A BETA FAMILY IN THE HOMOTOPY OF SPHERES

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(Communicated by Brooke Shipley)

Abstract. Let \( p \) be a prime number greater than three. In the \( p \)-component of stable homotopy groups of spheres, Oka constructed a beta family from a \( v_2 \)-periodic map on a four cell complex. In this paper, we construct another beta family in the groups at a prime \( p \) greater than five from a \( v_2 \)-periodic map on an eight cell complex.

1. Introduction

We fix a prime number \( p \) greater than three and work in the stable homotopy category \( S_\p \) of spectra localized at the prime \( p \). Let \( S \) and \( BP \) in \( S_\p \) denote the sphere and the Brown-Peterson spectra. It is important to understand the homotopy groups \( \pi_\ast(S) \), whose structure is little known. On the other hand, we know the structures of \( \pi_\ast(BP) = BP_\ast \) and \( BP_\ast(BP) \):

\[
BP_\ast = \mathbb{Z}_p[v_1, v_2, \ldots] \quad \text{and} \quad BP_\ast(BP) = BP_\ast[t_1, t_2, \ldots]
\]

and \( BP_\ast(BP) \) is a Hopf algebroid over \( BP_\ast \). Here, the generators have degrees \( |v_k| = |t_k| = 2(p^k - 1) \). Furthermore, we have the Adams-Novikov spectral sequence converging to the homotopy groups \( \pi_\ast(X) \) of a spectrum \( X \) with \( E_2 \)-term

\[
E_2^{s,t}(X) = \text{Ext}_{BP_\ast(BP)}^{s,t}(BP_\ast, BP_\ast(X)),
\]

and the spectral sequence for \( X = S \) acts as a go-between between \( BP \) and \( S \). Here we consider the homotopy groups \( \pi_\ast(S) \) through the spectral sequence. In the \( E_2 \)-term \( E_2^{s,t}(S) \), Miller, Ravenel and Wilson [1] defined the beta elements \( \beta_{s/t,r} \) for suitable triples \( (s, t, r) \) of positive integers. In this paper, we merge the methods of Oka and the authors to find permanent cycles among the beta elements and obtain a family of beta elements in the homotopy groups of the sphere spectrum.

Theorem. Let \( p \) be a prime number greater than five. We have following beta elements in the homotopy groups \( \pi_\ast(S) \) detected by the beta elements in the \( E_2 \)-term:

a) \( \beta_{sp^n/t} \in \pi(sp^n(p+1)-t)_{q-2}(S) \) for \( s \geq 1, n \geq 2, 1 \leq t \leq 2^{n-2}p^2 - 2 \) and

b) \( \beta_{sp^n/up,2} \in \pi(sp^n(p+1)-up)_{q-2}(S) \) for \( s \geq 1, n \geq 3, 1 \leq u \leq 2^{n-3}p - 1 \).

The orders of elements of a) and b) are no less than \( p \) and \( p^2 \), respectively.
From now on, we state the definitions, known results and theorems obtained in this paper. Consider the spectra and the maps defined by the cofiber sequences:

\[ S \xrightarrow{\partial_r} S \xrightarrow{j_r} M(r) \xrightarrow{j_r} \Sigma S \quad \text{and} \quad \Sigma^{up^{-1}}q M(r) \xrightarrow{A_v} M(r) \xrightarrow{i_{r,up^{-1}}} M(r, up^{-1}) \xrightarrow{j_r,up^{-1}} \Sigma^{up^{-1}}q M(r), \]

where \( A_v \) denotes an element such that \( BP_*(A_v) = v_r^{(r)} \) for \( r \geq 0 \) (cf. [3] Th. 6.2]; see also (2.6)), and \( A_0 \) is known as the Adams map and denoted by \( \alpha \). Hereafter, \( q = 2p - 2 \). We note that \( BP_*(M(r)) = BP_*(p^r) \) and \( BP_*(M(r, up^{-1})) = BP_*/(p^r, v_1^{up^{-1}}) \) as \( BP_*(BP) \)-comodules. The cofiber sequences in (1.1) induce the connecting homomorphisms \( \partial_r : E^{s,t}_2(M(r)) \rightarrow E^{s+1,t}_2(S) \) and \( \partial_{r,up^{-1}} : E^{s,t}_2(M(r, up^{-1})) \rightarrow E^{s+1,t-1}uq(M(r)) \) on the \( E_2 \)-terms. In [1], we modified the definition of the beta elements as follows: let \((s, t, r)\) be a triple of positive integers, and suppose that \( v_1^s v_2^t \in E_2^{0, (s(p+1)+c)q}(M(r, t+c)) \) with \( p^{r-1} \mid (t+c) \) for an integer \( c \). Then, the beta element for a triple \((s, t, r)\) is defined by

\[ \beta_{s/t,r} = \partial_r \partial_{r,t+c}(v_1^s v_2^t) \in E_2^{2, (s(p+1)-t)q}(S). \]

We abbreviate \( \beta_{s/t,1} \) and \( \beta_{s/1} \) to \( \beta_s \) and \( \beta_s \), respectively, as usual. In the case where \( c = 0 \), the definition is the ordinary one. Besides, these elements generate the \( E_2 \)-term by [1] Th. 2.6). It is an interesting problem asking which of them survives in the spectral sequence. So far, the following elements are known to be permanent cycles:

\[ \begin{align*}
\alpha & \beta_s \text{ for } s \geq 1 \text{ in [12],} \\
\beta_{s/p} & \text{ for } s \geq 1 \text{ and } t \leq p, \text{ and } t < p \text{ if } s = 1 \text{ in [2], [3],} \\
\beta_{s/p^2} & \text{ for } s \geq 1 \text{ and } t \leq 2p, \text{ and } t \leq 2p - 2 \text{ if } s = 1 \text{ in [2], [4],} \\
\beta_{s/p^2} & \text{ for } s \geq 1 \text{ and } t \leq p^2 - 2 \text{ in [1],} \\
\beta_{s/p^n} & \text{ for } s \geq 1, \text{ and } 1 \leq t \leq 2^{n-2}p, \text{ and } t \leq 2^{n-2}(p - 1) \text{ if } s = 1 \text{ in [6], [7],} \\
\beta_{s/p^n} & \text{ for } s \geq 2 \text{ in [4], and} \\
\beta_{s/p^n} & \text{ for } s \geq 1, \text{ and } 1 \leq u \leq 2^{n-2}, \text{ and } up \leq 2^{n-2}(p - 1) \text{ if } s = 1 \text{ in [6], [7].} 
\end{align*} \]

We note that we have \( \beta_{s/p^n} \) for \( t \leq p^n + p^{n-1} - 1 \) and \( t \leq p^n \) if \( s = 1 \) in the \( E_2 \)-term by [1], and Ravenel showed that \( \beta_{p^n/p} \) cannot be a permanent cycle for \( n \geq 1 \) (cf. [9] 6.4.2. Th.|). Thus, the beta elements \( \beta_{s/p^n} \) for \( 2^{n-2}p < t \leq p^n + p^{n-1} - 1 \) and \( t < p^n \) if \( s = 1 \) are left undetermined.

In this paper we modify the definition further.

**Definition 1.4.** Let \( b(s; t, r) \) denote a set of elements \( x \) of \( E_2^{0, (s(p+1)+c)q}(M(r, t+c)) \) such that \( x \equiv v_1^s v_2^t \mod (p, v_1^{c+1}) \) for a non-negative integer \( c \). We define the beta coset by

\[ \widehat{\beta_{s/t,r}} = \partial_r \partial_{r,t+c}(b(s; t, r)) \subset E_2^{2, (s(p+1)+c-up^{-1})q}(S). \]

We also abbreviate \( \widehat{\beta_{s/t,1}} \) to \( \widehat{\beta_{s/t}} \).

In [6] Th. II, Oka showed a possibility of many beta elements in the same dimension. This indicates that a difference of elements of \( \widehat{\beta_{s/t,r}} \) may be another beta element. In this paper, we study a beta element \( \beta_{s/t,r} \) with a larger \( t \), not the \( p \)-rank of \( \pi_s(S) \), and we introduce the notation. We further abuse a term.
Definition 1.5. We say that the beta coset \( \hat{\beta}_{s/t,r} \) survives to the homotopy groups \( \pi_*(S) \) if an element of \( \hat{\beta}_{s/t,r} \) is a permanent cycle. In this case, the \( \beta \)-element \( \beta_{s/t,r} \) of \( \pi_*(S) \) denotes one of survivors of \( \hat{\beta}_{s/t,r} \).

In this paper, we consider the beta cosets \( \hat{\beta}_{s/t,r} \) for \( r = 1,2 \), and so we consider the following spectra and maps of (1.1):

\[
M = M(1), \quad \overline{M} = M(2), \quad K_u = M(1,u) \quad \text{and} \quad \overline{K}_u = M(2,up); \quad \text{and}
\]

\[
k = k_1, \quad \overline{k} = k_2, \quad \alpha = A_0, \quad A = A_1, \quad k_u = k_{1,u} \quad \text{and} \quad \overline{k}_u = k_{2,up}
\]

for \( u > 0 \), where \( k \) stands for \( i \) and \( j \). (We use the notation \( M \) and \( K_u \) following those of Oka [7].) Thus, from now on, \( i_u \) and \( j_u \) denote \( i_{1,u} \) and \( j_{1,u} \), not the maps \( i_u \) and \( j_u \) in (1.1).

The above definitions make Oka’s method developed in [4] and [7] simple: Let \( f_{s,u} \in \pi_*(K_u) \) be an element such that \( \eta_*(f_{s,u}) = v_2^s \in BP_*(K_u) = BP_*/(p,v_u^n) \) for the unit map \( \eta: S \to BP \) of the ring spectrum \( BP \), and put

\[
\mathcal{B}_Oka(s,u) = \mathcal{B}_{Oka}^0(s,u) \cup \mathcal{B}_{Oka}^1(s,u),
\]

\[
\mathcal{B}_{Oka}^0(s,u) = \{ \hat{\beta}_{skp_{n}/t} : k \geq 1, \ n \geq 0, \ 1 \leq t \leq 2^n u \},
\]

\[
\mathcal{B}_{Oka}^1(s,u) = \{ \hat{\beta}_{skp_{n}/tp,2} : k \geq 1, \ n \geq 1, \ t \geq 1, \ tp \leq 2^{n-1} u \}.
\]

Theorem 1.7 (Oka [6], [7]). If \( f_{s,u} \in \pi_*(K_u) \) exists, then every element of \( \mathcal{B}_Oka(s,u) \) survives to \( \pi_*(S) \).

Consider \( \mathcal{B}_Oka((a,b),u) = \bigcup_{k,l \geq 0} \mathcal{B}_Oka(ak+bl,u) \). Since Oka also showed the existence of \( f_{sp,u} \in \pi_*(K_u) \) for \( s = 2,3 \) and \( u \leq p \) and for \( s = 1 \) and \( u < p \) in [2] Th. C] and [3] Ch. II], the theorem implies that every element of \( \mathcal{B}_Oka(p,p-1) \cup \mathcal{B}_Oka((2p,3p),p) \) survives to \( \pi_*(S) \), which is the theorem [6] Th. I, and yields elements in b), c), e), f) and g) in (1.3).

Let \( W \) be the cofiber of the generator \( \beta_1 \in \pi_{pq-2}(S) \), and we have a cofiber sequence

\[
S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{iW} W \xrightarrow{jW} S^{pq-1}.
\]

In [10], we introduce a method to obtain a beta family from \( f_{s,u} \in \pi_*(W \wedge K_u) \) such that \( \eta_*(f_{s,u}) = v_2^s \in BP_*(W \wedge K_u) \). In this paper, we merge these methods. For an element \( f_{p',u} \in \pi_*(W \wedge K_u) \), consider a family

\[
\mathcal{B}(p',u) = \mathcal{B}_Oka^0(p',u) \cup \mathcal{B}_Oka^1(p',u), \quad \text{where}
\]

\[
\mathcal{B}_Oka^0(p',u) = \{ \hat{\beta}_{sp_{n}/t} : s \geq 1, \ n \geq 0, \ 1 \leq t \leq 2^n u - 2 \},
\]

\[
\mathcal{B}_Oka^1(p',u) = \{ \hat{\beta}_{sp_{n}/tp,2} : s \geq 1, \ n \geq 1, \ t \geq 1, \ tp \leq 2^{n-1} u - p \}.
\]

Theorem 1.9. If \( f_{p',u} \in \pi_*(W \wedge K_u) \) exists, then every element of \( \mathcal{B}(p',u) \) survives to \( \pi_*(S) \).

In [11] Th. 1.7], we showed the existence of \( f_{p^2,p^2} \in \pi_*(W \wedge K_{p^2}) \) for \( p > 5 \), though there does not exist \( f_{p^2,p^2} \in \pi_*(K_{p^2}) \) shown by Ravenel.

Corollary 1.10. Let \( p > 5 \). Then, \( \mathcal{B}(p^2,p^2) \) yields a beta family of \( \pi_*(S) \).

This implies our main theorem stated above. This improves Oka’s results (1.5 e) and g) if the prime number \( p \) is greater than five.
2. Recollection of the finite ring spectrum

In this section, we recall some results of Oka. We call a spectrum $E$ a ring spectrum if it admits a multiplication $\mu: E \wedge E \to E$ and a unit $\iota: S \to E$ such that $\mu(1 \wedge 1) = 1 = \mu(1 \wedge \iota)$ and $\mu(\mu \wedge 1) = \mu(1 \wedge \mu)$. A ring spectrum $E$ is commutative if $\mu T = \mu$ for the switching map $T: E \wedge E \to E \wedge E$. The homotopy groups $E_* = \pi_*(E)$ of $E$ have a multiplication given by $ab = \mu(a \wedge b)$ for $a, b \in E_*$, which makes $E_*$ a ring. Oka \cite{Oka7} (cf. \cite{Oka8}) defined $\text{Mod}(E)$ and $\text{Der}(E)$ by

$$\text{Mod}(E) = \{ f \in [E, E]_* \mid \mu(f \wedge 1) = f \mu \} \quad \text{and} \quad \text{Der}(E) = \{ f \in [E, E]_* \mid \mu(f \wedge 1) + \mu(1 \wedge f) = f \mu \}.$$  

We call an element of $\text{Der}(E)$ a derivation of $E$.

**Theorem 2.1** (Oka \cite{Oka8} Lemma 1.3). For the unit $\iota$, the induced homomorphism $\iota^*: \text{Mod}(E) \to E_*$ is a ring isomorphism. Its inverse $\kappa: E_* \to \text{Mod}(E)$ is given by $\kappa(f) = \mu(f \wedge 1)$.

Consider a spectrum $K_u$ in (1.6). Then, Oka showed that

**Theorem 2.2** (Oka \cite{Oka7} Th. 2.5). $K_u$ has a commutative and associative multiplication $m_u$.

**Theorem 2.3** (Oka \cite{Oka7} Lemma 2.3). $\text{Mod}(K_u)$ is a commutative subring of $[K_u, K_u]_*$, and a commutator $[f, g]$ belongs to $\text{Mod}(K_u)$ for $f \in \text{Mod}(K_u)$ and $g \in \text{Der}(K_u)$. In particular,

$$f^p g = gf^p \quad \text{for } f \in \text{Mod}(K_u) \text{ and } g \in \text{Der}(K_u).$$  

Let $\delta'_u = i_u j_u \in [K_u, K_u]_{-uq-1}$. Then it fits into a cofiber sequence

$$\Sigma^{uq} K_u \xrightarrow{i_u} K_{2u} \xrightarrow{j_u} K_u \xrightarrow{\delta'_u} \Sigma^{uq+1} K_u$$

by (1.1) with a $3 \times 3$ Lemma.

**Theorem 2.5** (Oka \cite{Oka7} Th. 2.5). $\delta'_u \in \text{Der}(K_u)$.

It is well known that $\delta = ij \in \text{Der}(M)$ and $\alpha \in \text{Mod}(M)$, and so $\alpha^p \delta = \delta \alpha^p \in [M, M]_{pq-1}$. It gives rise to not only the element $A = A_1$ in (1.1) but also $\delta_u$ in the commutative diagram

$$\Sigma^{upq-1} M \xrightarrow{\delta} \Sigma^{upq} M \xrightarrow{\pi} \Sigma^{upq} M \xrightarrow{\delta} \Sigma^{upq} M$$

$$\begin{array}{ccc}
\Sigma^{1} M M & \xrightarrow{\delta} & M \\
\xrightarrow{i_u} & \xrightarrow{j_u} & \xrightarrow{i_u} \\
\Sigma^{1} K_{up} & \xrightarrow{\delta_u} & K_{up} \\
\xrightarrow{j_u} & \xrightarrow{i_u} & \xrightarrow{j_u} \\
\Sigma^{upq} M & \xrightarrow{-\delta} & \Sigma^{upq+1} M \xrightarrow{\pi} \Sigma^{upq+1} M \xrightarrow{\delta} \Sigma^{upq+1} M
\end{array}$$

in which rows and columns are cofiber sequences. By \cite{Oka9} Lemma 4.5, Th. 4.2, we have the following:
Theorem 2.7 (Oka [7 p. 425]). The map $\delta_u$ in the above diagram is a derivation of $K_{up}$.

Proof. The matrices for the map $\delta_u \wedge 1$ and the switching map $T$ are given by

$$\tau(\delta_u \wedge 1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta & -1 & 0 & 0 \\ \delta' & 0 & -1 & 0 \\ \delta \delta' & \delta' & -\delta & 1 \end{pmatrix}$$

by [6 Lemma 4.5] and [6 Th. 4.2], and so the first row of the matrix for $(\delta_u \wedge 1) + (\delta_u \wedge 1)T$ is $(\delta 0 0 0)$. Since the multiplication $m_u$ is the projection to the first summand, we see that $m_u(\delta_u \wedge 1 + \delta_u) = m_u((\delta_u \wedge 1) + T(\delta_u \wedge 1)T) = m_u((\delta_u \wedge 1) + (\delta_u \wedge 1)T) = \delta_u m_u$, as desired. \qed

The following lemma is a folklore:

Lemma 2.8. There exist self-maps $\tilde{\alpha} : \Sigma^q K_u \to K_u$ and $\tilde{A} : \Sigma^p K_u \to K_u$ such that $BP_*(\tilde{\alpha}) = \nu_1$ and $BP_*(\tilde{A}) = \nu'_1$.

Lemma 2.9. $A\tilde{\beta}_1 = 0 \in \pi_{2pq-2}(\overline{M})$.

Proof. Consider the cobar complex $\{C^s, d\}_{s \geq 0}$ whose cohomology is the $E_2$-term $E^2_3(\overline{M})$ of the Adams-Novikov spectral sequence converging to $\pi_*(\overline{M})$. Then, $C^s = \Gamma/(p^2) \otimes \Gamma^{s-1}$, where $(A, \Gamma) = (BP_*, BP_*BP)$, and the differential $d$ of the complex is given by derivation with $d(v) = \eta_R(v) - \eta_L(v) \in C^1 = \Gamma/(p^2)$ for $v \in C^0 = A/(p^2)$ and $d(x) = 1 \otimes x - \Delta(x) + x \otimes 1 \in C^2$ for $x \in C^1$. We also use the formulas on the structure maps of the Hopf algebra given by the formulas of Quillen and Hazewinkel:

$$\eta_R(v_1) = v_1 + pt_1, \quad \eta_R(v_2) \equiv v_2 + v_1 t_1^p + pt_2 - (p+1)v_1 t_1 \mod (p^2),$$

$$\Delta(t_1) = 1 \otimes t_1 + t_1 \otimes 1 \quad \text{and} \quad \Delta(t_2) = 1 \otimes t_2 + t_1 \otimes t_1^p + t_1^p \otimes 1 + v_1 b_{10}.$$

Here, $b_{10}$ denotes the cocycle defined by $d(t_1^p) = pb_{10}$. Let $b_0$ denote the cohomology class of $b_{10}$. Then, by definition, $\beta_1 = \delta_1 \delta_1(\nu_2) = b_0$. Therefore, $A\tilde{\beta}_1$ is detected by $v_1 b_0 \in E^{2,2pq}_2(\overline{M})$. We compute that $d(c) = v_1 b_{10} \in C^2$ for $c = -v_1^{p-3} t_2 + v_1^{p-3} (v_1 t_1 - p t_2) \eta_R(v_2) + v_1^{2p-3} \left( \frac{p+1}{2} v_1 t_1^2 - \frac{p}{3} t_1 \right)$. It follows that $v_1^p b_0 = 0 \in E^{2,2pq}_2(\overline{M})$. We further see that $E^{2+r+r+2pq}_2(\overline{M}) = 0$ for $r \geq 1$ by the vanishing line (cf. [11 Lemma 1.16, Remark 1.17]). Hence $A\tilde{\beta}_1 = 0 \in \pi_{2pq-2}(\overline{M})$. \qed

3. Arguments on $W \wedge K_u$

In this section, we fix an integer $u$, and $K$ denotes $K_u = (1, u)$, which is a commutative ring spectrum with multiplications $m = m_u$ by Theorem 2.2. The spectrum $W$ in [1,8] admits a multiplication $m_W : W \wedge W \to W$ such that $m_W(iW \wedge 1W) = 1W = m_W(1W \wedge iW)$ by [5 Example 2.9].

Consider the spectrum $WM = W \wedge M$ and the multiplication $m_{WM} = (m_W \wedge m_M)(1W \wedge T \wedge 1M) : WM \wedge WM \to WM$. Then, we see that

$$m_W(i' \wedge i') = i'm_W,$$

for $i' = 1W \wedge i$. We further see the following lemma by [6 Th. 4.13]:

Lemma 3.2. $WM$ is a commutative and associative ring spectrum with multiplication $m_{WM}$. 

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Proof. Since \( p = 0 \in [WM, WM]_0 \) and \( \beta_1 \cdot 1 = 0 \in [M \wedge WM, M \wedge WM]_{pq-2} \), \( WM \) is a split ring spectrum. (See [6, Def. 2.1] for the definition of a split ring spectrum.)

Put \( WMK = W \wedge M \wedge K \), and consider a multiplication \( m_{WMK} = (m_{WM} \wedge m)(1_{WM} \wedge T \wedge 1_K) : WMK \wedge WMK \xrightarrow{1_{WM} \wedge T \wedge 1_K} WM \wedge WM \wedge K \wedge K \xrightarrow{m_{WM} \wedge m} WMK \) on \( WMK \). Since the smash product of commutative ring spectra is a commutative ring spectrum, we have the following

**Corollary 3.3.** \( WMK \) is a commutative ring spectrum with multiplication \( m_{WMK} \).

Consider the spectrum \( WK = W \wedge K \) and a multiplication \( m_{WK} \) on \( WK \) defined by \( m_{WK} = (m_W \wedge m)(1_W \wedge T \wedge 1_K) \) for the switching map \( T : K \wedge W \to W \wedge K \). We have the split cofiber sequence

\[
(WK \xrightarrow{i} WMK \xrightarrow{j} \Sigma WK,)
\]

in which \( \hat{i} = 1_W \wedge i \wedge 1_K : WK \to WMK \) and \( \sigma \) denotes a splitting.

**Lemma 3.5.** \( \hat{i}m_{WK} = m_{WMK}(\hat{i} \wedge \hat{i}) \).

Proof. This follows from computation:

\[
\hat{i}m_{WK} = (i' \wedge 1_K)(m_W \wedge m)(1_W \wedge T \wedge 1_K) = (m_{WM} \wedge m)(i' \wedge i' \wedge 1_K)(1_W \wedge T \wedge 1_K) \quad \text{by (3.1)}
\]

\[
= (m_{WM} \wedge m)(1_{WM} \wedge T \wedge 1_K)(\hat{i} \wedge \hat{i}) = m_{WMK}(\hat{i} \wedge \hat{i}). \quad \square
\]

**Lemma 3.6.** The spectrum \( WK \) is a commutative ring spectrum with multiplication \( m_{WK} \).

Proof. Apply \( \sigma \) in (3.4) to Lemma 3.5 and we have

\[
m_{WK} = \sigma m_{WMK}(\hat{i} \wedge \hat{i}). \quad (3.7)
\]

Then, noticing that \( m_{WMK}T' = m_{WMK} \) by Corollary 3.3

\[
m_{WK}Tm_{WMK}(\hat{i} \wedge \hat{i})T = \sigma m_{WMK}T'(\hat{i} \wedge \hat{i}) = \sigma m_{WMK}(\hat{i} \wedge \hat{i}) = m_{WK}.
\]

Here, \( T : WK \wedge WK \to WK \wedge WK \) and \( T' : WMK \wedge WMK \to WMK \wedge WMK \) are the switching maps. The associativity of it is verified as follows:

\[
m_{WK}(m_{WK} \wedge 1_{WK}) = \sigma m_{WMK}(\hat{i} \wedge \hat{i})(m_{WK} \wedge 1_{WK}) \quad \text{by (3.7)}
\]

\[
= \sigma m_{WMK}(m_{WMK} \wedge 1_{WMK})(\hat{i} \wedge \hat{i} \wedge \hat{i}) \quad \text{(by Lemma 3.5)}
\]

\[
= \sigma m_{WMK}(1_{WMK} \wedge m_{WMK})(\hat{i} \wedge \hat{i} \wedge \hat{i}) \quad \text{(by Corollary 3.3)}
\]

\[
= \sigma m_{WMK}(\hat{i} \wedge \hat{i})(1_{WK} \wedge m_{WK}) \quad \text{(by Lemma 3.5)}
\]

\[
= m_{WK}(1_{WK} \wedge m_{WK}). \quad \square
\]

Consider the homomorphism \( \varphi_W : [K, K]_* \to [WK, WK]_* \) given by \( \varphi_W(f) = 1_W \wedge f \). Then, an easy computation shows

**Lemma 3.8.** The homomorphism \( \varphi_W \) induces those of \( \varphi_W : \text{Mod}(K) \to \text{Mod}(WK) \) and \( \varphi_W : \text{Der}(K) \to \text{Der}(WK) \).
4. CONSTRUCTION OF HOMOTOPY ELEMENTS

By definition, we see that $BP_\ast(W \wedge X) = BP_\ast(X) \oplus gBP_\ast(X)$ for a spectrum $X$ such that $BP_\ast(X) = 0$ unless $q \mid s$, in which the degree of the generator $g$ is $pq - 1$. This also implies the relation $E_2^s(W \wedge X) = E_2^s(X) \oplus gE_2^s(X)$ of the Adams-Novikov $E_2$-term. We apply this for $X = K_u$ and $\overline{K}_u$.

We begin with a general result corresponding to Oka’s theorems [6, Th. 7.1, Th. 7.2] and [7, Construction III].

Proposition 4.1. Let $f \in \pi_\ast(WK_u)$ be an element such that $\eta_\ast(f) \equiv v_2^s \mod (v_1)$ for the unit map $\eta$ of $BP$. Then,

1) for $n \geq 0$, there is an element $f_n \in \pi_\ast(WK_{2n+u})$ such that $\eta_\ast(f_n) \equiv v_2^{sp^n} \mod (v_1)$, and

2) for $n \geq 1$ and a positive integer $u'$ such that $u'p \leq 2^{n-1}u$, there is an element $\tilde{f}_n \in \pi_\ast(WK_{u'})$ such that $\eta_\ast(\tilde{f}_n) \equiv v_2^{sp^n} \mod (p, v_1)$. Here, $WK_u = W \wedge \overline{K}_u$.

Proof. Put $f_0 = f$, and suppose the existence of $f_n \in \pi_\ast(WK_{2n+u})$. Then, $\kappa(f_n) \in \text{Mod}(WK_{2n+u})$ for $\kappa$ in Theorem 2.1. By Theorem 2.2 and Lemma 3.8, $\delta' = \varphi_{W}(\delta_{2n+u}) \in \text{Der}(WK_{2n+u})$, and so we have $\kappa(f_n)^p \delta' = \delta' \kappa(f_n)^p$ by Theorem 2.3.

Thus we obtain a map $\tilde{f}_{n+1}$, which makes the diagram

$$
\begin{array}{ccc}
WK_{2n+u} & \longrightarrow & WK_{2n+u} \\
\tilde{f}_{n+1} \downarrow & \delta & \downarrow \kappa(f_n)^p \\
WK_{2n+u} & \longrightarrow & WK_{2n+u}
\end{array}
$$

commute. Here, the rows of the diagram are the cofiber sequence of (2.4). Now put $f_{n+1} = (i_W \wedge 2^{n+1}u)^* (\tilde{f}_{n+1})$ to complete the induction.

To prove the second part of Proposition 4.1. By 1), we have $f_{n-1} \in \pi_\ast(WK_{2n-1+u})$ for $n \geq 1$, and so an element $f'_{n-1} \in \pi_\ast(WK_{u'}p)$ for $u'p \leq 2^{n-1}u$ (cf. [7, Construction I]). Theorem 2.7, Lemma 3.8, and Theorem 2.3 imply that

$$
\kappa(f'_{n-1})^p \delta'' = \delta'' \kappa(f'_{n-1})^p
$$

for $\delta'' = \varphi_{W}(\delta_{u'})$, where $\delta_{u'}$ is the map in (2.6). We then have a map $\tilde{f}_n$ fitting into the commutative diagram

$$
\begin{array}{ccc}
WK_{u'} & \longrightarrow & WK_{u'} \\
\tilde{f}_n \downarrow & \delta'' & \downarrow \kappa(f'_{n-1})^p \\
WK_{u'} & \longrightarrow & WK_{u'}
\end{array}
$$

in which the rows of the diagram are the cofiber sequence in (2.6). Now the map $\tilde{f}_n$ is obtained by $(i_W \wedge i_{u'}i)^* (\tilde{f}_n)$. $\square$

In [10], we show the following lemma:

Lemma 4.2 ([10, Lemma 2.11]). For $u > 2$, there exists an element $\omega_u \in \pi_{(p+2)q-1}(WK_u)$ such that $(j_W)_\ast(\omega_u) = i_u \alpha^2 i \in \pi_{2q}(K_u)$. Moreover, $v_2^2 g \in E_2^q(WK_u) = E_2^q(K_u) \oplus gE_2^q(K_u)$ detects it.
A similar lemma follows from Lemma 2.9

**Lemma 4.3.** For \( u > 1 \), there exists an element \( \varpi_u \in \pi_{2pq-1}(W \mathcal{K}_u) \) such that \( (j_w)_*(\varpi_u) = \tilde{\eta}_u \alpha \tilde{\eta}_u \subset \pi_{pq}(\mathcal{K}_u) \). Moreover, \( v_2^1 g \in E_2^0(W \mathcal{K}_u) = E_2^0(\mathcal{K}_u) \oplus g E_2^0(\mathcal{K}_u) \) detects it.

**Proof of Theorem 1.9** By Proposition 4.1 we have elements \( (f_{p^* u})_n \in \pi_n(W \mathcal{K}_{u'}) \) for \( n \geq 0 \) and \( (f_{p^* u})_n \in \pi_n(W \mathcal{K}_{u'}) \) for \( n > 0 \) and \( u'p \leq 2^{n-1}u \) such that \( \eta_*(f_{p^* u})_n \equiv v_2^{p^* + n} \mod (v_1) \) and \( \eta_*(f_{p^* u})_n \equiv v_2^{p^* + n} \mod (p, v_1) \). Consider the composites

\[
B_{sp^{i+n}/2u-r} = \alpha^{-2}(j_w)_*(1_B) \kappa((f_{p^* u})_n) \omega^{2u} \quad (r \geq 2)
\]

\[
B_{sp^{i+n}/u'p-r'} = A^{-1}(j_w)_*(1_B) \kappa((f_{p^* u})_n) \omega^{u'} \quad (r \geq 1),
\]

where \( \alpha, \tilde{\alpha}, \omega_u \) and \( \varpi_u \) are the elements of Lemmas 2.8, 4.2 and 4.3, and \( \kappa \) is the homomorphism in Theorem 2.1. Then, \( \eta_*(B_{sp^{i+n}/2u-r}) \in b(s^{i+n}; 2^{u-r} - 1) \) and \( \eta_*(B_{sp^{i+n}/u'p-r'}) \in b(s^{i+n}; u'p - r' - 2) \). Since \( j, j_2, j_1, j_{u'} \) correspond to \( \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_u, \tilde{\eta}_2 \) respectively, the elements \( j_{u'} B_{sp^{i+n}/2u-r} \) and \( \tilde{j}_{u'} B_{sp^{i+n}/u'p-r'} \) are detected by elements of \( \beta_{sp^{i+n}/2u-r} \) and \( \tilde{\beta}_{sp^{i+n}/u'p-r'} \), as desired. \( \square \)

**REFERENCES**


