PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 8, August 2014, Pages 2921–2928
S 0002-9939(2014)12009-0
Article electronically published on April 25, 2014

A BETA FAMILY IN THE HOMOTOPY OF SPHERES

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(Communicated by Brooke Shipley)

ABSTRACT. Let p be a prime number greater than three. In the p-component of stable homotopy groups of spheres, Oka constructed a beta family from a v_2-periodic map on a four cell complex. In this paper, we construct another beta family in the groups at a prime p greater than five from a v_2-periodic map on an eight cell complex.

1. Introduction

We fix a prime number p greater than three and work in the stable homotopy category $\mathcal{S}_{(p)}$ of spectra localized at the prime p. Let S and BP in $\mathcal{S}_{(p)}$ denote the sphere and the Brown-Peterson spectra. It is important to understand the homotopy groups $\pi_*(S)$, whose structure is little known. On the other hand, we know the structures of $\pi_*(BP) = BP_*$ and $BP_*(BP)$:

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad BP_*(BP) = BP_*[t_1, t_2, \dots]$$

and $BP_*(BP)$ is a Hopf algebroid over BP_*. Here, the generators have degrees $|v_k| = |t_k| = 2(p^k - 1)$. Furthermore, we have the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$ of a spectrum X with E_2-term

$$E_2^{s,t}(X) = \operatorname{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*(X)),$$

and the spectral sequence for $X = S$ acts as a go-between between BP and S. Here we consider the homotopy groups $\pi_*(S)$ through the spectral sequence. In the E_2-term $E_2^{2,*}(S)$, Miller, Ravenel and Wilson [1] defined the beta elements $\beta_{s/t,r}$ for suitable triples (s, t, r) of positive integers. In this paper, we merge the methods of Oka and the authors to find permanent cycles among the beta elements and obtain a family of beta elements in the homotopy groups of the sphere spectrum.

Theorem. *Let p be a prime number greater than five. We have following beta elements in the homotopy groups $\pi_*(S)$ detected by the beta elements in the E_2-term:*

a) $\beta_{sp^n/t} \in \pi_{(sp^n(p+1)-t)q-2}(S)$ *for* $s \geq 1$, $n \geq 2$, $1 \leq t \leq 2^{n-2}p^2 - 2$ *and*
b) $\beta_{sp^n/up,2} \in \pi_{(sp^n(p+1)-up)q-2}(S)$ *for* $s \geq 1$, $n \geq 3$, $1 \leq u \leq 2^{n-3}p - 1$.

The orders of elements of a) *and* b) *are no less than p and p^2, respectively.*

Received by the editors March 20, 2012 and, in revised form, August 13, 2012 and August 30, 2012.

2010 *Mathematics Subject Classification.* Primary 55Q45; Secondary 55Q51.

From now on, we state the definitions, known results and theorems obtained in this paper. Consider the spectra and the maps defined by the cofiber sequences:

$$\begin{gathered} S \xrightarrow{p^r} S \xrightarrow{i_r} M(r) \xrightarrow{j_r} \Sigma S \quad \text{and} \\ \Sigma^{up^{r-1}q} M(r) \xrightarrow{A^u_{r-1}} M(r) \xrightarrow{i_{r,up^{r-1}}} M(r, up^{r-1}) \xrightarrow{j_{r,up^{r-1}}} \Sigma^{up^{r-1}q} M(r), \end{gathered} \tag{1.1}$$

where A_r denotes an element such that $BP_*(A_r) = v_1^{p^r}$ for $r \geq 0$ (*cf.* [6, Th. 6.2]; see also (2.6)), and A_0 is known as the Adams map and denoted by α. Hereafter, $q = 2p - 2$. We note that $BP_*(M(r)) = BP_*/(p^r)$ and $BP_*(M(r, up^{r-1})) = BP_*/(p^r, v_1^{up^{r-1}})$ as $BP_*(BP)$-comodules. The cofiber sequences in (1.1) induce the connecting homomorphisms $\partial_r \colon E_2^{s,t}(M(r)) \to E_2^{s+1,t}(S)$ and $\partial_{r,up^{r-1}} \colon E_2^{s,t}(M(r, up^{r-1})) \to E_2^{s+1,t-up^{r-1}q}(M(r))$ on the E_2-terms. In [11], we modified the definition of the beta elements as follows: let (s, t, r) be a triple of positive integers, and suppose that $v_1^c v_2^s \in E_2^{0,(s(p+1)+c)q}(M(r, t + c))$ with $p^{r-1} \mid (t + c)$ for an integer c. Then, the beta element for a triple (s, t, r) is defined by

$$\beta_{s/t,r} = \partial_r \partial_{r,t+c}(v_1^c v_2^s) \in E_2^{2,(s(p+1)-t)q}(S). \tag{1.2}$$

We abbreviate $\beta_{s/t,1}$ and $\beta_{s/1}$ to $\beta_{s/t}$ and β_s, respectively, as usual. In the case where $c = 0$, the definition is the ordinary one. Besides, these elements generate the E_2-term by [1, Th. 2.6]. It is an interesting problem asking which of them survives in the spectral sequence. So far, the following elements are known to be permanent cycles:

(1.3)

a) β_s for $s \geq 1$ in [12],
b) $\beta_{sp/t}$ for $s \geq 1$ and $t \leq p$, and $t < p$ if $s = 1$ in [2], [3],
c) $\beta_{sp^2/t}$ for $s \geq 1$ and $t \leq 2p$, and $t \leq 2p - 2$ if $s = 1$ in [2], [4],
d) $\beta_{sp^2/t}$ for $s \geq 1$ and $t \leq p^2 - 2$ in [11],
e) $\beta_{sp^n/t}$ for $s \geq 1$, $n \geq 3$, $1 \leq t \leq 2^{n-2}p$, and $t \leq 2^{n-2}(p-1)$ if $s = 1$ in [6], [7],
f) $\beta_{sp^2/p,2}$ for $s \geq 2$ in [4], and
g) $\beta_{sp^n/up,2}$ for $s \geq 1$, $n \geq 3$, $1 \leq u \leq 2^{n-2}$, and $up \leq 2^{n-2}(p-1)$ if $s = 1$ in [6], [7].

We note that we have $\beta_{sp^n/t}$ for $t \leq p^n + p^{n-1} - 1$ and $t \leq p^n$ if $s = 1$ in the E_2-term by [1], and Ravenel showed that β_{p^n/p^n} cannot be a permanent cycle for $n \geq 1$ (*cf.* [9, 6.4.2. Th.]). Thus, the beta elements $\beta_{sp^n/t}$ for $2^{n-2}p < t \leq p^n + p^{n-1} - 1$ and $t < p^n$ if $s = 1$ are left undetermined.

In this paper we modify the definition further.

Definition 1.4. Let $b(s; t, r)$ denote a set of elements x of $E_2^{0,(s(p+1)+c)q}(M(r, t+c))$ such that $x \equiv v_1^c v_2^s \mod (p, v_1^{c+1})$ for a non-negative integer c. We define the *beta coset* by

$$\widehat{\beta}_{s/t,r} = \partial_r \partial_{r,t+c}(b(s; t, r)) \subset E_2^{2,(s(p+1)+c-up^{r-1})q}(S).$$

We also abbreviate $\widehat{\beta}_{s/t,1}$ to $\widehat{\beta}_{s/t}$.

In [6, Th. II], Oka showed a possibility of many beta elements in the same dimension. This indicates that a difference of elements of $\widehat{\beta}_{s/t,r}$ may be another beta element. In this paper, we study a beta element $\beta_{s/t,r}$ with a larger t, not the p-rank of $\pi_*(S)$, and we introduce the notation. We further abuse a term.

Definition 1.5. We say that the beta coset $\widehat{\beta}_{s/t,r}$ *survives* to the homotopy groups $\pi_*(S)$ if an element of $\widehat{\beta}_{s/t,r}$ is a permanent cycle. In this case, the β-element $\beta_{s/t,r}$ of $\pi_*(S)$ denotes one of survivors of $\widehat{\beta}_{s/t,r}$.

In this paper, we consider the beta cosets $\widehat{\beta}_{s/t,r}$ for $r = 1, 2$, and so we consider the following spectra and maps of (1.1):

$$M = M(1), \quad \overline{M} = M(2), \quad K_u = M(1,u) \quad \text{and} \quad \overline{K}_u = M(2,up); \quad \text{and}$$

$$k = k_1, \quad \overline{k} = k_2, \quad \alpha = A_0, \quad A = A_1, \quad k_u = k_{1,u} \quad \text{and} \quad \overline{k}_u = k_{2,up} \tag{1.6}$$

for $u > 0$, where k stands for i and j. (We use the notation M and K_u following those of Oka [7].) Thus, from now on, i_u and j_u denote $i_{1,u}$ and $j_{1,u}$, not the maps i_u and j_u in (1.1).

The above definitions make Oka's method developed in [6] and [7] simple: Let $f_{s,u} \in \pi_*(K_u)$ be an element such that $\eta_*(f_{s,u}) = v_2^s \in BP_*(K_u) = BP_*/(p, v_1^u)$ for the unit map $\eta\colon S \to BP$ of the ring spectrum BP, and put

$$\begin{aligned}
\mathfrak{B}_{Oka}(s,u) &= \mathfrak{B}^0_{Oka}(s,u) \cup \mathfrak{B}^1_{Oka}(s,u), \text{ where}\\
\mathfrak{B}^0_{Oka}(s,u) &= \{\widehat{\beta}_{skp^n/t} : k \geq 1,\ n \geq 0,\ 1 \leq t \leq 2^n u\},\\
\mathfrak{B}^1_{Oka}(s,u) &= \{\widehat{\beta}_{skp^n/tp,2} : k \geq 1,\ n \geq 1,\ t \geq 1,\ tp \leq 2^{n-1}u\}.
\end{aligned}$$

Theorem 1.7 (Oka [6], [7]). *If $f_{s,u} \in \pi_*(K_u)$ exists, then every element of $\mathfrak{B}_{Oka}(s,u)$ survives to $\pi_*(S)$.*

Consider $\overline{\mathfrak{B}}_{Oka}((a,b),u) = \bigcup_{k,l\geq 0,\ k+l>0} \mathfrak{B}_{Oka}(ak+bl,u)$. Since Oka also showed the existence of $f_{sp,u} \in \pi_*(K_u)$ for $s = 2, 3$ and $u \leq p$ and for $s = 1$ and $u < p$ in [2, Th. C] and [3, Th. CII], the theorem implies that every element of $\mathfrak{B}_{Oka}(p, p-1) \cup \overline{\mathfrak{B}}_{Oka}((2p,3p),p)$ survives to $\pi_*(S)$, which is the theorem [6, Th. I], and yields elements in b), c), e), f) and g) in (1.3).

Let W be the cofiber of the generator $\beta_1 \in \pi_{pq-2}(S)$, and we have a cofiber sequence

$$S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{i_W} W \xrightarrow{j_W} S^{pq-1}. \tag{1.8}$$

In [10], we introduce a method to obtain a beta family from $f_{s,u} \in \pi_*(W \wedge K_u)$ such that $\eta_*(f_{s,u}) = v_2^s \in BP_*(W \wedge K_u)$. In this paper, we merge these methods. For an element $f_{p^i,u} \in \pi_*(W \wedge K_u)$, consider a family

$$\begin{aligned}
\mathfrak{B}(p^i,u) &= \mathfrak{B}^0(p^i,u) \cup \mathfrak{B}^1(p^i,u), \quad \text{where}\\
\mathfrak{B}^0(p^i,u) &= \{\widehat{\beta}_{sp^{i+n}/t} : s \geq 1,\ n \geq 0,\ 1 \leq t \leq 2^n u - 2\},\\
\mathfrak{B}^1(p^i,u) &= \{\widehat{\beta}_{sp^{i+n}/tp,2} : s \geq 1,\ n \geq 1,\ t \geq 1,\ tp \leq 2^{n-1}u - p\}.
\end{aligned}$$

Theorem 1.9. *If $f_{p^i,u} \in \pi_*(W \wedge K_u)$ exists, then every element of $\mathfrak{B}(p^i,u)$ survives to $\pi_*(S)$.*

In [11, Th. 1.7], we showed the existence of $f_{p^2,p^2} \in \pi_*(W \wedge K_{p^2})$ for $p > 5$, though there does not exist $f_{p^2,p^2} \in \pi_*(K_{p^2})$ shown by Ravenel.

Corollary 1.10. *Let $p > 5$. Then, $\mathfrak{B}(p^2,p^2)$ yields a beta family of $\pi_*(S)$.*

This implies our main theorem stated above. This improves Oka's results (1.3)e) and g) if the prime number p is greater than five.

2. Recollection of the finite ring spectrum

In this section, we recall some results of Oka. We call a spectrum E *a ring spectrum* if it admits a multiplication $\mu\colon E\wedge E\to E$ and a unit $\iota\colon S\to E$ such that $\mu(\iota\wedge 1)=1=\mu(1\wedge\iota)$ and $\mu(\mu\wedge 1)=\mu(1\wedge\mu)$. A ring spectrum E is *commutative* if $\mu T=\mu$ for the switching map $T\colon E\wedge E\to E\wedge E$. The homotopy groups $E_*=\pi_*(E)$ of E have a multiplication given by $ab=\mu(a\wedge b)$ for $a,b\in E_*$, which makes E_* a ring. Oka [7] (*cf.* [8]) defined Mod(E) and Der(E) by

$$\begin{array}{rcl}\mathrm{Mod}(E) & = & \{f\in[E,E]_*\mid \mu(f\wedge 1)=f\mu\}\quad\text{and}\\ \mathrm{Der}(E) & = & \{f\in[E,E]_*\mid \mu(f\wedge 1)+\mu(1\wedge f)=f\mu\}.\end{array}$$

We call an element of Der(E) a *derivation* of E.

Theorem 2.1 (Oka [8, Lemma 1.3])**.** *For the unit ι, the induced homomorphism $\iota^*\colon \mathrm{Mod}(E)\to E_*$ is a ring isomorphism. Its inverse $\kappa\colon E_*\to\mathrm{Mod}(E)$ is given by $\kappa(f)=\mu(f\wedge 1)$.*

Consider a spectrum K_u in (1.6). Then, Oka showed that

Theorem 2.2 (Oka [7, Th. 2.5])**.** *K_u has a commutative and associative multiplication m_u.*

Theorem 2.3 (Oka [7, Lemma 2.3])**.** Mod(K_u) *is a commutative subring of* $[K_u, K_u]_*$*, and a commutator* $[f,g]$ *belongs to* Mod(K_u) *for* $f\in\mathrm{Mod}(K_u)$ *and* $g\in\mathrm{Der}(K_u)$*. In particular,*

$$f^pg=gf^p\quad\text{for } f\in\mathrm{Mod}(K_u) \text{ and } g\in\mathrm{Der}(K_u).$$

Let $\delta'_u=i_uj_u\in[K_u,K_u]_{-uq-1}$. Then it fits into a cofiber sequence

$$\Sigma^{uq}K_u\xrightarrow{\widetilde{i}_u}K_{2u}\xrightarrow{\widetilde{j}_u}K_u\xrightarrow{\delta'_u}\Sigma^{uq+1}K_u \tag{2.4}$$

by (1.1) with a 3×3 Lemma.

Theorem 2.5 (Oka [7, Th. 2.5])**.** $\delta'_u\in\mathrm{Der}(K_u)$.

It is well known that $\delta=ij\in\mathrm{Der}(M)$ and $\alpha\in\mathrm{Mod}(M)$, and so $\alpha^p\delta=\delta\alpha^p\in[M,M]_{pq-1}$. It gives rise to not only the element $A=A_1$ in (1.1) but also δ_u in the commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{upq-1}M & \xrightarrow{\delta} & \Sigma^{upq}M & \xrightarrow{\pi} & \Sigma^{upq}\overline{M} & \longrightarrow & \Sigma^{upq}M\\
\downarrow{\scriptstyle\alpha^{up}} & & \downarrow{\scriptstyle\alpha^{up}} & & \downarrow{\scriptstyle A^u} & & \downarrow{\scriptstyle\alpha^{up}}\\
\Sigma^{-1}MM & \xrightarrow{\delta} & M & \xrightarrow{\pi} & \Sigma^{pq}\overline{M} & \longrightarrow & M\\
\downarrow{\scriptstyle i_{up}} & & \downarrow{\scriptstyle i_{up}} & & \downarrow{\scriptstyle \bar{i}_u} & & \downarrow{\scriptstyle i_{up}}\\
\Sigma^{-1}K_{up} & \xrightarrow{\delta_u} & K_{up} & \xrightarrow{\overline{\pi}} & \overline{K}_u & \longrightarrow & K_{up}\\
\downarrow{\scriptstyle j_{up}} & & \downarrow{\scriptstyle j_{up}} & & \downarrow{\scriptstyle \bar{j}_u} & & \downarrow{\scriptstyle j_{up}}\\
\Sigma^{upq}M & \xrightarrow{-\delta} & \Sigma^{upq+1}M & \xrightarrow{\pi} & \Sigma^{upq+1}\overline{M} & \longrightarrow & \Sigma^{upq+1}M
\end{array} \tag{2.6}$$

in which rows and columns are cofiber sequences. By [6, Lemma 4.5, Th. 4.2], we have the following:

Theorem 2.7 (Oka [7, p. 425]). *The map δ_u in the above diagram is a derivation of K_{up}.*

Proof. The matrices for the map $\delta_u \wedge 1$ and the switching map T are given by

$$\tau(\delta_u \wedge 1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta & -1 & 0 & 0 \\ \delta' & 0 & -1 & 0 \\ \delta\delta' & \delta' & -\delta & 1 \end{pmatrix}$$

by [6, Lemma 4.5] and [6, Th. 4.2], and so the first row of the matrix for $(\delta_u \wedge 1) + (\delta_u \wedge 1)T$ is $(\delta\ 0\ 0\ 0)$. Since the multiplication m_u is the projection to the first summand, we see that $m_u(\delta_u \wedge 1 + 1 \wedge \delta_u) = m_u((\delta_u \wedge 1) + T(\delta_u \wedge 1)T) = m_u((\delta_u \wedge 1) + (\delta_u \wedge 1)T) = \delta_u m_u$, as desired. □

The following lemma is a folklore:

Lemma 2.8. *There exist self-maps $\widetilde{\alpha}\colon \Sigma^q K_u \to K_u$ and $\widetilde{A}\colon \Sigma^{pq}\overline{K}_u \to \overline{K}_u$ such that $BP_*(\widetilde{\alpha}) = v_1$ and $BP_*(\widetilde{A}) = v_1^p$.*

Lemma 2.9. $A\bar{i}\beta_1 = 0 \in \pi_{2pq-2}(\overline{M})$.

Proof. Consider the cobar complex $\{(C^s, d)\}_{s\geq 0}$ whose cohomology is the E_2-term $E_2^*(\overline{M})$ of the Adams-Novikov spectral sequence converging to $\pi_*(\overline{M})$. Then, $C^s = \Gamma/(p^2) \otimes_A \Gamma^{s-1}$, where $(A, \Gamma) = (BP_*, BP_*BP)$, and the differential d of the complex is given by derivation with $d(v) = \eta_R(v) - \eta_L(v) \in C^1 = \Gamma/(p^2)$ for $v \in C^0 = A/(p^2)$ and $d(x) = 1 \otimes x - \Delta(x) + x \otimes 1 \in C^2$ for $x \in C^1$. We also use the formulas on the structure maps of the Hopf algebroid given by the formulas of Quillen and Hazewinkel:

$$\eta_R(v_1) = v_1 + pt_1, \quad \eta_R(v_2) \equiv v_2 + v_1 t_1^p + pt_2 - (p+1)v_1^p t_1 \mod (p^2),$$
$$\Delta(t_1) = 1 \otimes t_1 + t_1 \otimes 1 \quad \text{and} \quad \Delta(t_2) = 1 \otimes t_2 + t_1 \otimes t_1^p + t_1 \otimes 1 + v_1 b_{10}.$$

Here, b_{10} denotes the cocycle defined by $d(t_1^p) = pb_{10}$. Let b_0 denote the cohomology class of b_{10}. Then, by definition, $\beta_1 = \partial_1\partial_{1,1}(v_2) = b_0$. Therefore, $A\bar{i}\beta_1$ is detected by $v_1^p b_0 \in E_2^{2,2pq}(\overline{M})$. We compute that $d(c) = v_1^p b_{10} \in C^2$ for $c = -v_1^{p-1}t_2 + v_1^{p-3}(v_1 t_1 - pt_1^2)\eta_R(v_2) + v_1^{2p-3}\left(\frac{p+1}{2}v_1 t_1^2 - \frac{p}{3}t_1^3\right)$. It follows that $v_1^p b_0 = 0 \in E_2^{2,2pq}(\overline{M})$. We further see that $E_2^{2+rq,2pq+rq}(\overline{M}) = 0$ for $r \geq 1$ by the vanishing line (*cf.* [1, Lemma 1.16, Remark 1.17]). Hence $A\bar{i}\beta_1 = 0 \in \pi_{2pq-2}(\overline{M})$. □

3. Arguments on $W \wedge K_u$

In this section, we fix an integer u, and K denotes $K_u = M(1, u)$, which is a commutative ring spectrum with multiplication $m = m_u$ by Theorem 2.2. The spectrum W in (1.8) admits a multiplication $m_W\colon W \wedge W \to W$ such that $m_W(i_W \wedge 1_W) = 1_W = m_W(1_W \wedge i_W)$ by [5, Example 2.9].

Consider the spectrum $WM = W \wedge M$ and the multiplication $m_{WM} = (m_W \wedge m_M)(1_W \wedge T \wedge 1_M)\colon WM \wedge WM \to WM$. Then, we see that

$$m_{WM}(i' \wedge i') = i' m_W, \tag{3.1}$$

for $i' = 1_W \wedge i$. We further see the following lemma by [6, Th. 4.13]:

Lemma 3.2. *WM is a commutative and associative ring spectrum with multiplication m_{WM}.*

Proof. Since $p = 0 \in [WM, WM]_0$ and $\beta_1 \wedge 1 = 0 \in [M \wedge WM, M \wedge WM]_{pq-2}$, WM is a split ring spectrum. (See [6, Def. 2.1] for the definition of a split ring spectrum.) □

Put $WMK = W \wedge M \wedge K$, and consider a multiplication $m_{WMK} = (m_{WM} \wedge m)(1_{WM} \wedge T \wedge 1_K)\colon WMK \wedge WMK \xrightarrow{1_{WM} \wedge T \wedge 1_K} WM \wedge WM \wedge K \wedge K \xrightarrow{m_{WM} \wedge m} WMK$ on WMK. Since the smash product of commutative ring spectra is a commutative ring spectrum, we have the following

Corollary 3.3. *WMK is a commutative ring spectrum with multiplication m_{WMK}.*

Consider the spectrum $WK = W \wedge K$ and a multiplication m_{WK} on WK defined by $m_{WK} = (m_W \wedge m)(1_W \wedge T \wedge 1_K)$ for the switching map $T\colon K \wedge W \to W \wedge K$. We have the split cofiber sequence

$$WK \underset{\sigma}{\overset{\widehat{i}}{\rightleftarrows}} WMK \xrightarrow{\widehat{j}} \Sigma WK, \tag{3.4}$$

in which $\widehat{i} = 1_W \wedge i \wedge 1_K\colon WK \to WMK$ and σ denotes a splitting.

Lemma 3.5. *$\widehat{i} m_{WK} = m_{WMK}(\widehat{i} \wedge \widehat{i})$.*

Proof. This follows from computation:

$$\begin{aligned} \widehat{i} m_{WK} &= (i' \wedge 1_K)(m_W \wedge m)(1_W \wedge T \wedge 1_K) \\ &= (m_{WM} \wedge m)(i' \wedge i' \wedge 1_{K \wedge K})(1_W \wedge T \wedge 1_K) \quad \text{by (3.1)} \\ &= (m_{WM} \wedge m)(1_{WM} \wedge T \wedge 1_K)(\widehat{i} \wedge \widehat{i}) \;=\; m_{WMK}(\widehat{i} \wedge \widehat{i}). \end{aligned}$$ □

Lemma 3.6. *The spectrum WK is a commutative ring spectrum with multiplication m_{WK}.*

Proof. Apply σ in (3.4) to Lemma 3.5, and we have

$$m_{WK} = \sigma m_{WMK}(\widehat{i} \wedge \widehat{i}). \tag{3.7}$$

Then, noticing that $m_{WMK}T' = m_{WMK}$ by Corollary 3.3,

$$m_{WK}T \sigma m_{WMK}(\widehat{i} \wedge \widehat{i})T \;=\; \sigma m_{WMK}T'(\widehat{i} \wedge \widehat{i}) \;=\; \sigma m_{WMK}(\widehat{i} \wedge \widehat{i}) \;=\; m_{WK}.$$

Here, $T\colon WK \wedge WK \to WK \wedge WK$ and $T'\colon WMK \wedge WMK \to WMK \wedge WMK$ are the switching maps. The associativity of it is verified as follows:

$$\begin{aligned} m_{WK}(m_{WK} \wedge 1_{WK}) &= \sigma m_{WMK}(\widehat{i} \wedge \widehat{i})\,(m_{WK} \wedge 1_{WK}) \quad \text{by (3.7)} \\ &= \sigma m_{WMK}(m_{WMK} \wedge 1_{WMK})(\widehat{i} \wedge \widehat{i} \wedge \widehat{i}) \quad \text{(by Lemma 3.5)} \\ &= \sigma m_{WMK}(1_{WMK} \wedge m_{WMK})(\widehat{i} \wedge \widehat{i} \wedge \widehat{i}) \quad \text{(by Corollary 3.3)} \\ &= \sigma m_{WMK}(\widehat{i} \wedge \widehat{i})\,(1_{WK} \wedge m_{WK}) \quad \text{(by Lemma 3.5)} \\ &= m_{WK}(1_{WK} \wedge m_{WK}). \end{aligned}$$ □

Consider the homomorphism

$$\varphi_W\colon [K, K]_* \to [WK, WK]_*$$

given by $\varphi_W(f) = 1_W \wedge f$. Then, an easy computation shows

Lemma 3.8. *The homomorphism φ_W induces those of $\varphi_W\colon \mathrm{Mod}(K) \to \mathrm{Mod}(WK)$ and $\varphi_W\colon \mathrm{Der}(K) \to \mathrm{Der}(WK)$.*

4. Construction of homotopy elements

By definition, we see that $BP_*(W\wedge X) = BP_*(X)\oplus gBP_*(X)$ for a spectrum X such that $BP_s(X) = 0$ unless $q \mid s$, in which the degree of the generator g is $pq-1$. This also implies the relation $E_2^*(W\wedge X) = E_2^*(X)\oplus gE_2^*(X)$ of the Adams-Novikov E_2-term. We apply this for $X = K_u$ and $\overline{K}_u$.

We begin with a general result corresponding to Oka's theorems [6, Th. 7.1, Th. 7.2] and [7, Construction III].

Proposition 4.1. *Let $f \in \pi_*(WK_u)$ be an element such that $\eta_*(f) \equiv v_2^s \bmod (v_1)$ for the unit map η of BP. Then,*

1) *for $n \geq 0$, there is an element $f_n \in \pi_*(WK_{2^n u})$ such that $\eta_*(f_n) \equiv v_2^{sp^n} \bmod (v_1)$, and*
2) *for $n \geq 1$ and a positive integer u' such that $u'p \leq 2^{n-1}u$, there is an element $\overline{f}_n \in \pi_*(W\overline{K}_{u'})$ such that $\eta_*(\overline{f}_n) \equiv v_2^{sp^n} \bmod (p, v_1)$. Here, $W\overline{K}_u = W\wedge \overline{K}_u$.*

Proof. Put $f_0 = f$, and suppose the existence of $f_n \in \pi_*(WK_{2^n u})$. Then, $\kappa(f_n) \in \mathrm{Mod}(WK_{2^n u})$ for κ in Theorem 2.1. By Theorem 2.2 and Lemma 3.8, $\delta' = \varphi_W(\delta'_{2^n u}) \in \mathrm{Der}(WK_{2^n u})$, and so we have $\kappa(f_n)^p\delta' = \delta'\kappa(f_n)^p$ by Theorem 2.3. Thus we obtain a map $\widetilde{f}_{n+1}$, which makes the diagram

$$\begin{array}{ccccc}
WK_{2^{n+1}u} & \longrightarrow & WK_{2^n u} & \xrightarrow{\delta} & WK_{2^n u} \\
{\scriptstyle \widetilde{f}_{n+1}}\downarrow & & \downarrow{\scriptstyle \kappa(f_n)^p} & & \downarrow{\scriptstyle \kappa(f_n)^p} \\
WK_{2^{n+1}u} & \longrightarrow & WK_{2^n u} & \xrightarrow{\delta'} & WK_{2^n u}
\end{array}$$

commute. Here, the rows of the diagram are the cofiber sequence of (2.4). Now put $f_{n+1} = (i_W \wedge i_{2^n u} i)^*(\widetilde{f}_{n+1})$ to complete the induction.

To prove the second part of Proposition 4.1: By 1), we have $f_{n-1} \in \pi_*(WK_{2^{n-1}u})$ for $n \geq 1$, and so an element $f'_{n-1} \in \pi_*(WK_{u'p})$ for $u'p \leq 2^{n-1}u$ (*cf.* [7, Construction I]). Theorem 2.7, Lemma 3.8 and Theorem 2.3 imply that

$$\kappa(f'_{n-1})^p\delta'' = \delta''\kappa(f'_{n-1})^p$$

for $\delta'' = \varphi_W(\delta_{u'})$, where $\delta_{u'}$ is the map in (2.6). We then have a map $\widetilde{\overline{f}}_n$ fitting into the commutative diagram

$$\begin{array}{ccccc}
W\overline{K}_{u'} & \longrightarrow & WK_{u'p} & \xrightarrow{\delta''} & WK_{u'p} \\
{\scriptstyle \widetilde{\overline{f}}_n}\downarrow & & \downarrow{\scriptstyle \kappa(f'_{n-1})^p} & & \downarrow{\scriptstyle \kappa(f'_{n-1})^p} \\
W\overline{K}_{u'} & \longrightarrow & WK_{u'p} & \xrightarrow{\delta''} & WK_{u'p}
\end{array}$$

in which the rows of the diagram are the cofiber sequence in (2.6). Now the map $\overline{f}_n$ is obtained by $(i_W \wedge \overline{i}_{u'} i)^*(\widetilde{\overline{f}}_n)$. □

In [10], we show the following lemma:

Lemma 4.2 ([10, Lemma 2.11]). *For $u > 2$, there exists an element $\omega_u \in \pi_{(p+2)q-1}(WK_u)$ such that $(j_W)_*(\omega_u) = i_u\alpha^2 i \in \pi_{2q}(K_u)$. Moreover, $v_1^2 g \in E_2^0(WK_u) = E_2^0(K_u)\oplus gE_2^0(K_u)$ detects it.*

A similar lemma follows from Lemma 2.9.

Lemma 4.3. *For $u > 1$, there exists an element $\overline{\omega}_u \in \pi_{2pq-1}(W\overline{K}_u)$ such that $(j_W)_*(\overline{\omega}_u) = \bar{i}_u A\bar{i} \in \pi_{pq}(\overline{K}_u)$. Moreover, $v_1^p g \in E_2^0(W\overline{K}_u) = E_2^0(\overline{K}_u) \oplus gE_2^0(\overline{K}_u)$ detects it.*

Proof of Theorem 1.9. By Proposition 4.1, we have elements $(f_{p^i,u})_n \in \pi_*(WK_{2^n u})$ for $n \geq 0$ and $\overline{(f_{p^i,u})_n} \in \pi_*(W\overline{K}_{u'})$ for $n > 0$ and $u'p \leq 2^{n-1}u$ such that $\eta_*((f_{p^i,u})_n) \equiv v_2^{p^{i+n}} \bmod (v_1)$ and $\eta_*(\overline{(f_{p^i,u})_n}) \equiv v_2^{p^{i+n}} \bmod (p, v_1)$. Consider the composites

$$\begin{aligned} B_{sp^{i+n}/2^n u-r} &= \widetilde{\alpha}^{r-2}(j_W \wedge 1_K)\kappa((f_{p^i,u})_n)^s\omega_{2^n u} \quad (r \geq 2) \quad \text{and} \\ B_{sp^{i+n}/u'p-rp,2} &= \widetilde{A}^{r-1}(j_W \wedge 1_{\overline{K}})\kappa(\overline{(f_{p^i,u})_n})^s\overline{\omega}_{u'} \quad (r \geq 1), \end{aligned}$$

where $\widetilde{\alpha}$, $\widetilde{A}$, ω_u and $\overline{\omega}_u$ are the elements of Lemmas 2.8, 4.2 and 4.3, and κ is the homomorphism in Theorem 2.1. Then, $\eta_*(B_{sp^{i+n}/2^n u-r}) \in b(sp^{i+n}; 2^n u - r, 1)$ and $\eta_*(B_{sp^{i+n}/u'p-rp,2}) \in b(sp^{i+n}; u'p - rp, 2)$. Since j, j_{2^n}, $\overline{j}$ and $\overline{j}_{u'}$ correspond to ∂_1, $\partial_{1,2^n u}$, ∂_2 and $\partial_{2,u'p}$, respectively, the elements $jj_{2^n u}B_{sp^{i+n}/2^n u-r}$ and $\overline{j}\overline{j}_{u'}B_{sp^{i+n}/u'p-rp,2}$ are detected by elements of $\widehat{\beta}_{sp^{i+n}/2^n u-r}$ and $\widehat{\beta}_{sp^{i+n}/u'p-rp,2}$, as desired. $\square$

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