SINGULAR EQUIVALENCES
INDUCED BY HOMOLOGICAL EPIMORPHISMS

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Abstract. We prove that a certain homological epimorphism between two algebras induces a triangle equivalence between their singularity categories. Applying the result to a construction of matrix algebras, we describe the singularity categories of some non-Gorenstein algebras.

1. Introduction

Let $A$ be a finite dimensional algebra over a field $k$. Denote by $A$-mod the category of finitely generated left $A$-modules and by $\mathbf{D}^b(A$-mod) the bounded derived category. Following [20], the singularity category $\mathbf{D}_{\text{sg}}(A)$ of $A$ is the Verdier quotient triangulated category of $\mathbf{D}^b(A$-mod) with respect to the full subcategory formed by perfect complexes; see also [4, 6, 16, 17, 23].

The singularity category measures the homological singularity of an algebra: the algebra $A$ has finite global dimension if and only if its singularity category $\mathbf{D}_{\text{sg}}(A)$ is trivial. Meanwhile, the singularity category captures the stable homological features of an algebra ([6]).

A fundamental result of Buchweitz and Happel states that for a Gorenstein algebra $A$, the singularity category $\mathbf{D}_{\text{sg}}(A)$ is triangle equivalent to the stable category of (maximal) Cohen-Macaulay $A$-modules ([6, 14]), where the latter category is related to Tate cohomology theory ([2, 6]). This result specializes Rickard’s result ([23]) on self-injective algebras. For non-Gorenstein algebras, not much is known about their singularity categories ([7, 9]).

The following concepts might be useful in the study of singularity categories. Two algebras $A$ and $B$ are said to be singularly equivalent if there is a triangle equivalence between $\mathbf{D}_{\text{sg}}(A)$ and $\mathbf{D}_{\text{sg}}(B)$. Such an equivalence is called a singular equivalence; compare [21]. In this case, if $A$ is non-Gorenstein and $B$ is Gorenstein, then Buchweitz-Happel’s theorem applies to give a description of $\mathbf{D}_{\text{sg}}(A)$ in terms of (maximal) Cohen-Macaulay $B$-modules. We observe that a derived equivalence of two algebras, that is, a triangle equivalence between their bounded derived categories, naturally induces a singular equivalence. The converse is not true in general.
Let \( A \) be an algebra and let \( J \subseteq A \) be a two-sided ideal. Following [22], we call \( J \) a **homological ideal** provided that the canonical map \( A \to A/J \) is a homological epimorphism ([12]), meaning that the naturally induced functor \( D^b(A/J\text{-mod}) \to D^b(A\text{-mod}) \) is fully faithful.

The main observation we make is as follows.

**Theorem.** Let \( A \) be a finite dimensional \( k \)-algebra and let \( J \subseteq A \) be a homological ideal which has finite projective dimension as an \( A\text{-}A \)-bimodule. Then there is a singular equivalence between \( A \) and \( A/J \).

This paper is structured as follows. In Section 2, we recall some ingredients and then prove the Theorem. In Section 3, we apply the Theorem to a construction of matrix algebras and then describe the singularity categories of some non-Gorenstein algebras. In particular, we give two examples which extend in different manners an example considered by Happel in [14].

### 2. Proof of the Theorem

We will present the proof of the Theorem in this section. Before that, we recall from [25] and [15] some results on triangulated categories and derived categories.

Let \( T \) be a triangulated category. We will denote its translation functor by \( [1] \). For a triangulated subcategory \( N \), we denote by \( T/N \) the Verdier quotient triangulated category. The quotient functor \( q: T \to T/N \) has the property that \( q(X) \simeq 0 \) if and only if \( X \) is a direct summand of an object in \( N \). In particular, if \( N \) is a **thick** subcategory, that is, it is closed under direct summands, we have that \( \text{Ker } q = N \). Here, for a triangle functor \( F \), \( \text{Ker } F \) denotes its essential kernel, that is, the (thick) triangulated subcategory consisting of objects on which \( F \) vanishes.

The following result is well known.

**Lemma 2.1.** Let \( F: T \to T' \) be a triangle functor which allows a fully faithful right adjoint \( G \). Then \( F \) induces uniquely a triangle equivalence \( T/\text{Ker } F \simeq T' \).

**Proof.** The existence of the induced functor follows from the universal property of the quotient functor. The result is a triangulated version of [11, Proposition I. 1.3]. For details, see [5, Propositions 1.5 and 1.6]. \( \square \)

Let \( F: T \to T' \) be a triangle functor. Assume that \( N \subseteq T \) and \( N' \subseteq T' \) are triangulated subcategories satisfying \( FN \subseteq N' \). Then there is a uniquely induced triangle functor \( \bar{F}: T/N \to T'/N' \).

**Lemma 2.2** ([20, Lemma 1.2]). Let \( F: T \to T' \) be a triangle functor which has a right adjoint \( G \). Assume that \( N \subseteq T \) and \( N' \subseteq T' \) are triangulated subcategories satisfying the fact that \( FN \subseteq N' \) and \( GN' \subseteq N \). Then the induced functor \( \bar{F}: T/N \to T'/N' \) has a right adjoint \( \bar{G} \). Moreover, if \( G \) is fully faithful, so is \( \bar{G} \).

**Proof.** The unit and counit of \( (F,G) \) induce uniquely two natural transformations \( \text{Id}_{T/N} \to GF \) and \( FG \to \text{Id}_{T'/N'} \), which are the corresponding unit and counit of the adjoint pair \( (\bar{F}, \bar{G}) \); consult [19, Chapter IV, Section 1, Theorem 2(v)]. Note that the fully-faithfulness of \( G \) is equivalent to the fact that the counit of \( (F,G) \) is an isomorphism. It follows that the counit of \( (\bar{F}, \bar{G}) \) is also an isomorphism, which is equivalent to the fully-faithfulness of \( \bar{G} \); consult [19, Chapter IV, Section 3, Theorem 1]. \( \square \)
Let $k$ be a field and let $A$ be a finite dimensional $k$-algebra. Recall that $A$-$\text{mod}$ is the category of finite dimensional left $A$-modules. We write $_AA$ for the regular left $A$-module. Denote by $D(A$-$\text{mod}$) (resp. $D^b(A$-$\text{mod}$)) the (resp. bounded) derived category of $A$-$\text{mod}$. We identify $A$-$\text{mod}$ as the full subcategory of $D^b(A$-$\text{mod}$) consisting of stalk complexes concentrated at degree zero; see [15] Proposition I. 4.3.

A complex of $A$-modules is usually denoted by $X^\bullet = (X^n, d^n)_{n \in \mathbb{Z}}$, where $X^n$ are $A$-modules and the differentials $d^n : X^n \to X^{n+1}$ are homomorphisms of modules satisfying $d^{n+1} \circ d^n = 0$. Recall that a complex in $D^b(A$-$\text{mod}$) is perfect provided that it is isomorphic to a bounded complex consisting of projective modules. The full subcategory consisting of perfect complexes is denoted by $\text{perf}(A)$. Recall from [6] Lemma 1.2.1 that a complex $X^\bullet$ in $D^b(A$-$\text{mod}$) is perfect if and only if there is a natural number $n_0$ such that for each $A$-module $M$, $\text{Hom}_{D^b(A$-$\text{mod})}(X^\bullet, M[n]) = 0$ for all $n \geq n_0$. It follows that $\text{perf}(A)$ is a thick subcategory of $D^b(A$-$\text{mod}$). Indeed, it is the smallest thick subcategory of $D^b(A$-$\text{mod}$) containing $_AA$.

Let $\pi : A \to B$ be a homomorphism of algebras. The functor of restricting of scalars $\pi^* : B$-$\text{mod} \to A$-$\text{mod}$ is exact, and it extends to a triangle functor $D^b(B$-$\text{mod}) \to D^b(A$-$\text{mod}$), which will still be denoted by $\pi^*$. Following [12], we call the homomorphism $\pi$ a homological epimorphism provided that $\pi^* : D^b(B$-$\text{mod}) \to D^b(A$-$\text{mod}$) is fully faithful. By [12] Theorem 4.1(1)] this is equivalent to the fact that $\pi \otimes_A^L B : B \cong A \otimes_A^L B \to B \otimes_A^L B$ is an isomorphism in $D(A^e$-$\text{mod})$. Here, $A^e = A \otimes_k A^{op}$ is the enveloping algebra of $A$, and we identify $A^e$-$\text{mod}$ as the category of $A$-$A$-bimodules.

**Lemma 2.3** ([22] Proposition 2.2(a)]). Let $J \subseteq A$ be an ideal and let $\pi : A \to A/J$ be the canonical projection. Then $\pi$ is a homological epimorphism if and only if $J^2 = J$ and $\text{Tor}^A_i (J, A/J) = 0$ for all $i \geq 1$.

In the situation of the lemma, the ideal $J$ is called a homological ideal in [22]. As a special case, we call an ideal $J$ a hereditary ideal provided that $J^2 = J$ and $J$ is a projective $A$-$A$-bimodule; compare [22] Lemma 3.4).

**Proof.** The natural exact sequence $0 \to J \to A \xrightarrow{\pi} A/J \to 0$ of $A$-$A$-bimodules induces a triangle $J \to A \xrightarrow{\pi} A/J \to J[1]$ in $D^b(A^e$-$\text{mod})$. Applying the functor $- \otimes_A^L A/J$, we get a triangle $J \otimes_A^L A/J \to A/J \to A/J \otimes_A^L A/J \to J \otimes_A^L A/J[1]$. Then $\pi$ is a homological epimorphism or, equivalently, $\pi \otimes_A^L J$ is an isomorphism if and only if $J \otimes_A^L A/J = 0$; see [13] Lemma I.1.7]. This is equivalent to the fact that $\text{Tor}^A_i (J, A/J) = 0$ for all $i \geq 0$. We note that $\text{Tor}^A_i (J, A/J) \simeq J \otimes_A A/J \simeq J/J^2$. □

Now we are in the position to prove the Theorem. Recall that for an algebra $A$, its singularity category $D_{sg}(A) = D^b(A$-$\text{mod})/\text{perf}(A)$. Moreover, a complex $X^\bullet$ becomes zero in $D_{sg}(A)$ if and only if it is perfect. Here, we use the fact that $\text{perf}(A) \subseteq D^b(A$-$\text{mod})$ is a thick subcategory.

**Proof of the Theorem.** Write $B = A/J$. Since $J$, as an $A$-$A$-bimodule, has finite projective dimension, so it has finite projective dimension both as a left and right $A$-module. Consider the natural exact sequence $0 \to J \to A \to B \to 0$. It follows that $B$, both as a left and right $A$-module, has finite projective dimension. Moreover, for a complex $X^\bullet$ in $D^b(A$-$\text{mod})$, $J \otimes_A^L X^\bullet$ is perfect. Indeed, take a bounded projective resolution $P^\bullet \to J$ as an $A^e$-module. Then $J \otimes_A^L X^\bullet \simeq P^\bullet \otimes_A X^\bullet$. This is a perfect complex, since each left $A$-module $P^i \otimes_A X^j$ is projective.
Denote by $\pi: A \to B$ the canonical projection. By the assumption, the functor $\pi^*: \text{D}^b(B\text{-mod}) \to \text{D}^b(A\text{-mod})$ is fully faithful. Since $\pi^*(B) = _AB$ is perfect, the functor $\pi^*$ sends perfect complexes to perfect complexes. Then it induces a triangle functor $\pi^*: \text{D}_{sg}(B) \to \text{D}_{sg}(A)$. We will show that $\pi^*$ is an equivalence.

The functor $\pi^*: \text{D}^b(B\text{-mod}) \to \text{D}^b(A\text{-mod})$ has a left adjoint $F = B \otimes_A^L -$ : $\text{D}^b(A\text{-mod}) \to \text{D}^b(B\text{-mod})$. Here we use the fact that the right $A$-module $B_A$ has finite projective dimension. Since $F$ sends perfect complexes to perfect complexes, we have the induced triangle functor $\bar{F}: \text{D}_{sg}(A) \to \text{D}_{sg}(B)$. By Lemma 2.2 we have the adjoint pair $(\bar{F}, \pi^*)$; moreover, the functor $\pi^*$ is fully faithful. By Lemma 2.1 there is a triangle equivalence $\text{D}_{sg}(A)/\text{Ker} \bar{F} \simeq \text{D}_{sg}(B)$.

It remains to show that the essential kernel $\text{Ker} \bar{F}$ is trivial. For this, we assume that a complex $X^\bullet$ lies in $\text{Ker} \bar{F}$. This means that the complex $F(X^\bullet)$ in $\text{D}^b(B\text{-mod})$ is perfect. Since $\pi^*$ preserves perfect complexes, it follows that $\pi^*F(X^\bullet)$ is also perfect. The natural exact sequence $0 \to J \to A \to B \to 0$ induces a triangle $J \otimes_A^L X^\bullet \to X^\bullet \to \pi^*F(X^\bullet) \to J \otimes_A^L X^\bullet[1]$ in $\text{D}^b(A\text{-mod})$. Recall that $J \otimes_A^L X^\bullet$ is perfect. It follows that $X^\bullet$ is perfect, since $\text{perf}(A) \subseteq \text{D}^b(A\text{-mod})$ is a triangulated subcategory. The proves that $X^\bullet$ is zero in $\text{D}_{sg}(A)$. 

The following special case of the Theorem is of interest.

**Corollary 2.4.** Let $A$ be a finite dimensional algebra and $J \subseteq A$ a hereditary ideal. Then we have a triangle equivalence $\text{D}_{sg}(A) \simeq \text{D}_{sg}(A/J)$.

**Proof.** It suffices to observe by Lemma 2.3 that $J$ is a homological ideal. 

3. **Examples**

In this section, we will describe a construction of matrix algebras to illustrate Corollary 2.2. In particular, the singularity categories of some non-Gorenstein algebras are described.

The following construction is similar to [18, Section 4]. Let $A$ be a finite dimensional algebra over a field $k$. Let $M$ and $N$ be a left and right $A$-module, respectively. Then $M \otimes_k N$ becomes an $A$-$A$-bimodule. Consider an $A$-$A$-bimodule monomorphism $\phi: M \otimes_k N \to A$. Then $\text{Im} \phi$ is a two-sided ideal of $A$. We require further that $(\text{Im} \phi)M = 0$ and $N(\text{Im} \phi) = 0$. The matrix $\Gamma = \begin{pmatrix} A & M \\ N & k \end{pmatrix}$ becomes an associative algebra via the following multiplication:

$$\begin{pmatrix} a & m \\ n & \lambda \end{pmatrix} \begin{pmatrix} a' & m' \\ n' & \lambda' \end{pmatrix} = \begin{pmatrix} aa' + \phi(m \otimes n') & am' + \lambda m' \\ na' + \lambda n' & \lambda \lambda' \end{pmatrix}.$$  

For the associativity, we need the above requirement on $\text{Im} \phi$.

**Proposition 3.1.** Keep the notation and assumption as above. Then there is a triangle equivalence $\text{D}_{sg}(\Gamma) \simeq \text{D}_{sg}(A/\text{Im} \phi)$.

**Proof.** Set $J = \Gamma e \Gamma$ with $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Observe that $\Gamma/J = A/\text{Im} \phi$. The ideal $J$ is hereditary: $J^2 = J$ is clear, while the natural map $\Gamma e \otimes_k e \Gamma \to J$ is an isomorphism of $\Gamma$-$\Gamma$-bimodules and then $J$ is a projective $\Gamma$-$\Gamma$-bimodule. The isomorphism uses the assumption that $\phi$ is mono. Then we apply Corollary 2.4. 


Remark 3.2. The above construction contains the one-point extension and coextension of algebras, where $M$ or $N$ is zero. Hence Proposition 3.1 contains the results in [9, Section 4].

We will illustrate Proposition 3.1 by three examples. Two of these examples extend an example considered by Happel in [14]. In particular, based on results in [9], we obtain descriptions of the singularity categories of some non-Gorenstein algebras.

Recall from [14] that an algebra $A$ is Gorenstein provided that both as a left and right module, the regular module $A$ has finite injective dimension. It follows from [6, Theorem 4.4.1] and [14, Theorem 4.6] that in the Gorenstein case, the singularity category $\mathcal{D}_{sg}(A)$ is Hom-finite. This means that all Hom spaces in $\mathcal{D}_{sg}(A)$ are finite dimensional over $k$.

For algebras given by quivers and relations, we refer to [1, Chapter III].

Example 3.3. Let $\Gamma$ be the $k$-algebra given by the following quiver $Q$ with relations \{\[x^2, \delta x, \beta x, x\gamma, x\alpha, \beta\gamma, \delta\alpha, \beta\alpha, \delta\gamma, \alpha\beta - \gamma\delta\}\}. We write the concatenation of paths from right to left.

\[
\begin{array}{c}
1 \rightarrow x \rightarrow 2 \\
& \alpha \rightarrow \beta \\
\end{array}
\]

We have in $\Gamma$ that $1 = e_1 + e_* + e_2$, where the $e$'s are the primitive idempotents corresponding to the vertices. Set $\Gamma' = \Gamma/E_1 \Gamma$. It is an algebra with radical square zero, whose quiver is obtained from $Q$ by removing the vertex 1 and the adjacent arrows.

We identify $\Gamma$ with $A = \begin{pmatrix} A & k\alpha \\ k\beta & k \end{pmatrix}$, where the $k$ in the southeast corner is identified with $e_1 E_1$, and $A = (1 - e_1) \Gamma (1 - e_1)$. The corresponding $\text{Im} \phi$ equals $k\alpha \beta$, and we have $A/\text{Im} \phi = \Gamma'$; consult the proof of Proposition 3.1. Then Proposition 3.1 yields a triangle equivalence $\mathcal{D}_{sg}(\Gamma) \simeq \mathcal{D}_{sg}(\Gamma')$.

The triangulated category $\mathcal{D}_{sg}(\Gamma')$ is completely described in [9] (see also [24]); in particular, it is not Hom-finite. More precisely, it is equivalent to the category of finitely generated projective modules on a von Neumann regular algebra. The algebra $\Gamma'$, or rather its Koszul dual, is related to the noncommutative space of Penrose tilings via the work of Smith; see [24, Theorem 7.2 and Example]. We point out that the algebra $\Gamma$ is non-Gorenstein, since $\mathcal{D}_{sg}(\Gamma)$ is not Hom-finite.

Example 3.4. Let $\Gamma$ be the $k$-algebra given by the following quiver $Q$ with relations \{\[x_1^2, x_2^2, x_1x_1, x_2x_1, x_1\alpha_1, x_2\alpha_1, \beta_2\alpha_1, x_1\alpha_2, x_2\alpha_2, \beta_1\alpha_2, \beta_2\alpha_2, \alpha_1\beta_1 - x_1x_2, \alpha_2\beta_2 - x_2x_1\]\}.

We claim that there is a triangle equivalence $\mathcal{D}_{sg}(\Gamma) \simeq \mathcal{D}_{sg}(k(x_1, x_2)/(x_1, x_2)^2)$. Here, $k(x_1, x_2)$ is the free algebra with two variables.

We point out that the triangulated category $\mathcal{D}_{sg}(k(x_1, x_2)/(x_1, x_2)^2)$ is described completely in [9, Example 3.11], where related results are contained in [3, Section 10]. Similar to the example above, this algebra $\Gamma$ is non-Gorenstein.
To see the claim, we observe that the quiver $Q$ has two loops and two 2-cycles. The proof is done by “removing the 2-cycles”. We have a natural isomorphism $\Gamma = \left( \begin{array}{c} A \\ k \alpha_1 \\ k \end{array} \right)$, where $k = e_1 \Gamma e_1$ and $A = (1 - e_1) \Gamma (1 - e_1)$. We observe that Proposition 3.4 applies with the corresponding $\text{Im } \phi = k \alpha_1 \beta_1$. Set $A / \text{Im } \phi = \Gamma'$. So $D_{\text{sg}}(\Gamma) \simeq D_{\text{sg}}(\Gamma')$. The quiver of $\Gamma'$ is obtained from $Q$ by removing the vertex 1 and the adjacent arrows, while its relations are obtained from the ones of $\Gamma$ by replacing $\alpha_1 \beta_1 - x_1 x_2$ with $x_1 x_2$. Similarly, $\Gamma' = \left( \begin{array}{c} A' \\ k \alpha_2 \\ k \end{array} \right)$ with $k = e_2 \Gamma' e_2$ and $A' = e_\alpha \Gamma' e_\alpha$. Then Proposition 3.1 applies again, and we get the equivalence $D_{\text{sg}}(\Gamma') \simeq D_{\text{sg}}(k \langle x_1, x_2 \rangle / \langle x_1, x_2 \rangle^3)$.

This example generalizes directly to a quiver with $n$ loops and $n$ 2-cycles with similar relations. The corresponding statement for the case $n = 1$ is implicitly contained in [14] 2.3 and 4.8.

The last example is a Gorenstein algebra.

**Example 3.5.** Let $r \geq 2$. Consider the following quiver $Q$ consisting of three 2-cycles and a central 3-cycle $Z_3$. We identify $\gamma_3$ with $\gamma_0$ and denote by $p_i$ the path in the central cycle starting at vertex $i$ of length 3.

\[
\begin{array}{c}
1' \\
\alpha_1 \\
\beta_1 \\
1 \\
2' \\
\beta_2 \\
2 \\
\end{array}
\]

\[\begin{array}{c}
\alpha_3 \\
\beta_3 \\
3 \\
\end{array}
\]

Let $\Gamma$ be the $k$-algebra given by the quiver $Q$ with relations $\{\beta_i \alpha_i, \gamma_i \alpha_i, \beta_i \gamma_i - 1, \alpha_i \beta_i - p_i^r | i = 1, 2, 3\}$. We point out that in $\Gamma$ all paths in the central cycle of length strictly larger than $3r + 1$ vanish.

Set $A = kZ_3 / (\gamma_1, \gamma_2, \gamma_3)^{3r}$, where $kZ_3$ is the path algebra of the central 3-cycle $Z_3$. The algebra $A$ is self-injective and Nakayama ([11] p.111]). Denote by $A\text{-mod}$ the stable category of $A$-modules; it is naturally a triangulated category (see [13] Theorem 1.2.6).

We claim that there is a triangle equivalence $D_{\text{sg}}(\Gamma) \simeq A\text{-mod}$.

For the claim, we observe an isomorphism $A = \Gamma / (\Gamma e_1 + e_2 + e_3) \Gamma$. We argue as in Example 3.4 by removing the three 2-cycles and applying Proposition 3.1 repeatedly. Then we get a triangle equivalence $D_{\text{sg}}(\Gamma) \simeq D_{\text{sg}}(A)$. Finally, by [23] Theorem 2.1 we have a triangle equivalence $D_{\text{sg}}(A) \simeq A\text{-mod}$. Then we are done.

We point out that the algebra $\Gamma$ is Gorenstein with self-injective dimension two. Hence by [6] Theorem 4.4.1 and [14] Theorem 4.6 there is a triangle equivalence $D_{\text{sg}}(\Gamma) \simeq \text{MCM}(\Gamma)$, where $\text{MCM}(\Gamma)$ denotes the stable category of (maximal) Cohen-Macaulay $\Gamma$-modules. Then we have a triangle equivalence

$\text{MCM}(\Gamma) \simeq A\text{-mod}$.

We mention that $\Gamma$ is a special biserial algebra of finite representation type (by [10] Lemma II.8.1]). It would be interesting to identify (maximal) Cohen-Macaulay $\Gamma$-modules in the Auslander-Reiten quiver of $\Gamma$. 

This example generalizes directly to a quiver with $n$ 2-cycles and a central $n$-cycle with similar relations. The case where $n = 1$ and $r = 2$ coincides with the examples considered in [14] 2.3 and 4.8.

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