$.

(2) *Let $O(0,0)$ be a singular point of the Γ-non-degenerate vector field $\Omega(\nu)$ (i.e., $a_1=a_2=0$). Then $\tilde\rho\rho=1$ in* (2.5). *In x-coordinates the exponent ρ coincides with index α of one of the edges $\gamma^{(1)}\in\Gamma$ or coincides with index β of one of the vertices $\gamma^{(0)}\in\Gamma$; $\rho=\alpha$ if the function $F^\alpha(\omega)=-\alpha P_1^\alpha(1,\omega)+P_2^\alpha(1,\omega)$ has root $\omega=k\ne 0$, $\rho=\beta$ if the vertex is interior and β belongs to an interval composed by indexes of edges that are adjacent to this vertex.*

The rest of the paper is organized as follows. In Section 3 we show the connections between power changes of variables in system (1.1), (1.2) and transformations of its Newton polygon. In Section 4 we provide a complete proof of Theorem A.

3. Power substitutions in system (1.1) and transformations of NP

3.1. **Auxiliary statements.** Let us consider the question of how the change of the variables in systems (1.1), (1.2) affects NP Γ. Denote $\overline{\Gamma}$ as the transformed NP Γ, $\overline{\Omega}_\Gamma \equiv \Omega_{\overline{\Gamma}}$ the transformed vector field Ω_Γ, etc., keeping the "initial" denotation supplemented by the bar. The following statement holds:

Proposition 3.1. *The change of the independent variable in system* (1.1),

$$dt = x_1^{l_1} x_2^{l_2} d\tau, \quad l_1, l_2 \in \mathbf{R}, \tag{3.1}$$

induces the linear transformation $\mathbf{L} : \Gamma \to \overline{\Gamma}$ *shifting by the vector* (l_1, l_2) *in the plane* (μ, ν). $\mathbf{L}$ *is invertible and* $\mathbf{L}^{-1} : \overline{\Gamma} \to \Gamma$ *corresponds to the change* $d\tau = x_1^{-l_1} x_2^{-l_2}\, dt$.

Consider the power change of the variables in system (1.1):

$$(x_1, x_2) = (\xi_1, \xi_2)^E : x_1 = \xi_1^{e_1} \xi_2^{e_2},\ x_2 = \xi_1^{e_3} \xi_2^{e_4},\ E = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix},\ \Delta = \det(E)\ \ (x_1, x_2) \in \mathbf{R}_2^+. \tag{3.2}$$

If $\Delta \neq 0$, then (3.2) has the inverse $(\xi_1, \xi_2) = (x_1, x_2)^{E^{-1}}$ and converts (1.1) to the form:

$$\begin{aligned} \frac{d\xi_1}{d\tau} &= \Xi_1(\xi_1, \xi_2) \equiv \xi_1 H_1(\xi_1, \xi_2) = \xi_1(e_4 P_1(x_1, x_2) - e_2 P_2(x_1, x_2)) \equiv \xi_1 \sum_{\overline{\mu}, \overline{\nu}} p_{\overline{\mu}\overline{\nu}} \xi_1^{\overline{\mu}} \xi_2^{\overline{\nu}}, \\ \frac{d\xi_2}{d\tau} &= \Xi_2(\xi_1, \xi_2) \equiv \xi_2 H_2(\xi_1, \xi_2) = \xi_2(-e_3 P_1(x_1, x_2) + e_1 P_2(x_1, x_2)) \equiv \xi_2 \sum_{\overline{\mu}, \overline{\nu}} q_{\overline{\mu}\overline{\nu}} \xi_1^{\overline{\mu}} \xi_2^{\overline{\nu}}, \\ \overline{p}_{\mu\nu} &\equiv p_{\overline{\mu}\overline{\nu}} = e_4 p_{\mu\nu} - e_2 q_{\mu\nu}, \quad \overline{q}_{\mu\nu} \equiv q_{\overline{\mu}\overline{\nu}} = -e_3 p_{\mu\nu} + e_1 q_{\mu\nu}. \end{aligned} \tag{3.3}$$

Let $\overline{\mathrm{M}}$ be the support of system (3.3). Comparing powers (μ, ν), $(\overline{\mu}, \overline{\nu})$ of systems (1.1), (1.2) and (3.3) we see that the change (3.2) induces a linear transformation $\mathbf{G} : \mathrm{M} \to \overline{\mathrm{M}}$ with the matrix E^T: $(\mu, \nu) \to (\overline{\mu}, \overline{\nu})$: $\overline{\mu} = e_1 \mu + e_3 \nu, \overline{\nu} = e_2 \mu + e_4 \nu$. If $\Delta \neq 0$, then inverse to $\mathbf{G}$ is given by $\mathbf{G}^{-1}$, which has the matrix

$$(E^T)^{-1} = \frac{1}{\Delta} \begin{pmatrix} e_4 & -e_2 \\ -e_3 & e_1 \end{pmatrix} : \ (\overline{\mu}, \overline{\nu}) = \mathbf{G}^{-1}(\mu, \nu);$$

$\mathbf{G}^{-1}$ transforms the Newton polygon $\Gamma \to \overline{\Gamma}$ such that the vector fields Ω_Γ and $\overline{\Omega}_\Gamma \equiv \Omega_{\overline{\Gamma}}$ completely define one another. The systems can be non-equivalent only at their equilibria, in particular at the coordinate axes. To avoid this "non-equivalency" we use the operators $\mathbf{L} \circ \mathbf{G}$ and $(\mathbf{L} \circ \mathbf{G})^{-1}$.

Lemma 3.1 ([8], [9])**.** (1) *The indexes of the vertices and edges of* $\overline{\Gamma}$, *which are obtained from* Γ *by the transformation* $\mathbf{L}$, *are identical to those of* Γ.

(2) *If* $\Delta \neq 0$, *then the transformation* $\mathbf{G}^{-1}$: $\Gamma_{\Omega(x)} \to \Gamma_{\overline{\Omega}(\xi)}$ *converts any edge and any vertex of* $\Gamma_{\Omega(x)}$ *into the edge and the vertex of* $\Gamma_{\overline{\Omega}(\xi)}$. *Indexes* α, $\overline{\alpha}$ *of the edges* $\gamma^{(1)}$, $\overline{\gamma}^{(1)}$ *and* $\beta, \overline{\beta}$ *of vertices* $\gamma^{(0)}, \overline{\gamma}^{(0)}$ *are expressed by the formulas*

$$\overline{\alpha} = (\alpha e_1 - e_3)/(e_4 - \alpha e_2), \quad \overline{\beta} = (\beta e_1 - e_3)/(e_4 - \beta e_2). \tag{3.4}$$

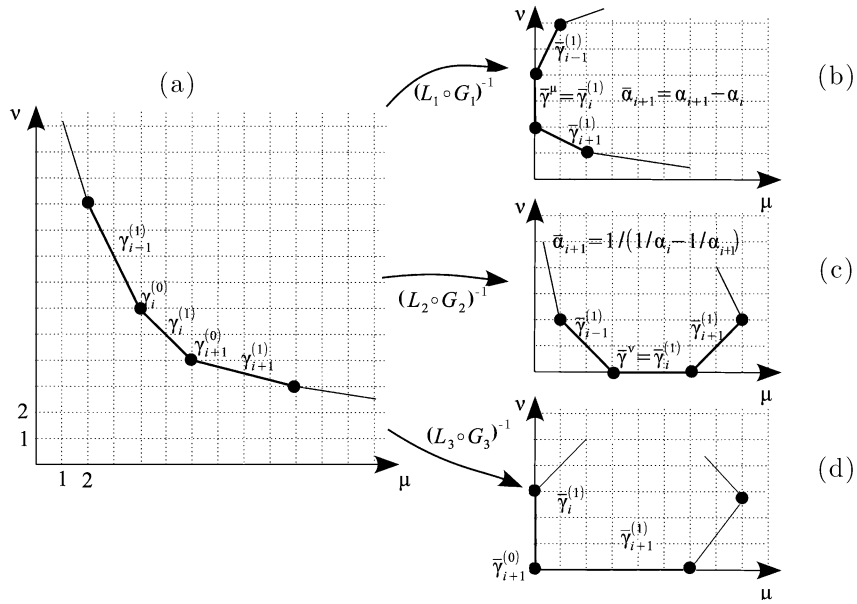

FIGURE 2. (a) A part of Newton diagram Γ_{00} (see Figure 1(b)); (b-d) results of changes of variables $(x_1, x_2) = (\xi_1, \xi_2)^E$, where

$$E = \begin{pmatrix} 1 & 0 \\ \alpha_i & 1 \end{pmatrix} \text{ (b)}, \quad E = \begin{pmatrix} 1 & 1/\alpha_i \\ 0 & 1 \end{pmatrix} \text{ (c)},$$

$$E = \begin{pmatrix} 1 & 1/(\alpha_{i+1}-\alpha_i) \\ \alpha_i & \alpha_{i+1}/(\alpha_{i+1}-\alpha_i) \end{pmatrix} \text{ (d)}$$

edges $\gamma_i^{(1)} \in \Gamma_{00} \subset \Gamma \to \gamma^{\mu} \in \overline{\Gamma}$ (b), $\gamma_i^{(1)} \in \Gamma_{00} \subset \Gamma \to \gamma^{\nu} \in \overline{\Gamma}$ (c), and the vertex $\gamma_{i+1}^{(0)} \to \overline{\gamma}_{i+1}^{(0)}(0,0) \in \overline{\Gamma}_{00}$ (d).

Important applications of Lemma 3.1 are given by the following statement.

Proposition 3.2. (1) *Let $\gamma^{(1)} \in \Gamma_{00}$ be a standard edge with index α. The linear transformation $(\mathbf{L}_1 \circ \mathbf{G}_1)^{-1}$ with matrix $E_1 = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$ and $(l_1, l_2) = (\mu + \alpha\nu, 0)$, $(\mu, \nu) \in \mathbf{M}_\gamma$ converts $\gamma^{(1)}$ into the special edge $\overline{\gamma}^{\mu}$ (see Figure 2(a), (b)). The linear transformation $(\mathbf{L}_2 \circ \mathbf{G}_2)^{-1}$ with matrix $E_2 = \left(\begin{smallmatrix} 1 & 1/\alpha \\ 0 & 1 \end{smallmatrix}\right)$ and $(l_1, l_2) = (0, (\mu + \alpha\nu)/\alpha)$, $(\mu, \nu) \in \mathbf{M}_\gamma$ converts $\gamma^{(1)}$ into the special edge $\overline{\gamma}^{\nu}$ (see Figure 2(a), (c)).*

(2) *Let $\gamma^{(0)}(\mu, \nu) \in \Gamma_{00}$ be an interior vertex, which is common for edges $\gamma_1^{(1)}, \gamma_2^{(1)}$ (having indexes $0 < \alpha_1 < \alpha_2$). The linear transformation $(\mathbf{L}_3 \circ \mathbf{G}_3)^{-1}$ with matrix $E_3 = \left(\begin{smallmatrix} 1 & 1/(\alpha_2-\alpha_1) \\ \alpha_1 & \alpha_2/(\alpha_2-\alpha_1) \end{smallmatrix}\right)$ and $(l_1, l_2) = (\mu + \alpha_1\nu, \nu)$ converts $\gamma^{(0)}(\mu, \nu) \to \overline{\gamma}^{(0)}(0,0) \in \overline{\Gamma}_{00}$ such that $\overline{\Gamma}_{00}$ consists of only this vertex (see Figure 2(a), (d)).*

3.2. **Newton diagrams and Poincaré coordinates.** Poincaré changes (1.3) and (1.5) are particular cases of changes (3.2) with $E_1 = \left(\begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and $E_2 = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$, respectively. Supplemented by the change of independent variable (3.1), where $l_1 = \sigma$, $l_2 = 0$ for $\mathbf{L}_1$ and $l_2 = \sigma$, $l_1 = 0$ for $\mathbf{L}_2$ with $\sigma \equiv \max_{(\mu,\nu)\in\mathbf{M}_\Gamma}(\mu + \nu)$, these changes transform the vector field $\Omega(x)$ defined by system (1.1), (1.2) to the vector

fields $\Omega(z)$ and $\Omega(u)$ defined by systems (1.1), (1.4) and (1.1), (1.5) respectively. Let Γ be a Newton polygon of (1.1), (1.2) and $\overline{\Gamma}^1$, $\overline{\Gamma}^2$ be Newton polygons of after transformations $(\mathbf{L}_1 \circ \mathbf{G}_1)^{-1}$ with $E = E_1$ and $(\mathbf{L}_2 \circ \mathbf{G}_2)^{-1}$ with $E = E_2$, and let $\overline{\Gamma}^1_{00} \in \overline{\Gamma}^1, \overline{\Gamma}^2_{00} \in \overline{\Gamma}^2$ be their Newton diagrams. To find the "preimages" of $\overline{\Gamma}^1_{00}$, $\overline{\Gamma}^2_{00}$ under transformations $\mathbf{G}_1^{-1}$, $\mathbf{G}_2^{-1}$ we introduce coherent open polygons $\widehat{\Gamma}_1 = (\Gamma_{\infty 0} \cup \gamma^{\mu^0} \cup \Gamma^{\alpha<1}_{\infty\infty}) \subset \Gamma$, $\widehat{\Gamma}_2 = (\Gamma_{0\infty} \cup \gamma^{\nu^0} \cup \Gamma^{\alpha>1}_{\infty\infty}) \subset \Gamma$ (see Figure 3(a),(b),(c)), where $\Gamma^{\alpha<1}_{\infty\infty}$ and $\Gamma^{\alpha>1}_{\infty\infty}$ are parts of $\Gamma_{\infty\infty}$ that contain edges γ with indexes $\alpha < 1$ and with indexes $\alpha > 1$, respectively, or $\widehat{\Gamma}_1 = \gamma^{(0)} = \widehat{\Gamma}_2$ if $\Gamma_{\infty 0}$, $\Gamma_{\infty\infty}$, $\Gamma_{0\infty}$ consist of one point $\gamma^{(0)}$.

According to Lemma 3.1 we have from formula (3.4) that for $E = E_1$ and $E = E_2$, respectively,

$$\overline{\alpha} = 1 - \alpha, \quad \overline{\beta} = 1 - \beta, \tag{3.5}$$

$$\overline{\alpha} = \alpha/(\alpha - 1), \quad \overline{\beta} = \beta/(\beta - 1), \tag{3.6}$$

where $\overline{\alpha} \equiv \alpha(\overline{\gamma}^{(1)} \in \overline{\Gamma})$, $\overline{\beta} \equiv \beta(\overline{\gamma}^{(0)} \in \overline{\Gamma})$, $\alpha \equiv \alpha(\gamma \in \Gamma)$, $\beta \equiv \beta(\gamma^{(0)} \in \Gamma)$. After additional algebraic verifications we prove the following statement.

Theorem 3.1 (see Figure 3(a),(b),(c))**.** *$\overline{\Gamma}_{00}$ is the one-to-one image of the union $\widehat{\Gamma}^1$ under transformation $(\mathbf{L}_1 \circ \mathbf{G}_1)^{-1}$ and the union $\widehat{\Gamma}^2$ under transformation $(\mathbf{L}_2 \circ \mathbf{G}_2)^{-1}$; the edge $\gamma^\alpha \in \Gamma_{\infty\infty}$ with $\alpha = 1$ transforms to the edge $\overline{\gamma}^\mu \in \overline{\Gamma}^1$ under $(\mathbf{L}_1 \circ \mathbf{G}_1)^{-1}$ and to the edge $\overline{\gamma}^\nu \in \overline{\Gamma}^2$ under $(\mathbf{L}_2 \circ \mathbf{G}_2)^{-1}$.*

It is straightforward to verify the following statement, which describes the *conditions of existence* of isolated singular points $A(a_1, a_2)$, $a_1 a_2 = 0$ in the non-degenerate vector field $\Omega_\Gamma(x)$.

Proposition 3.3. *Let Ω_Γ be a Γ-non-degenerate vector field given by* (1.1)*. Then*

(1) *point* $(0,0)$ *is the isp of vector field Ω_Γ if and only if it is isp of $\Omega_{\Gamma_{00}}(\nu)$;*

(2) *point $A(0, a_2), a_2 \neq 0$ $(A(a_1, 0), a_1 \neq 0)$ is the isp of Ω_Γ if and only if Γ contains a special edge $\gamma^\mu(\gamma^\nu)$ such that the function $F^\gamma(x_2) \equiv P_2^{\gamma^\mu}(0, x_2)$ has a root $a_2(F^{\gamma^\nu}(x_1) \equiv P_1^\gamma(x_1, 0))$ has a root a_1.*

According to Theorem 3.1 *analysis of all asymptotics of orbits of the Γ-non-degenerate vector field Ω_Γ is reduced to the analysis of asymptotics of orbits of the vector fields $\Omega_{\Gamma_{0,0}}$ and $\Omega_{\gamma^\mu}, \Omega_\gamma$.*

4. Proof of Theorem A. Description of asymptotics

4.1. **Andronov's statement. Orbits of the γ-non-degenerate vector field and its γ-truncation.** Further, we use the statement, based on Lemmas 1 and 2 in Chapter IV of [1].

Lemma 4.1. *Let $O(0,0)$ be an isolated simple equilibrium of the C^1-system* (1.1)*, which has a triangular Jordan matrix at O with non-zero eigenvalues λ_1, λ_2. If $\lambda_1 \lambda_2 > 0$, then O-orbits of the vector field defined by this system have asymptotics $x_2 = k x_1^{\lambda_2/\lambda_1}(1 + o(1)), k \neq 0$.*

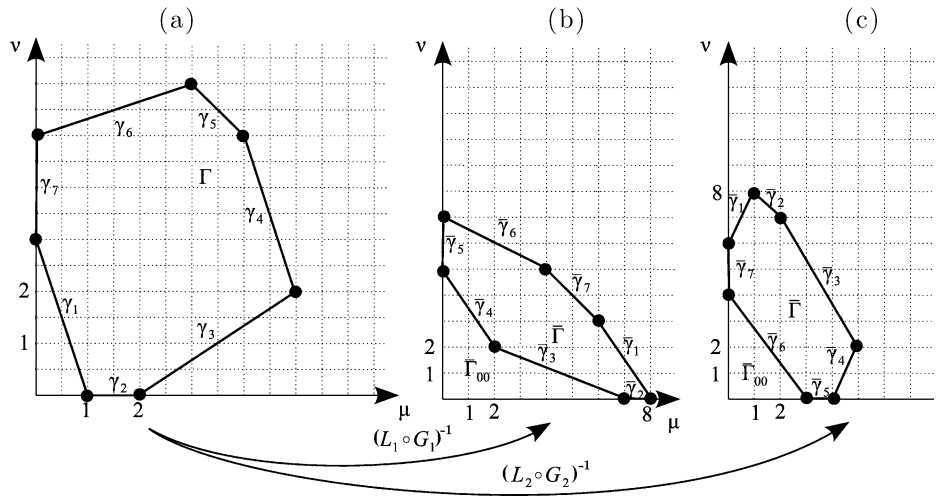

FIGURE 3. (a) Newton polygon Γ of the vector field $\Omega_\Gamma(x)$ which is presented and described in Figure 1(a), polygons $\widehat{\Gamma}_1 = (\gamma_3 \cup \gamma_4)$, $\widehat{\Gamma}_2 = (\gamma_6 \cup \gamma_7)$, edge $\gamma_5 \in \Gamma_{\infty\infty}$, $\alpha(\gamma_5) = 1$; (b) Newton polygon $\overline{\Gamma}$ of the vector field $\overline{\Omega}_\Gamma(z) \equiv \Omega_{\overline{\Gamma}}(z)$ in the Poincaré coordinates ($z_1 = 1/x_1, z_2 = x_2/x_1$), polygon $\widehat{\Gamma}_1 \in \Gamma \to (\overline{\gamma}_3 \cup \overline{\gamma}_4) \equiv \overline{\Gamma}_{00} \in \overline{\Gamma}$, $\gamma_5 \to \gamma^\mu \in \overline{\Gamma}$; (c) Newton polygon $\overline{\Gamma}$ of the vector field $\overline{\Omega}_\Gamma(u) \equiv \Omega_{\overline{\Gamma}}(u)$ in the Poincaré coordinates ($u_1 = x_2/x_1, u_2 = 1/x_2$), polygon $\widehat{\Gamma}_2 \in \Gamma \to (\overline{\gamma}_6 \cup \overline{\gamma}_7) \equiv \overline{\Gamma}_{00} \in \overline{\Gamma}$, edge $\gamma_5 \to \gamma^\nu \in \overline{\Gamma}$.

Further, we consider systems (1.1), (1.2) in the form of (2.1):

$$\begin{aligned} x_1' &= x_1 P_1(x_1, x_2) \equiv x_1(P_1^\gamma(x_1, x_2) + \phi_1(x_1, x_2)), \\ x_2' &= x_2 P_2(x_1, x_2) \equiv x_2(P_2^\gamma(x_1, x_2) + \phi_2(x_1, x_2)), \end{aligned} \tag{4.1}$$

where $P_1^\gamma(x_1, x_2), P_2^\gamma(x_1, x_2), \phi_1(x_1, x_2), \phi_2(x_1, x_2)$ present specifying functions. Let $\gamma \equiv \gamma^{(1)} \in \Gamma_{00}$.

4.2. **Asymptotics of the orbits tending to an equilibrium with one zero coordinate.**

Proposition 4.1. *Let the Γ-non-degenerate vector field Ω_Γ given by* (4.1), (1.2), *have an isolated singular point $A_2(x_1 = 0, x_2 = a_2 \neq 0)$ and/or $A_1(x_1 = a_1 \neq 0, x_2 = 0)$. Then orbits tending to A_2 (A_1) with $x_1 \to 0$ ($x_2 \to 0$) have power asymptotics* (2.5) *or trivial $x_1 = 0$, $x_2 = a_2$ and/or $x_1 = a_1, x_2 = 0$.*

Proof. Let us consider the first case: $P_2(0, a_2) = 0$, $P_1(0, a_2) \neq 0$. According to Proposition 3.3, $\gamma \equiv \gamma^\mu$, $\alpha(\gamma) = 0$, in (4.1) functions $P_1^\gamma(x_1, x_2) = \sum_\nu p_{\mu\nu} x_2^\nu$, $P_2^\gamma(x_1, x_2) = \sum_\nu q_{\mu\nu} x_2^\nu$ (P_1^γ, P_2^γ are of the form (2.3)), and $\phi_1(0, x_2) = \phi_2(0, x_2) = 0$ (see, for example, edge γ_7 in Figure 3(a)). Due to conditions of non-degeneracy (ii), (iii) (see Definition (6)), a_2 is a simple root of function $P_2^{\gamma^\mu}(0, x_2)$ and $P_1^{\gamma^\mu}(1, a_2) \neq 0$. Supposing that system (4.1) is continuously differential in both variables, we find eigenvalues at $(0, a_2)$: $\lambda_1 = P_1^{\gamma^\mu}(0, a_2)$, $\lambda_2 = a_2 P_{2x_2}^{\gamma^\mu}(0, a_2)$, $\lambda_1 \lambda_2 \neq 0$. According to

Lemma 4.1 the vector field defined by (4.1) has orbits of the form (2.5): $x_2 - a_2 = kx_1^{\lambda_2/\lambda_1}(1+o(1))$, where k is an arbitrary constant if $\lambda_1\lambda_2 > 0$ and $k = 0$ if $\lambda_1\lambda_2 < 0$.

The second case can be considered in an analogous way. Proposition 4.1 is proven.

4.3. **Asymptotics to O-orbits.** The following theorem has been formulated and proven for polynomial vector fields in [8] (see also [7]). The case Γ_{00} consisting of one standard edge was considered in [10].

Theorem 4.1. (i) *Asymptotics of O-orbits of the Γ-non-degenerate vector field $\Omega(x)$ coincide with the asymptotics of O-orbits of $\Omega_{\Gamma_{00}}(x)$ and have the form* (2.5): $x_2 = kx_1^{\rho}(1+o(1))$, *where the exponent $\rho > 0$ equals the index α of one of the edges $\gamma^{(1)} \in \Gamma_{00}$ or equals the index β of one of the vertices $\gamma^{(0)} \in \Gamma_{00}$.*

(ii) *$\rho = \alpha$ if function $F^{\alpha}(\xi_2) = -\alpha P_1^{\alpha}(1,\xi_2) + P_2^{\alpha}(1,\xi_2)$ has a root $\xi_2 = k \neq 0$, which serves the coefficient of* (2.5); *$\rho = \beta$ and coefficient $k \neq 0$ is any number if $\beta \in (\hat{\alpha}, \hat{\hat{\alpha}})$, where $0 \leq \hat{\alpha} < \hat{\hat{\alpha}} \leq \infty$ are indexes of edges that are adjacent to this vertex, $\hat{\alpha} = 0$ if vertex $\gamma^{(0)}(0,\nu) \in \Gamma_{00}$ and/or $\hat{\hat{\alpha}} = \infty$ if vertex $\gamma^{(0)}(\mu,0) \in \Gamma_{00}$.*

Below we prove Theorem 4.1 under the assumption that point O is the isp of the Γ-non-degenerate continuously differentiable vector field $\Omega(x)$ defined by (1.1), (1.2).

Proof of Theorem 4.1. (i) Let ND Γ_{00} consist of *one* vertex $\gamma^{(0)}(\mu_0,\nu_0)$. Because $O(0,0)$ is an isolated point, $\mu = \mu_0 = 0$, $\nu = \nu_0 = 0$. Therefore system (4.1) is of the form (4.1), (2.2): $P_1^{\gamma}(x_1,x_2) = p_{00} = \lambda_1, P_2^{\gamma}(x_1,x_2) = q_{00} = \lambda_2$ such that $p_{00}q_{00} \neq 0$, $\phi_1(0,x_2) = \phi_2(x_1,0) = 0$. According to Lemma 4.1 the system has O-orbits in the form (2.5) with $\rho \equiv \beta_0 = q_{00}/p_{00}$, where the coefficient k is an arbitrary constant if $p_{00}q_{00} > 0$ (i.e. O is a node) and $k = 0$ if $p_{00}q_{00} < 0$ (O is a saddle whose separatrices are $x_1 = 0$, $x_2 \neq 0$ and $x_2 = 0, x_1 \neq 0$).

(ii) Let ND Γ_{00} contain the unique edge $\gamma = \gamma^{(1)}$ (having index α) which is bounded by vertices $\gamma_1^{(0)}(\mu_1,\nu_1)$, $\gamma_2^{(0)}(\mu_2,\nu_2)$, $\mu_1 = 0 < \mu_2, \nu_1 > \nu_2 = 0$, whose indexes are β_1, β_2. Then system (4.1) is of the form (2.1), (2.4), where $P_1^{\alpha}(x_1,x_2) = \sum_{\mu+\alpha\nu=\sigma} p_{\mu\nu}x_1^{\mu}x_2^{\nu}$, $P_2^{\alpha}(x_1,x_2) = \sum_{\mu+\alpha\nu=\sigma} q_{\mu\nu}x_1^{\mu}x_2^{\nu}$.

Due to Proposition 3.2, change of variables (3.2): $x_1 = \xi_1, x_2 = \xi_1^{\alpha}\xi_2$, supplemented by (3.1) with $(l_1,l_2) = (\mu_2,0)$, converts system (4.1) to the system with

$$\overline{P}_1(\xi_1,\xi_2) = P_1^{\alpha}(1,\xi_2) + \overline{\phi}_1(\xi_1,\xi_2), \quad \overline{P}_2(\xi_1,\xi_2) = F^{\alpha}(\xi_2) + \overline{\phi}_2(\xi_1,\xi_2),$$
$$\overline{\phi}_1(0,\xi_2) = \overline{\phi}_2(0,\xi_2) = 0, \quad F^{\alpha}(\xi_2) = -\alpha P_1^{\alpha}(1,\xi_2) + P_2^{\alpha}(1,\xi_2).$$

With this change the edge $\gamma^{(1)} \in \Gamma_{00} \subset \Gamma$ is transformed into the special edge $\overline{\gamma}^{\mu} \in \overline{\Gamma}$ (see Figure 2(a),(b)), vertex $\gamma_2^{(0)}(\mu_2,0) \to \overline{\gamma}_2^{(0)}(0,0) \in \overline{\Gamma}_{00}$, where $\overline{\gamma}_{00}^{(0)}$ is the unique vertex of $\overline{\Gamma}_{00}$, and $\overline{\beta} = \beta_2 - \alpha$ is the index of $\overline{\gamma}_{00}^{(0)}$. According to Proposition 4.1 and conditions of non-degeneracy (ii), (iii) (see Definition (6)), if $F^{\alpha}(\xi_2)$ has root $\xi_2 = a_2 \neq 0$, then the system (3.3) has an orbit with asymptotics $\xi_2 - a_2 = o(1)$, $o(1) \to 0$ *for* $\xi_1 \to 0$. In such a case system (4.1) has an orbit with power asymptotics $x_2 = a_2x_1^{\alpha}(1+o(1))$. The vertex $\overline{\gamma}_{00}^{(0)} \in \overline{\Gamma}_{00}$ has index $\overline{\beta} \equiv \overline{q}_{00}/\overline{p}_{00} \neq 0, \infty$, where $\overline{p}_{00} = P_1^{\alpha}(1,0) = p_{\mu_2 0}, \overline{q}_{00} = F^{\alpha}(0) = -\alpha p_{\mu_2 0} + q_{\mu_2 0}$ (see Figure 2(a),(d)). Therefore, we have the case which has been discussed in part (i): system (3.3) has an orbit with asymptotics $\xi_2 = k\xi_1^{\overline{\beta}}(1+o(1))$ or trivial $\xi_2 = 0$. In these variables (x_1,x_2) these asymptotics obtain the form $x_2 = kx_1^{\beta}(1+o(1))$.

The second change of variables (3.2): $x_1 = \xi_1 \xi_1^{1/\alpha}$, $x_2 = \xi_2$, supplemented by (3.1) with $(l_1, l_2) = (0, \nu_1)$, lead to the transformation of edge $\gamma^{(1)}$ to the special edge $\overline{\gamma}^{\nu}$ (see Figure 2(a),(c),(d)) such that vertex $\gamma_1^{(0)}(0, \nu_1) \to \overline{\gamma}_{00}^{(0)} \in \overline{\Gamma}_{00}$, where $\overline{\gamma}_{00}^{(0)}$ is the unique vertex of $\overline{\Gamma}_{00}$. Due to conditions of non-degeneracy: $\beta_1 \neq \alpha, \infty$, index of $\overline{\gamma}_{00}^{(0)} : \overline{\beta} = \beta_1 - \alpha \neq 0, \infty$. Repeating the above arguments we finish the proof of statements considering the case.

Actually, we have proven that O-orbits of the Γ_{00}-non-degenerated vector field Ω_Γ can have *only* power or trivial asymptotics and described exponents and coefficients of these asymptotics in the form given in Theorem 4.1.

(iii) Note now that according to Lemma 3.1 and Proposition 3.2 any system (4.1) whose ND Γ_{00} contains $N > 1$ standard edges by changes of variables (3.2) and (3.1) can be reduced to the system whose ND $\overline{\Gamma}_{00}$ contains the only standard edge, and the "new" system will be $\overline{\Gamma}_{00}$-non-degenerate if the initial one was Γ_{00}-non-degenerate. Thus, all asymptotics of O-orbits are the same as described in Theorem 4.1. Theorem 4.1 is proven.

(iv) Taking into consideration the statements of Theorem 3.1, we can state that orbits tending to singular points of vector fields defined by systems (1.1), (1.4) and (1.6) have power asymptotics as well. For example, let an orbit of system (1.4) be of the form $z_2 = k z_1^{\overline{\alpha}}(1 + o(1))$; i.e., $\overline{\alpha} = \alpha(\overline{\gamma})$, $\overline{\gamma} \in \overline{\Gamma}_{00}$ and $k \neq 0$ is a root of the function $F^{\overline{\alpha}}(\omega)$. Due to (1.5) and (3.4) $x_2 = k x_1^{1-\overline{\alpha}}(1 + o(1)) = k x_1^{\alpha}(1 + o(1))$, where $\alpha = \alpha(\gamma)$ and γ is the preimage of edge $\overline{\gamma}$ under the transformation with matrix E_1 (see (3.3)). In an analogous way one can verify other cases and show that exponents and coefficients of the mentioned asymptotics are expressed in terms of indexes of vertices and edges of the Newton polygon. This finishes the proof of Theorem A.

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