A BETA FAMILY IN THE HOMOTOPY OF SPHERES

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(Communicated by Brooke Shipley)

Abstract. Let $p$ be a prime number greater than three. In the $p$-component of stable homotopy groups of spheres, Oka constructed a beta family from a $v_2$-periodic map on a four cell complex. In this paper, we construct another beta family in the groups at a prime $p$ greater than five from a $v_2$-periodic map on an eight cell complex.

1. Introduction

We fix a prime number $p$ greater than three and work in the stable homotopy category $S(p)$ of spectra localized at the prime $p$. Let $S$ and $BP$ in $S(p)$ denote the sphere and the Brown-Peterson spectra. It is important to understand the homotopy groups $\pi_*(S)$, whose structure is little known. On the other hand, we know the structures of $\pi_*(BP) = BP_*$ and $BP_*(BP)$:

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \quad \text{and} \quad BP_*(BP) = BP_*[t_1, t_2, \ldots]$$

and $BP_*(BP)$ is a Hopf algebroid over $BP_*$. Here, the generators have degrees $|v_k| = |t_k| = 2(p^k - 1)$. Furthermore, we have the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$ of a spectrum $X$ with $E_2$-term

$$E_2^{s,t}(X) = \text{Ext}_{BP_*^{BP}}^{s,t}(BP_*, BP_*(X)),$$

and the spectral sequence for $X = S$ acts as a go-between between $BP$ and $S$. Here we consider the homotopy groups $\pi_*(S)$ through the spectral sequence. In the $E_2$-term $E_2^{s,t}(S)$, Miller, Ravenel and Wilson [11] defined the beta elements $\beta_{s,t,r}$ for suitable triples $(s, t, r)$ of positive integers. In this paper, we merge the methods of Oka and the authors to find permanent cycles among the beta elements and obtain a family of beta elements in the homotopy groups of the sphere spectrum.

Theorem. Let $p$ be a prime number greater than five. We have following beta elements in the homotopy groups $\pi_*(S)$ detected by the beta elements in the $E_2$-term:

a) $\beta_{sp^n/t} \in \pi_{(sp^n(p+1)-t)q-2}(S)$ for $s \geq 1$, $n \geq 2$, $1 \leq t \leq 2^{n-2}p^2 - 2$ and

b) $\beta_{sp^n/up,2} \in \pi_{(sp^n(p+1)-up)q-2}(S)$ for $s \geq 1$, $n \geq 3$, $1 \leq u \leq 2^{n-3}p - 1$.

The orders of elements of a) and b) are no less than $p$ and $p^2$, respectively.
From now on, we state the definitions, known results and theorems obtained in this paper. Consider the spectra and the maps defined by the cofiber sequences:

\begin{equation}
S^r \xrightarrow{r} S \xrightarrow{1} M(r) \xrightarrow{j} \Sigma S \quad \text{and} \quad \Sigma^p q M(r) \xrightarrow{j} \Sigma M(r) \xrightarrow{i} \Sigma^{p-1} q M(r),
\end{equation}

where \( A_r \) denotes an element such that \( BP_r(A_r) = v^r_1 \) for \( r \geq 0 \) (cf. [9 Th. 6.2]); see also (2.6)), and \( A_0 \) is known as the Adams map and denoted by \( \alpha \). Hereafter, \( q = 2p - 2 \). We note that \( BP_r(M(r)) = BP_r(p^r) \) and \( BP_r(M(r, up^{-1})) = BP_r((p^r, v^r_1)) \) as \( BP_r(BP) \)-comodules. The cofiber sequences in (1.1) induce the connecting homomorphisms \( \partial_r : E^s_{2,t}(M(r)) \rightarrow E^{s+1}_{2,t}(S) \) and \( \partial_{r, up^{-1}} : E^s_{2,t}(M(r, up^{-1})) \rightarrow E^{s+1}_{2,t-1}(q(M(r))) \) on the \( E_2 \)-terms. In [11], we modified the definition of the beta elements as follows: let \( (s, t, r) \) be a triple of positive integers, and suppose that \( v_1^s v_2^q \in E^{0,(s(p+1)+c)q}_{2}(M(r, t+c)) \) with \( p^{r-1} \mid (t + c) \) for an integer \( c \). Then, the beta element for a triple \( (s, t, r) \) is defined by

\begin{equation}
\beta_{s/t,r} = \partial_r \partial_{r,t+c}(v_1^s v_2^q) \in E^{2,(s(p+1)-t)q}_{2}(S).
\end{equation}

We abbreviate \( \beta_{s/t,1} \) and \( \beta_{s/1} \) to \( \beta_s \) and \( \beta_t \), respectively, as usual. In the case where \( c = 0 \), the definition is the ordinary one. Besides, these elements generate the \( E_2 \)-term by [1 Th. 2.6]. It is an interesting problem asking which of them survives in the spectral sequence. So far, the following elements are known to be permanent cycles:

\begin{align}
&\text{a)} \quad \beta_s \text{ for } s \geq 1 \text{ in [12]}, \\
&\text{b)} \quad \beta_{s/p} \text{ for } s \geq 1 \text{ and } t \leq p, \text{ and } t < p \text{ if } s = 1 \text{ in [2], [3],} \\
&\text{c)} \quad \beta_{s/p^2} \text{ for } s \geq 1 \text{ and } t \leq 2p, \text{ and } t < 2p - 2 \text{ if } s = 1 \text{ in [2], [4],} \\
&\text{d)} \quad \beta_{s/p^2} \text{ for } s \geq 1 \text{ and } t \leq p^2 - 2 \text{ in [11],} \\
&\text{e)} \quad \beta_{s/p^n} \text{ for } s \geq 1, n \geq 3, 1 \leq t \leq 2^{n-2} - 2, \text{ and } t < 2^{n-2} - (p - 1) \text{ if } s = 1 \text{ in [6], [7],} \\
&\text{f)} \quad \beta_{s/p^n/p_2} \text{ for } s \geq 2 \text{ in [4],} \\
&\text{g)} \quad \beta_{s/p^n/p_2} \text{ for } s \geq 1, n \geq 3, 1 \leq u \leq 2^{n-2}, \text{ and } up < 2^{n-2} - (p - 1) \text{ if } s = 1 \text{ in [6], [7].}
\end{align}

We note that we have \( \beta_{s/p^n} \text{ for } t \leq p^n + p^{n-1} - 1 \) and \( t \leq p^n \) if \( s = 1 \) in the \( E_2 \)-term by [1], and Ravenel showed that \( \beta_{p^n}/p^n \) cannot be a permanent cycle for \( n \geq 1 \) (cf. [9 6.4.2. Th.]). Thus, the beta elements \( \beta_{s/p^n} \text{ for } 2^{n-2}p < t \leq p^n + p^{n-1} - 1 \) and \( t < p^n \) if \( s = 1 \) are left undetermined.

In this paper we modify the definition further.

**Definition 1.4.** Let \( b(s; t, r) \) denote a set of elements \( x \) of \( E^{0,(s(p+1)+c)q}_{2}(M(r, t+c)) \) such that \( x \equiv v_1^s v_2^q \mod (p, v_1^{c+1}) \) for a non-negative integer \( c \). We define the beta coset by

\[
\tilde{\beta}_{s/t,r} = \partial_r \partial_{r,t+c}(b(s; t, r)) \in E^{2,(s(p+1)+c-up^{r-1})q}_{2}(S).
\]

We also abbreviate \( \tilde{\beta}_{s/t,1} \) to \( \tilde{\beta}_{s/t} \).

In [6 Th. II], Oka showed a possibility of many beta elements in the same dimension. This indicates that a difference of elements of \( \tilde{\beta}_{s/t,r} \) may be another beta element. In this paper, we study a beta element \( \tilde{\beta}_{s/t,r} \) with a larger \( t \), not the \( p \)-rank of \( \pi_s(S) \), and we introduce the notation. We further abuse a term.
Definition 1.5. We say that the beta coset $\hat{\beta}_{s/t,r}$ survives to the homotopy groups $\pi_*(S)$ if an element of $\hat{\beta}_{s/t,r}$ is a permanent cycle. In this case, the beta-element $\beta_{s/t,r}$ of $\pi_*(S)$ denotes one of survivors of $\hat{\beta}_{s/t,r}$.

In this paper, we consider the beta cosets $\hat{\beta}_{s/t,r}$ for $r = 1, 2$, and so we consider the following spectra and maps of (1.1):

$$M = M(1), \quad M = M(2), \quad K_u = M(1, u) \quad \text{and} \quad K_u = M(2, up); \quad \text{and}$$

$$k = k_1, \quad k = k_2, \quad \alpha = A_0, \quad A = A_1, \quad k_u = k_{1,u} \quad \text{and} \quad k_u = k_{2,u}$$

for $u > 0$, where $k$ stands for $i$ and $j$. (We use the notation $M$ and $K_u$ following those of Oka [7].) Thus, from now on, $i_u$ and $j_u$ denote $i_{1,u}$ and $j_{1,u}$, not the maps $i_u$ and $j_u$ in (1.1).

The above definitions make Oka’s method developed in [6] and [7] simple: Let $f_{s,u} \in \pi_*(K_u)$ be an element such that $\eta_*(f_{s,u}) = v_s^u \in BP_*(K_u) = BP_*/(p, v_s^u)$ for the unit map $\eta$: $S \to BP$ of the ring spectrum $BP$, and put

$$\mathcal{B}_{Oka}(s,u) = \mathcal{B}_{Oka}^0(s,u) \cup \mathcal{B}_{Oka}^1(s,u),$$

$$\mathcal{B}_{Oka}^0(s,u) = \{\beta_{s,kp^n/t} : k \geq 1, \ n \geq 0, \ 1 \leq t \leq 2^n u\},$$

$$\mathcal{B}_{Oka}^1(s,u) = \{\beta_{s,kp^n/tp_2} : k \geq 1, \ n \geq 1, \ t \geq 1, \ tp \leq 2^{n-1} - u\}.$$  

Theorem 1.7 (Oka [6], [7]). If $f_{s,u} \in \pi_*(K_u)$ exists, then every element of $\mathcal{B}_{Oka}(s,u)$ survives to $\pi_*(S)$.

Consider $\mathcal{B}_{Oka}((a,b),u) = \bigcup_{k,l \geq 0, \ k+t > 0} \mathcal{B}_{Oka}(ak+bl, u)$. Since Oka also showed the existence of $f_{sp,u} \in \pi_*(K_u)$ for $s = 2, 3$ and $u \leq p$ and for $s = 1$ and $u < p$ in [2] Th. C] and [3] Th. CII], the theorem implies that every element of $\mathcal{B}_{Oka}(p,p-1) \cup \mathcal{B}_{Oka}((2p,3p), p)$ survives to $\pi_*(S)$, which is the theorem [6] Th. I], and yields elements in b), c), e), f) and g) in (13).

Let $W$ be the cofiber of the generator $\beta_1 \in \pi_{pq-2}(S)$, and we have a cofiber sequence

$$S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{iw} W \xrightarrow{jw} S^{pq-1}. \tag{1.8}$$

In [10], we introduce a method to obtain a beta family from $f_{s,u} \in \pi_*(W \wedge K_u)$ such that $\eta_*(f_{s,u}) = v_s^u \in BP_*(W \wedge K_u)$. In this paper, we merge these methods. For an element $f_{p^i,u} \in \pi_*(W \wedge K_u)$, consider a family

$$\mathcal{B}(p^i,u) = \mathcal{B}_0(p^i,u) \cup \mathcal{B}_1(p^i,u), \quad \text{where}$$

$$\mathcal{B}_0(p^i,u) = \{\beta_{sp^{i+n}/t} : s \geq 1, \ n \geq 0, \ 1 \leq t \leq 2^n u - 2\},$$

$$\mathcal{B}_1(p^i,u) = \{\beta_{sp^{i+n}/tp_2} : s \geq 1, \ n \geq 1, \ t \geq 1, \ tp \leq 2^{n-1} - u\}.$$  

Theorem 1.9. If $f_{p^i,u} \in \pi_*(W \wedge K_u)$ exists, then every element of $\mathcal{B}(p^i,u)$ survives to $\pi_*(S)$.

In [11] Th. 1.7], we showed the existence of $f_{p^2.,p^2} \in \pi_*(W \wedge K_{p^2})$ for $p > 5$, though there does not exist $f_{p^2.,p^2} \in \pi_*(K_{p^2})$ shown by Ravenel.

Corollary 1.10. Let $p > 5$. Then, $\mathcal{B}(p^2,p^2)$ yields a beta family of $\pi_*(S)$.

This implies our main theorem stated above. This improves Oka’s results (1.3 e) and g) if the prime number $p$ is greater than five.
2. Recollection of the finite ring spectrum

In this section, we recall some results of Oka. We call a spectrum $E$ a ring spectrum if it admits a multiplication $\mu: E \wedge E \rightarrow E$ and a unit $\iota: S \rightarrow E$ such that $\mu(\iota \wedge 1) = 1 = \mu(1 \wedge \iota)$ and $\mu(\mu \wedge 1) = \mu(1 \wedge \mu)$. A ring spectrum $E$ is commutative if $\mu T = \mu$ for the switching map $T: E \wedge E \rightarrow E \wedge E$. The homotopy groups $E_* = \pi_*(E)$ of $E$ have a multiplication given by $ab = \mu(a \wedge b)$ for $a, b \in E_*$, which makes $E_*$ a ring. Oka [7] (cf. [8]) defined $\text{Mod}(E)$ and $\text{Der}(E)$ by

\[
\begin{align*}
\text{Mod}(E) &= \{ f \in [E,E]_* \mid \mu(f \wedge 1) = f \mu \} \quad \text{and} \\
\text{Der}(E) &= \{ f \in [E,E]_* \mid \mu(f \wedge 1) + \mu(1 \wedge f) = f \mu \}.
\end{align*}
\]

We call an element of $\text{Der}(E)$ a derivation of $E$.

**Theorem 2.1** (Oka [8] Lemma 1.3]). For the unit $\iota$, the induced homomorphism $\iota^*: \text{Mod}(E) \rightarrow E_*$ is a ring isomorphism. Its inverse $\kappa: E_* \rightarrow \text{Mod}(E)$ is given by $\kappa(f) = \mu(f \wedge 1)$.

Consider a spectrum $K_u$ in (1.6). Then, Oka showed that

**Theorem 2.2** (Oka [7] Th. 2.5]). $K_u$ has a commutative and associative multiplication $m_u$.

**Theorem 2.3** (Oka [7] Lemma 2.3]). $\text{Mod}(K_u)$ is a commutative subring of $[K_u, K_u]_*$, and a commutator $[f, g]$ belongs to $\text{Mod}(K_u)$ for $f \in \text{Mod}(K_u)$ and $g \in \text{Der}(K_u)$. In particular,

$$f^p g = g f^p \quad \text{for} \ f \in \text{Mod}(K_u) \text{ and } g \in \text{Der}(K_u).$$

Let $\delta'_u = i_u j_u \in [K_u, K_u]_{-uq-1}$. Then it fits into a cofiber sequence

\[
\Sigma^{uq} K_u \xrightarrow{i_u} K_{2u} \xrightarrow{j_u} K_u \xrightarrow{\delta'_u} \Sigma^{uq+1} K_u
\]

by (1.11) with a $3 \times 3$ Lemma.

**Theorem 2.5** (Oka [7] Th. 2.5]). $\delta'_u \in \text{Der}(K_u)$.

It is well known that $\delta = ij \in \text{Der}(M)$ and $\alpha \in \text{Mod}(M)$, and so $\alpha^p \delta = \delta \alpha^p \in [M, M]_{pq-1}$. It gives rise to not only the element $A = A_1$ in (1.11) but also $\delta_u$ in the commutative diagram

\[
\begin{align*}
\Sigma^{upq-1} M &\xrightarrow{\delta} \Sigma^{upq} M \xrightarrow{\pi} \Sigma^{upq} M \\
\Sigma^{-1} MM &\xrightarrow{\delta} M \xrightarrow{\pi} \Sigma^{pq} M \\
\Sigma^{-1} K_{up} &\xrightarrow{\delta_u} K_{up} \xrightarrow{\pi} K_u \\
\Sigma^{upq} M &\xrightarrow{-\delta} \Sigma^{upq+1} M \xrightarrow{\pi} \Sigma^{upq+1} M
\end{align*}
\]

in which rows and columns are cofiber sequences. By [5] Lemma 4.5, Th. 4.2], we have the following:
Lemma 2.8. There exist self-maps $\tau$ of $\Sigma K_u$.

Proof. The matrices for the map $\tau$ are given by

$$
\tau_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta & -1 & 0 & 0 \\ \delta' & 0 & -1 & 0 \\ \delta' & \delta' & -\delta & 1 \end{pmatrix}
$$

by [6, Lemma 4.5] and [6, Th. 4.2], and so the first row of the matrix for $\tau$ is $(\delta \ 0 0 0)$. Since the multiplication $m_u$ is the projection to the first summand, we see that $m_u(\delta \ 1 + \delta_u) = m_u((\delta_u \ 1) + T(\delta_u \ 1)) = m_u((\delta_u \ 1) + (\delta_u \ 1)T) = \delta_u m_u$, as desired.

The following lemma is folklore:

**Lemma 2.9.** There exist self-maps $\tilde{\alpha}_i: \Sigma^k K_u \to K_u$ and $\tilde{A}_i: \Sigma^p K_u \to K_u$ such that $BP_*(\tilde{\alpha}_i) = v_1$ and $BP_*(\tilde{A}_i) = v_2^p$.

**Lemma 2.10.** $A_i \tilde{\beta}_j = 0 \in \pi_{2pq-2}(M)$.

Proof. Consider the cobar complex $\{\{C^*, d\}\}_{s \geq 0}$ whose cohomology is the $E_2$-term $E_2^s(\Sigma M)$ of the Adams-Novikov spectral sequence converging to $\pi_*(M)$. Then, $C^s = \Gamma/\langle q \rangle \otimes \Delta \Gamma^{-1}$, where $(A, \Delta) = (BP_*, BP_* BP)$, and the differential $d$ of the complex is given by derivation with $d(v) = \eta_R(v) - \eta_L(v) \in C^1 = \Gamma/\langle q \rangle$ for $v \in C^0 = C/A(\langle q \rangle)$ and $d(x) = 1 \otimes x - \Delta(x) + x \otimes 1 \in C^2$ for $x \in C^1$. We also use the formulas on the structure maps of the Hopf algebroid given by the formulas of Quillen and Hazewinkel:

$$
\eta_R(v_1) = v_1 + pt_1, \quad \eta_R(v_2) = v_2 + v_1 t_1^2 + pt_2 - (p + 1)v_1^2 t_1 \mod (q),
$$

$$
\Delta(t_1) = 1 \otimes t_1 + t_1 \otimes 1 \quad \text{and} \quad \Delta(t_2) = 1 \otimes t_2 + t_1 \otimes t_1 + t_1 \otimes 1 + v_1 b_{10}.
$$

Here, $b_{10}$ denotes the cocycle defined by $d(t_1^2) = pb_{10}$. Let $b_0$ denote the cohomology class of $b_{10}$. Then, by definition, $\tilde{\beta}_j = \partial_1 \beta_1(\tilde{v}_2) = b_0$. Therefore, $A_i \tilde{\beta}_j$ is detected by $v_1^3 b_0 \in E_2^{2,2pq}(\Sigma M)$. We compute that $d(c) = v_1^3 b_{10} \in C^2$ for $c = -v_1^{-3} t_2 + v_1^{-3} (v_1 t_1 - pt_1^2) \eta_R(v_2) + v_1^{-2} (p+1) v_1^3 t_1^2 - \frac{p}{3} t_1^2$. It follows that $v_1^3 b_0 \in E_2^{2,2pq}(\Sigma M)$. We further see that $E_2^{2,2pq+r}(M) = 0$ for $r \geq 1$ by the vanishing line (cf. [1] Lemma 1.16, Remark 1.17). Hence $A_i \tilde{\beta}_1 = 0 \in \pi_{2pq-2}(M)$.

3. Arguments on $W \wedge K_u$

In this section, we fix an integer $u$, and $K_u$ denotes $K_u = M(1, u)$, which is a commutative ring spectrum with multiplication $m_u = m_u$ by Theorem 2.2. The spectrum $W$ in [14, 15] admits a multiplication $m_W: W \wedge W \to W$ such that $m_W(i_W \wedge 1_W) = 1_W = m_W(1_W \wedge i_W)$ by [5, Example 2.9].

Consider the spectrum $W*M = W \wedge M$ and the multiplication $m_{WM} = (m_W \wedge m_M)(1_W \wedge T \wedge 1_M): WM \wedge WM \to WM$. Then, we see that

$$
m_{WM}(i' \wedge i') = i' m_W,
$$

for $i' = 1_W \wedge i$. We further see the following lemma by [6, Th. 4.13]:

**Lemma 3.2.** $WM$ is a commutative and associative ring spectrum with multiplication $m_{WM}$.
Lemma 3.8. We have the split cofiber sequence given by
\[ \text{WMK} \xrightarrow{\beta} \text{WM} \xrightarrow{\phi} \text{WK} \xrightarrow{\gamma} \Sigma \text{WMK}, \]
in which \( \beta \) and \( \gamma \) are the splitting maps of \( \text{WMK} \) and \( \text{WM} \), respectively.

Proof. Since \( p = 0 \in [\text{WM}, \text{WM}]_0 \) and \( \beta_1 \land 1 = 0 \in [\text{M}, \text{WM}, \text{M} \land \text{WM}]_{pq-2} \), \( \text{WM} \) is a split ring spectrum. (See [6, Def. 2.1] for the definition of a split ring spectrum.) \( \square \)

Put \( \text{WMK} = W \land M \land K \), and consider a multiplication \( m_{\text{WMK}} = (m_{\text{WM}} \land m)(1_{\text{WM}} \land T \land 1_K) : \text{WMK} \land \text{WMK} \xrightarrow{1_{\text{WM}} \land T \land 1_K} \text{WM} \land \text{WM} \land K \land K \xrightarrow{m_{\text{WM}} \land m} \text{WMK} \) on \( \text{WMK} \). Since the smash product of commutative ring spectra is a commutative ring spectrum, we have the following

Corollary 3.3. \( \text{WMK} \) is a commutative ring spectrum with multiplication \( m_{\text{WMK}} \).

Consider the spectrum \( \text{WK} = W \land K \) and a multiplication \( m_{\text{WK}} \) on \( \text{WK} \) defined by \( m_{\text{WK}} = (m_{\text{W}} \land m)(1_{\text{W}} \land T \land 1_K) \) for the switching map \( T : K \land W \rightarrow W \land K \).

We have the split cofiber sequence
\[ (3.4) \quad \text{WK} \xrightarrow{\hat{\iota}} \text{WMK} \xrightarrow{\hat{j}} \Sigma \text{WK}, \]
in which \( \hat{\iota} = 1_{\text{W}} \land \iota \land 1_K : \text{WK} \rightarrow \text{WMK} \) and \( \sigma \) denotes a splitting.

Lemma 3.5. \( \hat{\iota} m_{\text{WK}} = m_{\text{WMK}}(\hat{\iota} \land \hat{\iota}) \).

Proof. This follows from computation:
\[ \hat{\iota} m_{\text{WK}} = (i' \land 1_K)(m_{\text{W}} \land m)(1_{\text{W}} \land T \land 1_K) \]
\[ = (m_{\text{WM}} \land m)(i' \land i' \land 1_{K \land K})(1_{\text{W}} \land T \land 1_K) \quad \text{by (3.1)} \]
\[ = (m_{\text{WM}} \land m)(1_{\text{WM}} \land T \land 1_K)(\hat{\iota} \land \hat{\iota}) = m_{\text{WMK}}(\hat{\iota} \land \hat{\iota}). \quad \square \]

Lemma 3.6. The spectrum \( \text{WK} \) is a commutative ring spectrum with multiplication \( m_{\text{WK}} \).

Proof. Apply \( \sigma \) in (3.4) to Lemma 3.5 and we have
\[ (3.7) \quad m_{\text{WK}} = \sigma m_{\text{WMK}}(\hat{\iota} \land \hat{\iota}). \]

Then, noticing that \( m_{\text{WMK}} T' = m_{\text{WMK}} \) by Corollary 3.3
\[ m_{\text{WK}} T m_{\text{WMK}}(\hat{\iota} \land \hat{\iota}) = m_{\text{WMK}}(\hat{\iota} \land \hat{\iota}). \]

Here, \( T : \text{WK} \land \text{WK} \rightarrow \text{WK} \land \text{WK} \) and \( T' : \text{WMK} \land \text{WMK} \rightarrow \text{WMK} \land \text{WMK} \) are the switching maps. The associativity of it is verified as follows:
\[ m_{\text{WK}}(m_{\text{WK}} \land 1_{\text{WK}}) = \sigma m_{\text{WMK}}(\hat{\iota} \land \hat{\iota})(m_{\text{WK}} \land 1_{\text{WK}}) \quad \text{by (3.7)} \]
\[ = \sigma m_{\text{WMK}}(m_{\text{WMK}} \land 1_{\text{WMK}})(\hat{\iota} \land \hat{\iota} \land \hat{\iota}) \quad \text{by Lemma 3.5} \]
\[ = \sigma m_{\text{WMK}}(1_{\text{WMK}} \land m_{\text{WMK}})(\hat{\iota} \land \hat{\iota} \land \hat{\iota}) \quad \text{by Corollary 3.3} \]
\[ = m_{\text{WMK}}(\hat{\iota} \land \hat{\iota})(1_{\text{WK}} \land m_{\text{WK}}) \quad \text{by Lemma 3.5} \]
\[ = m_{\text{WK}}(1_{\text{WK}} \land m_{\text{WK}}). \quad \square \]

Consider the homomorphism
\[ \varphi_{\text{W}} : [K, K]_* \rightarrow [\text{WK}, \text{WK}]_* \]
given by \( \varphi_{\text{W}}(f) = 1_{\text{W}} \land f \). Then, an easy computation shows

Lemma 3.8. The homomorphism \( \varphi_{\text{W}} \) induces those of \( \varphi_{\text{W}} : \text{Mod}(K) \rightarrow \text{Mod}(\text{WK}) \) and \( \varphi_{\text{W}} : \text{Der}(K) \rightarrow \text{Der}(\text{WK}) \).
4. CONSTRUCTION OF HOMOTOPIE ELEMENTS

By definition, we see that $BP_s(W \wedge X) = BP_s(X) \oplus gBP_s(X)$ for a spectrum $X$ such that $BP_s(X) = 0$ unless $q \mid s$, in which the degree of the generator $q$ is $pq - 1$. This also implies the relation $E_2^*(W \wedge X) = E_2^*(X) \oplus gE_2^*(X)$ of the Adams-Novikov $E_2$-term. We apply this for $X = K_u$ and $\overline{K}_u$.

We begin with a general result corresponding to Oka’s theorems [6, Th. 7.1, Th. 7.2] and [7, Construction III].

**Proposition 4.1.** Let $f \in \pi_*(WK_u)$ be an element such that $\eta_*(f) \equiv v_s^p \mod (v_1)$ for the unit map $\eta$ of $BP$. Then,

1. for $n \geq 0$, there is an element $f_n \in \pi_*(WK_{2^n-u})$ such that $\eta_*(f_n) \equiv v_s^{sp^n} \mod (v_1)$, and
2. for $n \geq 1$ and a positive integer $u'$ such that $u'p \leq 2^{n-1}u$, there is an element $f_n \in \pi_*(WK_{u'})$ such that $\eta_*(f_n) \equiv v_s^{sp^n} \mod (p, v_1)$. Here, $WK_u = W \wedge K_u$.

**Proof.** Put $f_0 = f$, and suppose the existence of $f_n \in \pi_*(WK_{2^n-u})$. Then, $\kappa(f_n) \in \text{Mod}(WK_{2^n-u})$ for $\kappa$ in Theorem 2.1. By Theorem 2.2 and Lemma 3.8, $\delta' = \varphi_W(\delta_{2^n}) \in \text{Der}(WK_{2^n-u})$, and so we have $\kappa(f_n)^p \delta' = \delta' \kappa(f_n)^p$ by Theorem 2.3.

Thus we obtain a map $\tilde{f}_{n+1}$, which makes the diagram

$$
\begin{array}{ccc}
WK_{2^n+1-u} & \longrightarrow & WK_{2^n-u} \\
\downarrow \tilde{f}_{n+1} & & \downarrow \kappa(f_n)^p \\
WK_{2^n-u} & \longrightarrow & WK_{2^n-u}
\end{array}
$$

commute. Here, the rows of the diagram are the cofiber sequence of (2.4). Now put $f_{n+1} = (i_W \wedge 2^n i)^*(\tilde{f}_{n+1})$ to complete the induction.

To prove the second part of Proposition 4.1. By 1), we have $f_{n-1} \in \pi_*(WK_{2^{n-1}-u})$ for $n \geq 1$, and so an element $f'_{n-1} \in \pi_*(WK_{u'})$ for $u'p \leq 2^{n-1}u$ (cf. [7, Construction I]). Theorem 2.7, Lemma 3.8, and Theorem 2.3 imply that

$$
\kappa(f'_{n-1})^p \delta'' = \delta'' \kappa(f'_{n-1})^p
$$

for $\delta'' = \varphi_W(\delta_{u'})$, where $\delta_{u'}$ is the map in (2.6). We then have a map $\tilde{f}_n$ fitting into the commutative diagram

$$
\begin{array}{ccc}
WK_{u'} & \longrightarrow & WK_{u'} \\
\downarrow \tilde{f}_n & & \downarrow \kappa(f'_{n-1})^p \\
WK_{u'} & \longrightarrow & WK_{u'}
\end{array}
$$

in which the rows of the diagram are the cofiber sequence in (2.6). Now the map $\tilde{f}_n$ is obtained by $(i_W \wedge \tilde{f}_{n})(\tilde{f}_n)$.

In [10], we show the following lemma:

**Lemma 4.2 ([10, Lemma 2.1])**. For $u > 2$, there exists an element $\omega_u \in \pi_{(p+2)q-1}(WK_u)$ such that $(j_W)_*(\omega_u) = i_u \omega q - 2 \in \pi_{2q}(K_u)$. Moreover, $v_2^2 g \in E_2^0(WK_u) = E_2^0(K_u) \oplus gE_2^0(K_u)$ detects it.
A similar lemma follows from Lemma 4.3.

**Lemma 4.3.** For \( u > 1 \), there exists an element \( \pi \in \pi_{2pq-1}(W/K_u) \) such that \( jW_\ast(\pi) = \tilde{\pi} \tilde{A}^2 \in \pi_{pq}(K_u) \). Moreover, \( v^n_0 g \in E_2^0(W/K_u) = E_2^0(K_u) \oplus gE_2^0(K_u) \) detects it.

**Proof of Theorem 1.9** By Proposition 4.1, we have elements \( (f_{p',u})_n \in \pi_s(W/K_u) \) for \( n \geq 0 \) and \( (f_{p',u})_n \in \pi_s(W/K_u) \) for \( n > 0 \) and \( u'p \leq 2^{n-1}u \) such that \( \eta_*(f_{p',u})_n \equiv v_2^{p+n} \) mod \( (v_1) \) and \( \eta_*(f_{p',u})_n \equiv v_2^{p+n} \) mod \( (p,v_1) \). Consider the composites

\[
\begin{align*}
B_{sp'^{+}/2^nu-r} & = \tilde{\alpha} r^{-2}(jW \wedge 1_K) \kappa((f_{p',u})_n) \omega v_2^{u} (r \geq 2) \\
B_{sp'^{+}/u'p-rp,2} & = A^{-1}(jW \wedge 1_K) \kappa((f_{p',u})_n) \omega \nu (r \geq 1),
\end{align*}
\]

where \( \tilde{\alpha}, \tilde{A}, \omega_u \) and \( \varpi_u \) are the elements of Lemmas 2.8, 4.2 and 4.3, and \( \kappa \) is the homomorphism in Theorem 2.1. Then, \( \eta_*(B_{sp'^{+}/2^nu-1}) \in b(sp'^{+};2^nu - r, 1) \) and \( \eta_*(B_{sp'^{+}/u'p-rp,2}) \in b(sp'^{+};u'p - rp, 2) \). Since \( j, j_2, W \) and \( j, j' \) correspond to \( \partial_1, \partial_2, 2u, \partial_2 \) and \( \partial_2, u'p, \) respectively, the elements \( j_2, j_2'uB_{sp'^{+}/2^nu-r} \) and \( j_2'j uB_{sp'^{+}/u'p-rp,2} \) are detected by elements of \( \hat{\beta}_{sp'^{+}/2^nu-r} \) and \( \hat{\beta}_{sp'^{+}/u'p-rp,2} \), as desired. \( \square \)

**References**


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