PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 8, August 2014, Pages 2837–2848
S 0002-9939(2014)11995-2
Article electronically published on May 7, 2014

EULERIAN RELATIVE EQUILIBRIA OF THE CURVED 3-BODY PROBLEMS IN $\mathbf{S}^2$

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(Communicated by Walter Craig)

ABSTRACT. We consider the gravitational motion of n point particles with masses $m_1, m_2, \ldots, m_n > 0$ on surfaces of constant Gaussian curvature. Based on the work of Diacu and his co-authors, we derive the law of universal gravitation in spaces of constant curvature. Using the results, we examine all possible 3-body configurations that can generate geodesic relative equilibria. We prove the existence of all acute triangle Eulerian relative equilibria and get a necessary and sufficient condition for the existence of obtuse triangle Eulerian relative equilibria. We also show that any three positive masses can generate Eulerian relative equilibria.

1. INTRODUCTION

The n-body problem in spaces of constant curvature has its roots in the 2-body case, independently proposed by Bolyai, [2], and Lobachevsky, [10]. Their geometric ideas were expressed analytically by Schering, [12], who showed that the natural extension of the Newtonian potential is given by the cotangent of the distance, as we will explain later. Like the Kepler problem in the Euclidean space, the potential is a harmonic function in 3-dimensional space and generates a central field in which bounded orbits are closed.

Kozlov and Harin, [9], as well as Cariñena, Rañada, and Santander, [3], revisited the idea of the cotangent potential, wrote the Lagrangian function in generalized coordinates, and solved the Kepler problem with the methods of Lagrangian mechanics. Diacu, Pérez-Chavela, and Santoprete, [5], viewed the problem as a mechanical system with holonomic constraints, wrote the Lagrangian function in Cartesian coordinates, obtained the equations of motion using constrained Lagrangian dynamics, and studied relative equilibria in $\mathbf{S}^2$ and $\mathbf{H}^2$. Basing our results on their work, we analyze the attractive force between bodies in a geometrical way and obtain the law of universal gravitation in spaces of constant curvature κ, which leads to Newton's equations in Euclidean space when $\kappa \to 0$.

Using the formula of the attractive force, we study geodesic relative equilibria of the 3-body problem in $\mathbf{S}^2$ and especially focus on Eulerian relative equilibria. Diacu, Pérez-Charela, and Manuele Santoprete, [5], as well as Pérez-Chavela and Reyes-Victoria, [11], found solutions of this kind. They mainly discussed whether

Received by the editors July 23, 2012.
2010 *Mathematics Subject Classification.* Primary 70-XX; Secondary 70F07, 70F15.
Key words and phrases. 3-body problem, constant curvature spaces, law of universal gravitation, Eulerian relative equilibria, acute triangle configuration, obtuse triangle configuration, equilateral triangle configuration, isosceles triangle configuration.

or not three given masses can generate an Eulerian relative equilibrium. Instead, we examine all possible configurations that can generate Eulerian relative equilibria. We prove the existence of acute triangle Eulerian relative equilibria and obtain a necessary and sufficient condition for the existence of Eulerian relative equilibria given by obtuse triangles. We also show that any three positive masses can generate Eulerian relative equilibria.

2. Equations of motion

Let us first introduce some notation and formulas.

1. To set the 2-dimensional manifolds of constant curvature κ in which the bodies move, we endow $\mathbb{R}^3$ with two structures: $\mathbb{E}^3$, i.e. $(\mathbb{R}^3,\ ds^2 = dx_1^2 + dx_2^2 + dx_3^2)$, and $\mathbb{M}^3$, i.e., $(\mathbb{R}^3,\ ds^2 = dx_1^2 + dx_2^2 - dx_3^2)$. These metrics lead to the "inner"products: $\mathbf{a} \odot \mathbf{b} := a_{x_1}b_{x_1} + a_{x_2}b_{x_2} + \sigma a_{x_3}b_{x_3}$, where $\sigma = 1$, if $\kappa > 0$, and $\sigma = -1$, if $\kappa < 0$. We define two kinds of spheres.

$$\begin{cases} \mathbf{S}_\kappa^2 := \{\mathbf{a} \in \mathbb{R}^3 \mid \mathbf{a} \odot \mathbf{a} = \kappa^{-1},\ \kappa > 0\}, \\ \mathbf{H}_\kappa^2 := \{\mathbf{a} \in \mathbb{R}^3 \mid \mathbf{a} \odot \mathbf{a} = \kappa^{-1}, a_{x_3} > 0,\ \kappa < 0\}. \end{cases}$$

2. Following [3], let us consider the following *trigonometric κ-functions*, which unify elliptic and hyperbolic trigonometry. We define the κ-sine, sn_κ, as

$$sn_\kappa(x) := \begin{cases} \kappa^{-1/2} \sin \kappa^{1/2} x & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ (-\kappa)^{-1/2} \sinh(-\kappa)^{1/2} x & \text{if } \kappa < 0, \end{cases}$$

the κ-cosine, csn_κ, as

$$csn_\kappa(x) := \begin{cases} \cos \kappa^{1/2} x & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(-\kappa)^{1/2} x & \text{if } \kappa < 0, \end{cases}$$

as well as the κ-tangent, tn_κ, and κ-cotangent, ctn_κ, as

$$tn_\kappa(x) := \frac{sn_\kappa(x)}{csn_\kappa(x)} \quad \text{and} \quad ctn_\kappa(x) := \frac{csn_\kappa(x)}{sn_\kappa(x)},$$

respectively. The entire trigonometry can be rewritten in this unified context, but the only identity we will further need is the fundamental formula

$$ctn_\kappa'(x) = \frac{-1}{sn_\kappa^2(x)}.$$

3. The position of the i-th mass m_i on the surface of curvature κ is $\mathbf{q}_i$, and the distances between masses m_i and m_j are $d_\kappa(\mathbf{q}_i, \mathbf{q}_j)$, $i,\ j = 1, \dots, n$.

The curvature of $\mathbf{S}_\kappa^2$ and $\mathbf{H}_\kappa^2$ is κ, so the n-body problem in spaces of constant curvature κ can now be formulated as follows.

Consider a Lagrangian system in $\mathbb{R}^3$ given by

$$L_\kappa(\mathbf{q}, \dot{\mathbf{q}}) = T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) + U_\kappa(\mathbf{q})$$

where

$$T_\kappa(\mathbf{q},\dot{\mathbf{q}}) := \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)(\kappa\mathbf{q}_i \odot \mathbf{q}_i),$$

$$U_\kappa(\mathbf{q}) := \sum_{i,j=1,j\neq i}^{n} m_i m_j ctn_\kappa(d_\kappa(\mathbf{q}_i,\mathbf{q}_j)),$$

and the holonomic constraints are $\mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}$, $i = 1,\ldots,n$.

Using the method of Lagrange multipliers, we obtain the Euler-Lagrange equations:

$$\begin{cases} m_i\ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i}U_\kappa(\mathbf{q}) - m_i\kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i,\ \mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1},\ i = 1,\ldots,n,\ \kappa \neq 0, \\ \nabla_{\mathbf{q}_i}U_\kappa(\mathbf{q}) = \sum_{j=1,j\neq i}^{n} \frac{m_i m_j(\sigma\kappa)^{3/2}[\mathbf{q}_j - \kappa(\mathbf{q}_i\odot\mathbf{q}_j)\mathbf{q}_i]}{[\sigma-\sigma(\kappa\mathbf{q}_i\odot\mathbf{q}_j)^2]^{3/2}}. \end{cases}$$

Diacu pointed out that this formalism allows a straightforward generalization to the n-body problem in $\mathbf{S}^\mu_\kappa$ and $\mathbf{H}^\mu_\kappa$ for any integer $\mu \geq 1$. The equation of motion in the μ-dimensional space of constant curvature has the same form.

3. The generalized law of universal gravitation

Notice that two kinds of forces act on the mass m_i: $-m_i\kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i$, due to the constraints, and the attractive force arising from other masses m_j:

$$\mathbf{F}_{ji} := \nabla_{\mathbf{q}_i} m_i m_j ctn_\kappa(d_\kappa(\mathbf{q}_i,\mathbf{q}_j)) = \frac{m_i m_j(\sigma\kappa)^{3/2}[\mathbf{q}_j - \kappa(\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\kappa\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}},\ j \neq i.$$

Although the attractive force looks complicated at first glance, some geometric considerations make it look simpler.

We use the generic notation $\mathbf{M}^2_\kappa$ for $\mathbf{S}^2_\kappa$ if $\kappa > 0$ and for $\mathbf{H}^2_\kappa$ if $\kappa < 0$.

Lemma 3.1. *Consider the n-body problem in $\mathbf{M}^2_\kappa$, and assume $\mathbf{q}_i \neq \mathbf{q}_j$ for $\kappa > 0$, then the magnitude of $\mathbf{F}_{ji}$ is $\frac{m_i m_j}{sn^2_\kappa d_\kappa(\mathbf{q}_i,\mathbf{q}_j)}$, and the direction of $\mathbf{F}_{ji}$ is the same as the direction of $\dot{\varphi}(0)$, where φ is the minimal geodesic connecting $\mathbf{q}_i$ and $\mathbf{q}_j$ on $\mathbf{M}^2_\kappa$, with $\varphi(0) = \mathbf{q}_i$.*

Before we prove the above lemma, let us recall some geometrical facts. We define a distance function on $\mathbf{M}^2_\kappa$,

$$r_\kappa(\mathbf{q}) = d_\kappa(\mathbf{q}, A),$$

where $\mathbf{q}$, A are in $\mathbf{M}^2_\kappa$, $\mathbf{q} \neq A$, and $\mathbf{q} \neq -A$, if $\kappa > 0$.

Proposition 3.2. *Consider the gradient field of $r_\kappa(\mathbf{q})$ on $\mathbf{M}^2_\kappa$. Then the magnitude of $\nabla r_\kappa(\mathbf{q})$ is 1, and the direction of $\nabla r_\kappa(\mathbf{q})$ is the same as the direction of $-\dot{\varphi}(0)$, where φ is the minimal geodesic connecting $\mathbf{q}$ and A, with $\varphi(0) = \mathbf{q}$.*

Proof. According to [7], we can choose τ in the linear isometry group of $\mathbf{M}^2_\kappa$ such that $\tau(A) = (0,0,(\sigma\kappa)^{-1/2})$ and $\tau(\varphi)$ is the intersection of the $x - z$ plane and $\mathbf{M}^2_\kappa$. Hence we can assume that A is $(0,0,(\sigma\kappa)^{-1/2})$ and $\mathbf{q} = (x,0,z)$. Recalling $csn^2_\kappa(x) + \kappa sn^2_\kappa(x) = 1$, we can parameterize $\mathbf{M}^2_\kappa(x^2 + y^2 + \sigma z^2 = \kappa^{-1})$ as

$$(x,y,z) = (sn_\kappa(r)\cos\phi, sn_\kappa(r)\sin\phi, (\sigma\kappa)^{-\frac{1}{2}}csn_\kappa(r)).$$

Then

$$ds^2 = dx^2 + dy^2 + dx^2 = dr^2 + sn^2_\kappa(r)d\phi^2.$$

Since $\mathbf{q}$ is $(sn_\kappa(r), 0, (\sigma\kappa)^{-\frac{1}{2}} csn_\kappa(r))$, we find $r_\kappa(\mathbf{q}) = r$. Then

$$\nabla r_\kappa(\mathbf{q}) = \partial_r,$$
$$\langle \nabla r_\kappa(\mathbf{q}), \nabla r_\kappa(\mathbf{q})\rangle = \langle \partial_r, \partial_r\rangle = 1. \qquad \square$$

Proof of Lemma 3.1.

$$\begin{aligned}\mathbf{F}_{ji} &= \nabla m_i m_j ctn_\kappa(r(\mathbf{q}))|_{\mathbf{q}=\mathbf{q}_i}\\ &= m_i m_j \frac{dctn_\kappa(r(\mathbf{q}))}{dr} \nabla r(\mathbf{q})|_{\mathbf{q}=\mathbf{q}_i}\\ &= m_i m_j \frac{1}{sn_\kappa^2 r(\mathbf{q})}(-\nabla r(\mathbf{q}))|_{\mathbf{q}=\mathbf{q}_i}\\ &= \frac{m_i m_j}{sn_\kappa^2 d_\kappa(\mathbf{q}_i, \mathbf{q}_j)}(-\nabla r(\mathbf{q})|_{\mathbf{q}=\mathbf{q}_i}).\end{aligned}$$

By Proposition 3.2, this completes the proof. $\square$

The above computation does not depend on the dimension of the space, so the lemma holds in $\mathbf{S}^\mu_\kappa$ and $\mathbf{H}^\mu_\kappa$ for any integer $\mu \geq 1$. For $\kappa \to 0$, we have

$$\frac{m_i m_j}{sn_\kappa^2 d_\kappa(\mathbf{q}_i, \mathbf{q}_j)} \to \frac{m_i m_j}{d_0^2(\mathbf{q}_i, \mathbf{q}_j)},$$
$$-m_i\kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i \to 0;$$

i.e., we recover Newton's law of universal gravitation in Euclidean space.

4. Geodesic relative equilibria for 3-body problems in $\mathbf{S}^2$

As pointed out in [4], by a change of coordinates and a rescaling of time, we can eliminate the parameter κ, up to its sign, from the equations of motion, such that the problem reduces to $\mathbf{S}^2$ and $\mathbf{H}^2$. In this section, we apply Lemma 3.1 to study the geodesic relative equilibria for 3-body problems in $\mathbf{S}^2$, especially the Eulerian relative equilibria.

Taking $\kappa = 1,\ \sigma = 1$, we get the equation in $\mathbf{S}^2$:

$$(*) \quad \begin{cases} m_i\ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_1(\mathbf{q}) - m_i(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,\ \mathbf{q}_i \cdot \mathbf{q}_i = 1,\ \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0,\ i = 1, \ldots, n,\\ \nabla_{\mathbf{q}_i} U_1(\mathbf{q}) = \sum_{j=1, j\neq i}^n \frac{m_i m_j[\mathbf{q}_j - (\mathbf{q}_i\cdot\mathbf{q}_j)\mathbf{q}_i]}{[1-(\mathbf{q}_i\cdot\mathbf{q}_j)^2]^{3/2}}. \end{cases}$$

A *relative equilibrium* is a solution of $(*)$ which is invariant to $SO(3)$ rotations. Since every element of the group $SO(3)$ can be written, in an orthonormal basis, as a rotation about the z-axis, following [5], we define elliptic relative equilibria as follows.

Definition 4.1. An *elliptic relative equilibrium* in $\mathbf{S}^2$ is a solution of the form $\mathbf{q}_i = (x_i, y_i, z_i),\ i = 1, \ldots, n$, of equations $(*)$ with $x_i = r_i\cos(\omega t + \alpha_i),\ y_i = r_i \sin(\omega t + \alpha_i),\ z_i = constant$, where ω, α_i, and r_i with $0 \leq r_i = (1 - z_i^2)^{1/2} \leq 1,\ i = 1, \ldots, n$, are constants.

If the bodies lie on a geodesic curve, the corresponding solution is called a *geodesic relative equilibrium*. If the bodies lie on a rotating geodesic curve, the corresponding solution is called an *Eulerian relative equilibrium*. Diacu, Pérez-Charela, and Santoprete, [5], found one kind of Eulerian relative equilibria, namely when the bodies lie on a meridian rotating around the z-axis. They also found relative equilibria that rotate on the equator $z = 0$. We will further show that these are the only possible cases.

Theorem 4.2. *Consider n bodies lying initially on a geodesic of γ of $\mathbf{S}^2$. If this configuration can generate a relative equilibrium, then γ is either the equator or a meridian rotating around the z-axis.*

Proof. The i-th body's circular motion requires that the total force exerted on m_i lies in the plane spanned by the z-axis and $\mathbf{q}_i$ and is parallel to the plane $z = 0$. The force of constraint lies in the plane spanned by the z-axis and $\mathbf{q}_i$, and is not parallel to the plane $z = 0$, except when γ is the equator. Hence the total attractive force exerted on m_i lies in the plane spanned by the z-axis and $\mathbf{q}_i$ if γ is not the equator, which, by Lemma 3.1, implies that γ is a meridian. □

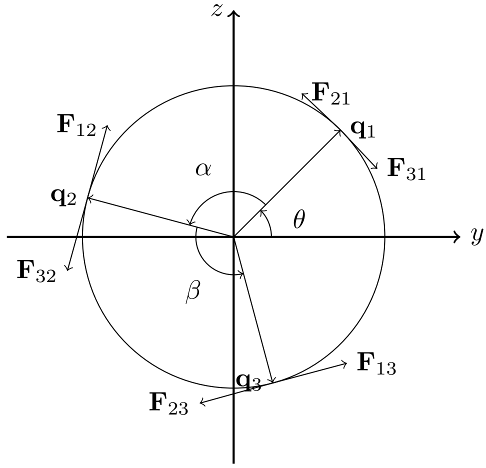
Hence, in the geodesic 3-body problem, we only have to study relative equilibria moving on the equator and Eulerian relative equilibria lying on a rotating meridian. Different from other authors, [5], [11], our aim is to determine whether or not a certain configuration exists and to get the detailed positions in some special cases. Since the right triangle leads to singularities, we divide our analysis into four cases, according to whether the bodies move on the equator or lie on a rotating meridian, and whether they form an acute or an obtuse triangle. The point is that the attractive force can be written in terms of the positions of the bodies in these cases. We will see that the existence of these relative equilibria reduces to the existence of the positive solution of some linear equations. In the following, we assume $\omega \neq 0$.

Case I. **Acute Triangle Inscribed in a Meridian**.

Without loss of generality, we set

$$\begin{aligned}
&\mathbf{q}_1(0) = (0, \cos\theta, \sin\theta), && \dot{\mathbf{q}}_1(0) = (-\omega\cos\theta, 0, 0),\\
&\mathbf{q}_2(0) = (0, \cos(\theta+\alpha), \sin(\theta+\alpha), && \dot{\mathbf{q}}_2(0) = (-\omega\cos(\theta+\alpha), 0, 0),\\
&\mathbf{q}_3(0) = (0, \cos(\theta+\alpha+\beta), \sin(\theta+\alpha+\beta)), && \dot{\mathbf{q}}_3(0) = (-\omega\cos(\theta+\alpha+\beta), 0, 0),
\end{aligned}$$

with $0 \le \theta \le \frac{\pi}{2}$, $0 < \alpha < \pi$, $0 < \beta < \pi$, $\pi < \alpha + \beta < 2\pi$. We have $d_1(\mathbf{q}_1, \mathbf{q}_2) = \alpha$, $d_1(\mathbf{q}_2, \mathbf{q}_3) = \beta$, $d_1(\mathbf{q}_1, \mathbf{q}_3) = 2\pi - \alpha - \beta$.



Theorem 4.3. *Consider three bodies with the above initial conditions. The configuration can generate an Eulerian relative equilibrium if and only if the following equation holds:*

$$f(\alpha,\beta,\theta) := \sin 2\theta\frac{\sin^2\alpha}{\sin^2\beta} + \sin 2(\theta+\alpha)\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \sin 2(\theta+\alpha+\beta) = 0.$$

Proof. The symmetry implies that we only have to check whether or not the initial conditions provide the necessary force for the circular motion of each body. It's convenient to use complex coordinates on the y-z plane, so $\mathbf{q}_1 = e^{i\theta}$, $\mathbf{q}_2 = e^{i(\theta+\alpha)}$, $\mathbf{q}_3 = e^{i(\theta+\alpha+\beta)}$. By Lemma 3.1, we obtain

$$\begin{aligned}
\mathbf{F}_{21} &= ie^{i\theta}\frac{m_1m_2}{\sin^2\alpha}, & \mathbf{F}_{31} &= -ie^{i\theta}\frac{m_1m_3}{\sin^2(\alpha+\beta)},\\
\mathbf{F}_{12} &= -ie^{i(\theta+\alpha)}\frac{m_1m_2}{\sin^2\alpha}, & \mathbf{F}_{32} &= ie^{i(\theta+\alpha)}\frac{m_2m_3}{\sin^2\beta},\\
\mathbf{F}_{13} &= ie^{i(\theta+\alpha+\beta)}\frac{m_3m_1}{\sin^2(\alpha+\beta)}, & \mathbf{F}_{23} &= -ie^{i(\theta+\alpha+\beta)}\frac{m_3m_2}{\sin^2\beta}.
\end{aligned}$$

Assuming that the initial conditions can provide the force for the circular motion of each body, we obtain

$$\begin{cases}
-m_1\omega^2\cos\theta = e^{i\theta}(i(\frac{m_1m_2}{\sin^2\alpha} - \frac{m_1m_3}{\sin^2(\alpha+\beta)}) - m_1\omega^2\cos^2\theta),\\
-m_2\omega^2\cos(\theta+\alpha) = e^{i(\theta+\alpha)}(i(-\frac{m_1m_2}{\sin^2\alpha} + \frac{m_2m_3}{\sin^2\beta}) - m_2\omega^2\cos^2(\theta+\alpha)),\\
-m_3\omega^2\cos(\theta+\alpha+\beta) = e^{i(\theta+\alpha+\beta)}(i(\frac{m_3m_1}{\sin^2(\alpha+\beta)} - \frac{m_3m_2}{\sin^2\beta}) - m_3\omega^2\cos^2(\theta+\alpha+\beta)).
\end{cases}$$

By straightforward computation, we obtain

$$\begin{cases}
1 = e^{i\theta}(\cos\theta - i\frac{\frac{m_2}{\sin^2\alpha} - \frac{m_3}{\sin^2(\alpha+\beta)}}{\omega^2\cos\theta}),\\
1 = e^{i(\theta+\alpha)}(\cos(\theta+\alpha) - i\frac{-\frac{m_1}{\sin^2\alpha} + \frac{m_3}{\sin^2\beta}}{\omega^2\cos(\theta+\alpha)}),\\
1 = e^{i(\theta+\alpha+\beta)}(\cos(\theta+\alpha+\beta) - i\frac{\frac{m_1}{\sin^2(\alpha+\beta)} - \frac{m_2}{\sin^2\beta}}{\omega^2\cos(\theta+\alpha+\beta)}).
\end{cases}$$

We get

$$\begin{cases}
\frac{m_2}{\sin^2\alpha} - \frac{m_3}{\sin^2(\alpha+\beta)} = \frac{\omega^2\sin 2\theta}{2}, & (1)\\
-\frac{m_1}{\sin^2\alpha} + \frac{m_3}{\sin^2\beta} = \frac{\omega^2\sin 2(\theta+\alpha)}{2}, & (2)\\
\frac{m_1}{\sin^2(\alpha+\beta)} - \frac{m_2}{\sin^2\beta} = \frac{\omega^2\sin 2(\theta+\alpha+\beta)}{2}. & (3)
\end{cases}$$

This is a non-homogeneous linear system,

$$\begin{bmatrix} 0 & \frac{1}{\sin^2\alpha} & -\frac{1}{\sin^2(\alpha+\beta)}\\ -\frac{1}{\sin^2\alpha} & 0 & \frac{1}{\sin^2\beta}\\ \frac{1}{\sin^2(\alpha+\beta)} & -\frac{1}{\sin^2\beta} & 0 \end{bmatrix}\begin{bmatrix} m_1\\ m_2\\ m_3\end{bmatrix} = \begin{bmatrix}\frac{\omega^2\sin 2\theta}{2}\\ \frac{\omega^2\sin 2(\theta+\alpha)}{2}\\ \frac{\omega^2\sin 2(\theta+\alpha+\beta)}{2}\end{bmatrix}.$$

We apply the row elementary operation to the augmented matrix,

$$\begin{bmatrix} 0 & \frac{1}{\sin^2\alpha} & -\frac{1}{\sin^2(\alpha+\beta)} & \frac{\omega^2\sin 2\theta}{2} \\ -\frac{1}{\sin^2\alpha} & 0 & \frac{1}{\sin^2\beta} & \frac{\omega^2\sin 2(\theta+\alpha)}{2} \\ \frac{1}{\sin^2(\alpha+\beta)} & -\frac{1}{\sin^2\beta} & 0 & \frac{\omega^2\sin 2(\theta+\alpha+\beta)}{2} \end{bmatrix}$$
$$\to \begin{bmatrix} 0 & \frac{1}{\sin^2\alpha} & -\frac{1}{\sin^2(\alpha+\beta)} & \frac{\omega^2\sin 2\theta}{2} \\ -\frac{1}{\sin^2\alpha} & 0 & \frac{1}{\sin^2\beta} & \frac{\omega^2\sin 2(\theta+\alpha)}{2} \\ 0 & 0 & 0 & \omega^2 f(\alpha,\beta,\theta)/2 \end{bmatrix},$$

which implies that the linear system admits a one-dimensional solution space if and only if

$$f(\alpha,\beta,\theta) = \sin 2\theta\frac{\sin^2\alpha}{\sin^2\beta} + \sin 2(\theta+\alpha)\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \sin 2(\theta+\alpha+\beta) = 0.$$

If $f(\alpha,\beta,\theta) = 0$, the solution space is

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} \frac{\lambda\sin^2(\alpha+\beta)}{\sin^2\alpha} - \frac{\omega^2\sin 2\theta\sin^2(\alpha+\beta)}{2} \\ \lambda \\ \frac{\lambda\sin^2(\alpha+\beta)}{\sin^2\beta} + \frac{\omega^2\sin 2(\theta+\alpha+\beta)\sin^2(\alpha+\beta)}{2} \end{bmatrix}, \quad \lambda \in \mathbb{R},$$

which admits positive solutions. This completes the proof. □

We can now prove the existence of an Eulerian relative equilibrium for any acute triangle configuration.

Corollary 4.4. *For any acute triangle lying on a rotating meridian of* $\mathbf{S}^2$*, there exist infinitely many values for the masses* $m_1,\ m_2,\ m_3 > 0$ *such that the bodies can be placed at the vertices of the triangle that generates an Eulerian relative equilibrium.*

Proof. It suffices to show the existence of θ in $[0, \pi/2]$ such that $f(\alpha,\beta,\theta) = 0$ for given α, β.

For $f = 0$, we get

$$\sin 2\theta\left[\frac{\sin^2\alpha}{\sin^2\beta} + \cos 2\alpha\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \cos 2(\alpha+\beta)\right]$$
$$= -\cos 2\theta\left[\sin 2(\alpha+\beta) + \sin 2\alpha\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)}\right],$$
$$\tan 2\theta = -\frac{\sin 2(\alpha+\beta) + \sin 2\alpha\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)}}{\frac{\sin^2\alpha}{\sin^2\beta} + \cos 2\alpha\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \cos 2(\alpha+\beta)},$$

which admits a solution, since the range of $\tan 2\theta$ is $(-\infty, \infty)$. □

When the configuration is an acute isosceles triangle, i.e., $\alpha = 2\pi - \alpha - \beta$, we can get some detailed information about the configuration and the masses.

Corollary 4.5. *Consider three bodies with the above initial conditions, and assume that they form an isosceles triangle, i.e.,* $\alpha = 2\pi - \alpha - \beta$, $\alpha \neq 2\pi/3$. *The configuration can generate an Eulerian relative equilibrium if and only if* $\theta = 0$ *or* $\pi/2$. *In this case, we have* $m_2 = m_3$.

Proof. Let $\alpha = 2\pi - \alpha - \beta$; then $\sin 2(\alpha+\beta) = \cos 2\alpha$, $\cos 2(\alpha+\beta) = -\sin 2\alpha$, $\sin(\alpha+\beta) = -\sin\alpha$, $\sin\beta = -\sin 2\alpha$. Hence

$$\tan 2\theta = -\frac{-\sin 2\alpha + \sin 2\alpha}{\frac{1}{4\cos^2\alpha} + 2\cos 2\alpha}.$$

Since $\alpha \neq 2\pi/3$, so that the denominator $\frac{1}{4\cos^2\alpha} + 2\cos 2\alpha \neq 0$, we get $\tan 2\theta = 0$, which implies $\theta = 0$, or $\pi/2$. Recalling equation (1),

$$m_2 - m_3 = \frac{3\omega^2 \sin 2\theta}{8},$$

we have $m_2 = m_3$. □

When the configuration is an equilateral triangle, we can get an interesting result.

Corollary 4.6. *Consider the 3-body problem in $\mathbf{S}^2$. There exists an equilateral Eulerian relative equilibrium for any three positive masses with $m_2 \geq m_3$.*

Proof. Let $\alpha = \beta = \frac{2\pi}{3}$; we get an equilateral triangle. Since

$$f(\alpha,\beta,\theta) = \sin 2\theta + \sin 2(\theta + \frac{4\pi}{3}) + \sin 2(\theta + \frac{8\pi}{3}) = 0,$$

independent of θ, we get an equilateral Eulerian relative equilibrium for any θ. By equations (1) and (2),

$$m_2 - m_3 = \frac{3\omega^2 \sin 2\theta}{8},$$
$$m_3 - m_1 = \frac{3\omega^2 \sin 2(\theta + \frac{2\pi}{3})}{8},$$

we have $m_2 \geq m_3$. Actually, we are free to choose m_1, m_2, m_3 with $m_2 \geq m_3$.

First, we are free to choose $m_3 > 0$. Since $m_2 - m_3 = \frac{3\omega^2 \sin 2\theta}{8}$, we are free to choose $m_2 \geq m_3$ by choosing ω. Finally, since $m_1 = m_3 - \frac{\sin 2(\theta + \frac{2\pi}{3})}{\sin 2\theta}(m_2 - m_3)$ and the range of $\frac{\sin 2(\theta + \frac{2\pi}{3})}{\sin 2\theta}$ is $(-\infty, \infty)$, we are free to choose $m_1 > 0$ by choosing θ. □

Case II. **Acute Triangle Rotating on the Equator**.

According to Theorem 4 in [6], we have infinitely many fixed points for any acute triangle in the equator of $\mathbf{S}^2$. According to Proposition 1 in [5], a fixed point on a great circle of $\mathbf{S}^2$ corresponds to a relative equilibrium along the geodesic, so we have obtained Diacu's theorem.

Theorem 4.7 (Theorem 5 in [6])**.** *For any acute triangle inscribed in the equator of $\mathbf{S}^2$, there exist infinitely bodies of masses m_1, m_2, $m_3 > 0$ that can be placed at the vertices of the triangle such that they can generate a relative equilibrium.*

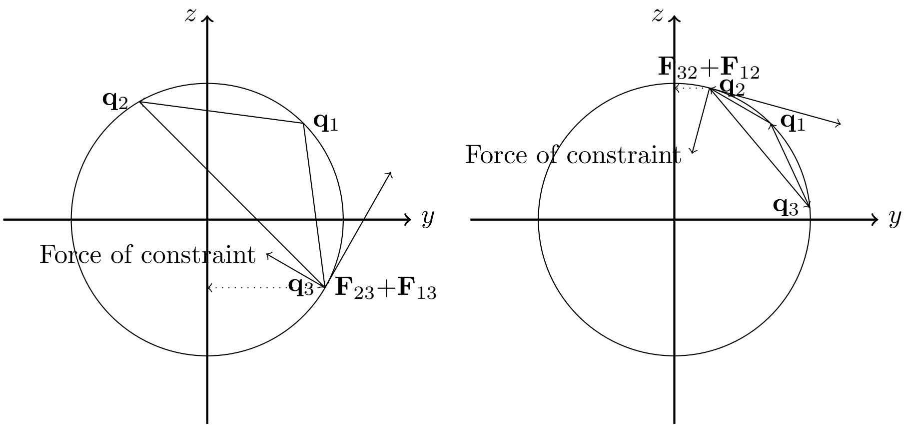
Case III. **Obtuse Triangle Rotating on the Equator**.

If the three bodies in the equator form an obtuse triangle, then they lie within a semicircle. By Lemma 3.1, the force exerted on two of these bodies can never be zero, hence they cannot generate a fixed point. According to Proposition 1 in [5], a fixed point on a great circle of $\mathbf{S}^2$ corresponds to a relative equilibrium along the geodesic, so we have obtained the following theorem.

Theorem 4.8. *For any obtuse triangle inscribed in the equator of* $\mathbf{S}^2$*, there exist no bodies of masses* $m_1,\ m_2,\ m_3 > 0$ *that can be placed at the vertices of the triangle such that they can generate a relative equilibrium.*

Case IV. **Obtuse Triangle Inscribed in a Meridian**.

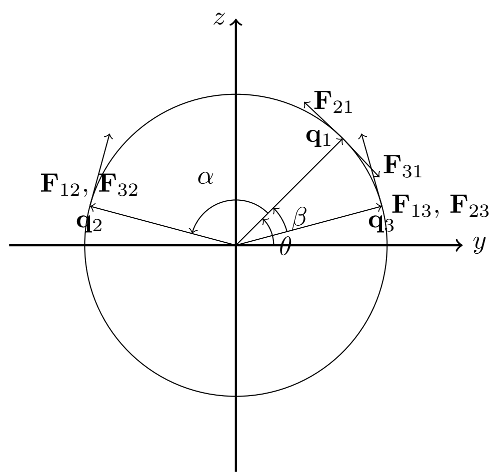
Without loss of generality, we assume that the three bodies are initially placed in the $y-z$ plane, and $\mathbf{q}_1$ is in the first quadrant; the angle $\mathbf{q}_1$ is obtuse. We can exclude the cases as shown in the figure below, where $\mathbf{q}_3$ is in the fourth quadrant or $\mathbf{q}_2$ is in the first quadrant, because in these cases, the sum of the force of constraint and the total attractive force exerted on $\mathbf{q}_3$ or $\mathbf{q}_2$ cannot make the centripetal force (the dashed line) needed. Thus we can assume that $\mathbf{q}_3$ is in the first quadrant and $\mathbf{q}_2$ is in the second quadrant. It is easy to exclude the cases when $\mathbf{q}_3$ or $\mathbf{q}_2$ is on the coordinate axis.



Then we set

$$\begin{aligned}
\mathbf{q}_1(0) &= (0, \cos\theta, \sin\theta), & \dot{\mathbf{q}}_1(0) &= (-\omega\cos\theta, 0, 0),\\
\mathbf{q}_2(0) &= (0, \cos(\theta+\alpha), \sin(\theta+\alpha)), & \dot{\mathbf{q}}_2(0) &= (-\omega\cos(\theta+\alpha), 0, 0),\\
\mathbf{q}_3(0) &= (0, \cos(\theta-\beta), \sin(\theta-\beta)), & \dot{\mathbf{q}}_3(0) &= (-\omega\cos(\theta-\beta), 0, 0),
\end{aligned}$$

with $0 < \theta < \frac{\pi}{2}$, $\pi/2 < \theta+\alpha < \pi$, $0 < \beta < \theta$. We have $d_1(\mathbf{q}_1, \mathbf{q}_2) = \alpha$, $d_1(\mathbf{q}_2, \mathbf{q}_3) = \alpha+\beta$, $d_1(\mathbf{q}_1, \mathbf{q}_3) = \beta$.



Theorem 4.9. *Consider three bodies with the above initial conditions. Then they can generate an Eulerian relative equilibrium if and only if the following equation and inequality hold:*

$$\begin{cases} g(\alpha,\beta,\theta) := -\sin 2\theta \frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \sin 2(\theta+\alpha)\frac{\sin^2\alpha}{\sin^2\beta} + \sin 2(\theta-\beta) = 0, \\ \sin^2\alpha \sin 2\theta < \sin^2(\alpha+\beta)\sin^2(\theta-\beta). \end{cases}$$

Proof. The symmetry implies that we only have to check whether or not the initial conditions provide the necessary force for the circular motion of each body. It's convenient to use complex coordinates in the y-z plane, so $\mathbf{q}_1 = e^{i\theta}$, $\mathbf{q}_2 = e^{i(\theta+\alpha)}$, $\mathbf{q}_3 = e^{i(\theta+\alpha+\beta)}$. By Lemma 3.1, we obtain

$$\begin{aligned} \mathbf{F}_{21} &= ie^{i\theta}\frac{m_1m_2}{\sin^2\alpha}, & \mathbf{F}_{31} &= -ie^{i\theta}\frac{m_1m_3}{\sin^2\beta}, \\ \mathbf{F}_{12} &= -ie^{i(\theta+\alpha)}\frac{m_1m_2}{\sin^2\alpha}, & \mathbf{F}_{32} &= -ie^{i(\theta+\alpha)}\frac{m_2m_3}{\sin^2(\alpha+\beta)}, \\ \mathbf{F}_{13} &= ie^{i(\theta-\beta)}\frac{m_3m_1}{\sin^2\beta}, & \mathbf{F}_{23} &= ie^{i(\theta-\beta)}\frac{m_3m_2}{\sin^2(\alpha+\beta)}. \end{aligned}$$

Assuming that the initial conditions can provide the force for the circular motion of each body, by similar computation as in the proof of Theorem 4.3 we obtain

$$\begin{cases} \frac{m_2}{\sin^2\alpha} - \frac{m_3}{\sin^2\beta} = \frac{\omega^2\sin 2\theta}{2}, \\ -\frac{m_1}{\sin^2\alpha} - \frac{m_3}{\sin^2(\alpha+\beta)} = \frac{\omega^2\sin 2(\theta+\alpha)}{2}, \\ \frac{m_1}{\sin^2\beta} + \frac{m_2}{\sin^2(\alpha+\beta)} = \frac{\omega^2\sin 2(\theta-\beta)}{2}. \end{cases}$$

This is a non-homogeneous linear system,

$$\begin{bmatrix} 0 & \frac{1}{\sin^2\alpha} & -\frac{1}{\sin^2\beta} \\ -\frac{1}{\sin^2\alpha} & 0 & -\frac{1}{\sin^2(\alpha+\beta)} \\ \frac{1}{\sin^2\beta} & \frac{1}{\sin^2(\alpha+\beta)} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} \frac{\omega^2\sin 2\theta}{2} \\ \frac{\omega^2\sin 2(\theta+\alpha)}{2} \\ \frac{\omega^2\sin 2(\theta-\beta)}{2} \end{bmatrix}.$$

We apply the row elementary operation to the augmented matrix,

$$\begin{bmatrix} 0 & \frac{1}{\sin^2\alpha} & -\frac{1}{\sin^2\beta} & \frac{\omega^2\sin 2\theta}{2} \\ -\frac{1}{\sin^2\alpha} & 0 & -\frac{1}{\sin^2(\alpha+\beta)} & \frac{\omega^2\sin 2(\theta+\alpha)}{2} \\ \frac{1}{\sin^2\beta} & \frac{1}{\sin^2(\alpha+\beta)} & 0 & \frac{\omega^2\sin 2(\theta-\beta)}{2} \end{bmatrix}$$
$$\to \begin{bmatrix} 0 & \frac{1}{\sin^2\alpha} & -\frac{1}{\sin^2\beta} & \frac{\omega^2\sin 2\theta}{2} \\ -\frac{1}{\sin^2\alpha} & 0 & -\frac{1}{\sin^2(\alpha+\beta)} & \frac{\omega^2\sin 2(\theta+\alpha)}{2} \\ 0 & 0 & 0 & \omega^2 g(\alpha,\beta,\theta)/2 \end{bmatrix},$$

which implies that the linear system allows a one-dimensional solution space if and only if

$$g(\alpha,\beta,\theta) := -\sin 2\theta\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \sin 2(\theta+\alpha)\frac{\sin^2\alpha}{\sin^2\beta} + \sin 2(\theta-\beta) = 0.$$

If $g(\alpha,\beta,\theta)=0$, the solution space is

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda\sin^2\beta}{\sin^2(\alpha+\beta)} + \frac{\omega^2\sin 2(\theta-\beta)\sin^2\beta}{2} \\ \lambda \\ \frac{\lambda\sin^2\beta}{\sin^2\alpha} - \frac{\omega^2\sin 2\theta\sin^2\beta}{2} \end{bmatrix}, \qquad \lambda\in\mathbb{R},$$

which admits positive solutions for positive λ if and only if

$$\begin{cases} -\frac{\lambda}{\sin^2(\alpha+\beta)} + \frac{\omega^2\sin 2(\theta-\beta)}{2} > 0 \\ \frac{\lambda}{\sin^2\alpha} - \frac{\omega^2\sin 2\theta}{2} > 0, \end{cases}$$

i.e., $\sin^2\alpha\sin 2\theta < \sin^2(\alpha+\beta)\sin 2(\theta-\beta)$. This completes the proof. □

Unlike the acute triangle case, there may be some obtuse triangle configuration that cannot generate an Eulerian relative equilibrium. For example, we can check that there exist no θ for the configuration: $\alpha=\pi/3$, $\beta=\pi/6$. We state the sufficient and necessary condition below.

Corollary 4.10. *For an obtuse triangle inscribed in a meridian with $d_1(\mathbf{q}_1,\mathbf{q}_2) = \alpha$, $d_1(\mathbf{q}_2,\mathbf{q}_3)=\alpha+\beta$, $d_1(\mathbf{q}_1,\mathbf{q}_3)=\beta$, there exist bodies of masses $m_1,\ m_2,\ m_3>0$ that can be placed at the vertices of the triangle such that they can generate an Eulerian relative equilibrium if and only if*

$$\frac{-\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \cos 2\alpha\frac{\sin^2\alpha}{\sin^2\beta} + \cos 2\beta}{\sin 2\beta - \sin 2\alpha\frac{\sin^2\alpha}{\sin^2\beta}} < \frac{\cos 2\beta\sin^2(\alpha+\beta) - \sin^2\alpha}{\sin 2\beta\sin^2(\alpha+\beta)}$$

or $\sin 2\beta\sin^2\beta = \sin 2\alpha\sin^2\alpha$.

Proof. Expanding the two relations in Theorem 4.9, we get

$$\sin 2\theta[-\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \cos 2\alpha\frac{\sin^2\alpha}{\sin^2\beta} + \cos 2\beta] = \cos 2\theta[\sin 2\beta - \sin 2\alpha\frac{\sin^2\alpha}{\sin^2\beta}],$$
$$\cos 2\theta[\sin 2\beta\sin^2(\alpha+\beta)] < \sin 2\theta[\cos 2\beta\sin^2(\alpha+\beta) - \sin^2\alpha].$$

If $\sin 2\beta\sin^2\beta \neq \sin 2\alpha\sin^2\alpha$, since $\sin 2\beta\sin^2(\alpha+\beta)>0$, we get

$$\frac{-\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + \cos 2\alpha\frac{\sin^2\alpha}{\sin^2\beta} + \cos 2\beta}{\sin 2\beta - \sin 2\alpha\frac{\sin^2\alpha}{\sin^2\beta}} < \frac{\cos 2\beta\sin^2(\alpha+\beta) - \sin^2\alpha}{\sin 2\beta\sin^2(\alpha+\beta)}.$$

Conversely, if the above inequality holds, then we can choose θ such that the equality in Theorem 4.9 holds, and the above inequality implies the inequality in Theorem 4.9; thus the two relations hold.

If $\sin 2\beta\sin^2\beta = \sin 2\alpha\sin^2\alpha$, we get $\theta=\pi/2$, and the two relations hold. This completes the proof. □

For example, we can prove the existence of any isosceles obtuse Eulerian relative equilibrium.

Corollary 4.11. *Consider three bodies with the above initial conditions, and assume they form an isosceles triangle, i.e, $\alpha=\beta$. Then the configuration can generate an Eulerian relative equilibrium if and only if $\theta=\pi/2$. In this case, we have $m_2=m_3$.*

Remark 4.12. The equation $\sin 2\beta \sin^2 \beta = \sin 2\alpha \sin^2 \alpha$ does not necessarily mean $\alpha = \beta$. One can see this easily from the graph of the function $h(x) = \sin 2x \sin^2 x$, $x \in (0, \pi/2)$. Hence m_2 and m_3 are not necessarily equal for these Eulerian relative equilibria. In Theorem 5.2 of [11], Pérez-Chavela and Reyes-Victoria have also reached similar results using stereographic projection.

Remark 4.13. We can generalize the above computation to study the Eulerian relative equilibria for the n-body problem in $\mathbf{S}^2$ and $\mathbf{H}^2$.

Acknowledgements

The author sincerely thanks his advisor, Professor Zhang Shiqing, for leading him into the area of celestial mechanics and for his deep reviews of the original manuscript, and for his valuable comments and suggestions to improve this work. The author would also like to thank his friends for enlightening discussions.

References

[1] V. I. Arnol′d, *Mathematical methods of classical mechanics*, 2nd ed., Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein. MR997295 (90c:58046)

[2] W. Bolyai and J. Bolyai, *Geometrische Untersuchungen*, Hrsg. P. Stäckel, Teubner, Leipzig-Berlin, 1913.

[3] José F. Cariñena, Manuel F. Rañada, and Mariano Santander, *Central potentials on spaces of constant curvature: the Kepler problem on the two-dimensional sphere S^2 and the hyperbolic plane H^2*, J. Math. Phys. **46** (2005), no. 5, 052702, 25, DOI 10.1063/1.1893214. MR2143000 (2006d:70019)

[4] Florin Diacu, *Relative equilibria of the curved N-body problem*, Atlantis Studies in Dynamical Systems, vol. 1, Atlantis Press, Paris, 2012. MR2976223

[5] Florin Diacu, Ernesto Pérez-Chavela, and Manuele Santoprete, *The n-body problem in spaces of constant curvature. Part I: Relative equilibria*, J. Nonlinear Sci. **22** (2012), no. 2, 247–266, DOI 10.1007/s00332-011-9116-z. MR2912328

[6] Florin Diacu, *Polygonal homographic orbits of the curved n-body problem*, Trans. Amer. Math. Soc. **364** (2012), no. 5, 2783–2802, DOI 10.1090/S0002-9947-2011-05558-3. MR2888228

[7] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern geometry—methods and applications. Part I. The geometry of surfaces, transformation groups, and fields,* 2nd ed., translated from the Russian by Robert G. Burns. Graduate Texts in Mathematics, vol. 93, Springer-Verlag, New York, 1992. MR1138462 (92h:53001)

[8] Herbert Goldstein, *Classical Mechanics*, Addison-Wesley Press, Inc., Cambridge, Mass., 1951. MR0043608 (13,291a)

[9] Valeriĭ V. Kozlov and Alexander O. Harin, *Kepler's problem in constant curvature spaces*, Celestial Mech. Dynam. Astronom. **54** (1992), no. 4, 393–399, DOI 10.1007/BF00049149. MR1188291 (93i:70010)

[10] N. I. Lobachevsky, *The new foundations of geometry with full theory of parallels* [in Russian], 1835-1838, in Collected Works, V. 2, GITTL, Moscow, 1949, p. 159.

[11] Ernesto Pérez-Chavela and J. Guadalupe Reyes-Victoria, *An intrinsic approach in the curved n-body problem. The positive curvature case*, Trans. Amer. Math. Soc. **364** (2012), no. 7, 3805–3827, DOI 10.1090/S0002-9947-2012-05563-2. MR2901235

[12] E. Schering, *Die Schwerkraft im Gaussischen Raume*, Nachr. Königl. Gesell. Wiss. Göttingen 13 July, **15** (1870), 311-321.

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