PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 9, September 2014, Pages 2953–2964
S 0002-9939(2014)12021-1
Article electronically published on May 1, 2014

ARTIN-WHAPLES APPROXIMATIONS OF BOUNDED DEGREE IN ALGEBRAIC VARIETIES

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(Communicated by Matthew A. Papanikolas)

Abstract. The celebrated Artin-Whaples approximation theorem (which is a generalization of the Chinese remainder theorem) asserts that, given a field K, distinct places $v_1, \dots, v_n$ of K, and points $x_1, \dots, x_n \in \mathbb{P}_1(K)$, it is possible to find an $x \in \mathbb{P}_1(K)$ simultaneously near x_i w.r.t. v_i with any prescribed accuracy. If we replace $\mathbb{P}_1$ with other algebraic varieties V, the analogous conclusion does not generally hold, e.g., because V may contain too few points over K. However, it has been proved by a number of authors that, at least in the case of global fields, it holds if we allow x to be algebraic over K. These results do not directly contain either the case of $\mathbb{P}_1$ or the case of general fields, and above all they do not control the degree of x.

In this paper we offer different arguments leading to a general approximation theorem properly generalizing that of Artin-Whaples. This works for every V, K as above, and not only asserts the existence of a suitable $x \in V(\overline{K})$, but bounds explicitly the degree $[K(x) : K]$ in terms only of geometric invariants of V. It shall also be seen that such a bound is in a sense close to being best-possible.

1. Introduction

A rather classical statement is the following approximation theorem [1]:

Theorem 1.1 (Artin-Whaples). *Let K be a field and $|\cdot|_1, \dots, |\cdot|_n$ be independent absolute values of K. If $(\alpha_1, \dots, \alpha_n)$ is a sequence of elements of K, then for all $\varepsilon > 0$ there is an element $\beta \in K$ such that $|\beta - \alpha_i|_i < \varepsilon$ for each $1 \le i \le n$.*[1]

This can be generalized by taking α_i in the completion of K at $|\cdot|_i$, and rephrased in topological fashion as a density statement (also including $\infty \in \mathbb{P}_1$ as a possibility for α_i). For this, it shall be convenient to speak of *places* instead of absolute values (where a "place" is a class of absolute values under topological equivalence). We will denote by K_v the completion of K at the place v.

Theorem 1.1′. *Let K be a field and S be a finite set of places of K. Then the diagonal embedding*

$$\mathbb{P}_1(K) \hookrightarrow \prod_{v \in S} \mathbb{P}_1(K_v)$$

has dense image.

Received by the editors March 8, 2012 and, in revised form, September 7, 2012.
2010 *Mathematics Subject Classification.* Primary 11G99.

[1] This is generally referred to as the "weak approximation theorem". There is also a stronger version for global fields, which asserts that given an extra absolute value $|\cdot|_0$, β can also be taken such that $|\beta| \le 1$ for all the remaining absolute values outside of $|\cdot|_0, \dots, |\cdot|_n$.

It is a natural question whether an analogous statement holds for algebraic varieties V over K other than $\mathbb{P}_1$: *given points $Q_v \in V(K_v)$ for $v \in S$, can we always find a point $P \in V(K)$ simultaneously 'near' to Q_v with respect to each v?*

The first evident obstacle is that a variety can have too few rational points, e.g., $V(K)$ can well be finite (think e.g. of curves of genus ≥ 2 over a number field) or even empty, whereas in the completions we often find plenty of points. Hence, it is sensible to extend with some freedom the field of definition of P, for instance allowing P to lie in $V(\overline{K})$.

However, this enlargement has the side effect of creating many possibilities for extending the given places of K to a field of definition for the point; then *for all $v \in S$, we fix once and for all an extension $\overline{v}$ to $\overline{K}$, and we ask that P is near Q_v with respect to $\overline{v}$.*

For this question, let us concentrate on the basic case when $V = \mathcal{C}$ is a curve. Then one could proceed by taking a non-constant function $x \in K(\mathcal{C})$, approximating (by Theorem 1.1) the values $x(Q_v)$ with an $\alpha \in \mathbb{P}_1(K)$, and finally choosing α so that the fiber $x^{-1}(\alpha)$ contains a point which, for all $v \in S$, is simultaneously near Q_v w.r.t. *the given* extensions $\overline{v}$ of v.

This is certainly possible for a single absolute value v; however, already for two absolute values it is not at all clear that the respective choices in the fiber $x^{-1}(\alpha)$ can be compatible (see also the beginning of the proof in subsection 2.1). Indeed, arithmetical obstructions may arise and, as illustrated by Example 1.7 below (detailed in the Appendix), there are cases where this procedure is bound to fail.

Anyway, in spite of these difficulties, it is proved in a number of papers that, at least when K is a global field, the sought approximation theorem is true if we allow the whole $\overline{K}$ as a field of definition for the approximating point. Such statements are purely qualitative and do not bound the degree over K of the approximating point except for a few special cases. Also, strictly speaking they cannot be considered as generalizations of Theorem 1.1. The purpose of this paper is to give instead a quantitative version for arbitrary fields, with a completely different (and simpler) proof, which 'logically' is a generalization of Theorem 1.1.

More precisely, we will prove the following result, which we state in the fashion of Theorem 1.1.

Let V be an algebraic variety over K, and for each place v of K let us choose a distance d_v on V (e.g., the sup norm in a system of coordinates) which plays the role of the norm $|\cdot|_v$ of K.

Theorem 1.2. *Let V be an absolutely irreducible subvariety of $\mathbb{P}_N$ of degree d defined over K. Let S be a finite set of places v of K, extended in some way to $\overline{K}$, with associated distances d_v on V. Also, for $v \in S$ let $Q_v \in V(K_v)$.*

Then for all $\varepsilon > 0$ there is a point $P \in V(\overline{K})$ such that $d_v(P, Q_v) < \varepsilon$ for all $v \in S$ and, moreover,

$$[K(P) : K] \leq d^{|S|\cdot(\min\{2,\dim V\}+1)-1}.$$

Remark 1.3. (i) The proof in fact will show that we can replace the above bound $d^{|S|\cdot(\min\{2,\dim V\}+1)-1}$ with the more cumbersome, but smaller, term

$$d^{|S|\cdot(\min\{2,\dim V\}-1)+1}\left(\frac{(d-1)(d-2)}{2}+1\right)^{|S|-1}.$$

In turn, for $d \geq 2$ this can be bounded by $d^{|S|\cdot(\min\{2,\dim V\}+1)-1}/2^{|S|-1}$.

(ii) Note that in the statement we can take d as the minimal degree such that V can be embedded in $\mathbb{P}_N$ over K, for some N, so the bound really depends only on V as a variety over K. In particular, when $V = \mathbb{P}_1$ our bound implies $P \in \mathbb{P}_1(K)$, and we recover precisely Theorem 1.1.

(iii) The bound on $[K(P) : K]$ is rather uniform in the data; for instance, it does not depend on ε, and only on the cardinality $|S|$ rather than S.

An immediate qualitative corollary, phrased similarly to Theorem 1.1′ is that: *the diagonal embedding*

$$V(\overline{K}) \hookrightarrow \prod_{v\in S} V(\overline{K}_v)$$

has dense image.

As alluded to above, for global fields this qualitative form in fact is not new: it follows from a density result isolated first by Rumely [10], who used capacity theory; then other proofs were provided by Moret-Bailly[2] [8], using Picard groups, and by Green, Pop and Roquette [6], using generalized Jacobians. The methods of these papers are delicate and lead to further results regarding *integral points*, such as a kind of strong approximation theorem and Rumely's local-global principle.

However, in spite of being more sophisticated, such proofs do not seem to lead to the same kind of uniform bound as in the above statement. Also, the qualitative statement that $V(\overline{K})$ is dense in $\prod_{v\in S} V(\overline{K}_v)$ does not itself imply Theorem 1.1 taking $V = \mathbb{P}_1$; so, strictly speaking, the quoted results cannot be considered generalizations of Theorem 1.1. (We also remark that some quantitative work based on the method of [8] has been carried out in [5] and [7]; however, this goes in a direction different from ours, in that it involves heights and yields bounds for the degree of *integral* points for certain varieties.)

We will now give the relevant statement for the special case of curves, which contains all the basic ingredients; now the bound is given in terms of the (arithmetic) genus and the gonality, rather than the degree. Theorem 1.2 will be deduced from the case of curves in the next sections.

The possible presence of singularities requires the use of the arithmetic genus, which we denote by p_a. Moreover, we shall denote by $G(K, \mathcal{C})$ the *gonality* of $\mathcal{C}$ over the field K, i.e., the minimum degree of a non-constant function defined over K and regular at the singular points of $\mathcal{C}$.

Theorem 1.4. *Let $\mathcal{C}$ be an absolutely irreducible projective curve of arithmetic genus p_a defined over K. Let S be a finite set of places of K, extended in some way to $\overline{K}$, with associated distances d_v on $\mathcal{C}$. Also, for $v \in S$ let $Q_v \in \mathcal{C}(K_v)$.*

Then for all $\varepsilon > 0$ there is a point $P \in \mathcal{C}(\overline{K})$ such that $d_v(Q_v, P) < \varepsilon$ for all $v \in S$, and moreover $[K(P) : K] \leq G(K, \mathcal{C}) \cdot (p_a + 1)^{|S|-1}$.

Remark 1.5. (i) The gonality can often be bounded in terms of other quantities; e.g.:

- When $\mathcal{C}$ contains at least one non-singular K-rational point, then by the Riemann-Roch formula we have $G(K, \mathcal{C}) \leq (p_a + 1)$.
- If $p_a \neq 1$, the canonical divisor has degree $2p_a - 2 \neq 0$; by Riemann-Roch again, $G(K, \mathcal{C}) \leq |2p_a - 2|$.

[2]As remarked by Moret-Bailly himself, part of the ideas are due to Szpiro.

Finally, in the case $p_a = 1$, either the curve is singular, and it is easy to see that the gonality is either 1 or 2, or $p_a = g = 1$. In the latter case, $G(K, \mathcal{C})$ cannot be bounded only in terms of the other data we are using; in fact, for any given $d > 0$ there are examples of smooth curves of genus 1 not containing rational points of degree $\leq d$ (see [2, Sec. 10]).

(ii) The dependence on the gonality of the bound for $[K(P) : K]$ cannot be removed in general. For instance, in [3] it is proved that if $\mathcal{C}$ is a smooth plane curve of degree $d \geq 7$ defined over a number field K, then $\mathcal{C}$ has only finitely many points P such that $[K(P) : K] \leq d - 2$, and in some cases there are actually only finitely many points such that $[K(P) : K] \leq d-1$. Since in these cases $G(K, \mathcal{C}) = d$, it follows that for these curves and fields K the bound of the theorem cannot be lowered below $G(K, \mathcal{C})$. All of this shows that already in the simplest case of curves, and $|S| = 1$, the order of magnitude of our bound is sharp.

An interesting issue raised by Peter Roquette is how far the bound is from being sharp for $|S| > 1$. For instance: *does a bound hold which is independent of* $|S|$*?* We do not know an answer, but it seems to be related to deep arithmetic questions on the field, as suggested by Example 1.7.

On the other hand, the proof will show that the factor $G(K, \mathcal{C})$ can be removed if at least one point Q_v is non-singular and lies in $\mathcal{C}(K)$.

(iii) We observe that in the singular case the inequality would not hold in general if the arithmetic genus p_a were replaced by the geometric genus of a non-singular model $\tilde{\mathcal{C}}$ of $\mathcal{C}$. Here is a very simple example which illustrates this:

Example 1.6. We take $\mathcal{C} := \{y^2 = x^3 - x^2\}$, $K = \mathbb{Q}$, $S = \{v\}$, where v is either the real place of $\mathbb{Q}$ or a p-adic place with $p \equiv 3 \mod 4$. Here $g(\tilde{\mathcal{C}}) = 0$, whereas $p_a(\mathcal{C}) = 1$. Taking $Q_v = (0, 0)$, the sup norm as the distance, and a sufficiently small $\varepsilon > 0$, it is easily seen that the conditions $\|P - Q_v\|_v < \varepsilon$ and $P \in \mathcal{C}(\mathbb{Q})$ imply $P = Q_v$. Hence, any further approximation condition w.r.t. places other than v cannot be eventually fulfilled with points $P \in \mathcal{C}(\mathbb{Q})$; on the other hand, the theorem would predict this false conclusion if p_a were replaced by 0.

(iv) At least in the case $|S| = 2$, and if some of the target points Q_v are in $\mathcal{C}(K)$, the proof will show that we can gain further precision. Namely, we even get a non-constant function x, dependent only on the said target point, but not on ε or on the other target point, such that the approximating point $P \in \mathcal{C}(\overline{K})$ may be gotten with $x(P) \in K$. The following example shows that such a function x cannot be chosen arbitrarily.

Example 1.7. Let K be a field of characteristic different from 2 and $p(x)$ be a polynomial in $K[x]$ such that the values $p(\alpha)$ for $\alpha \in K$ are infinitely many, nonzero, and mutually multiplicatively independent modulo squares in K. (It seems likely that such a polynomial exists, maybe already over $K = \mathbb{Q}$; however, this existence question seems extremely hard to decide.) If the curve $\mathcal{C}$ is defined by the equation $y^2 = p(x)$, it is not too difficult to see that we can construct two absolute values v_1, v_2 on $\overline{K}$, and two points Q_1, Q_2 in $\mathcal{C}(K_{v_1})$ and $\mathcal{C}(K_{v_2})$ respectively, such that if any approximating point $P \in \mathcal{C}(\overline{K})$ is sufficiently near Q_1 and Q_2 w.r.t. v_1 and v_2, then $x(P) \notin K$. We shall give the details of the construction of v_1 and v_2 in the Appendix.

This example is conditional on the existence of a suitable polynomial $p(x)$. A similar but unconditional example can be built using a plane curve defined by a more complicated equation. (See the Appendix for more.)

(v) This statement is a bit sharper than the curve case of Theorem 1.2, since it is well known that $p_a + 1 \leq d^2$, and actually with strict inequality for $d \geq 2$.

(vi) Again, when $\mathcal{C} = \mathbb{P}_1$ we have $p_a = g = 0$, and we recover Theorem 1.1.

In the next section we will give the proof of Theorems 1.2 and 1.4. Some of the ingredients are just known general elementary statements about algebraic varieties; they will be given in the Appendix for completeness.

2. The proofs

The proofs shall start with the special case of Theorem 1.4 obtained when there are only two (target) points to approximate, and moreover one of them is defined over the ground field K. This case will be treated in detail in the next subsection, where we shall also briefly anticipate the basic strategy. After this step, we shall deduce the general case by induction.

2.1. **The case of two points on a curve.** Let us anticipate a sketch of our strategy in a simpler case with redundant assumptions so that the subsequent details shall be hopefully clearer.

Let $V = \mathcal{C}$ be a plane absolutely irreducible curve, say defined by an equation $f(x, y) = 0$, with $f \in K[x, y]$. Let us take two points $Q_1, Q_2 \in \mathcal{C}(K)$ on it with coordinates (x_1, y_1) and (x_2, y_2), and let v_1, v_2 be two independent absolute values. We want to find a point $P \in \mathcal{C}$ which is near Q_1 w.r.t. v_1 and near Q_2 w.r.t. v_2.

In this specific case, we can easily approximate Q_1 with respect to v_1 by first choosing an $\alpha \in K$ near x_1 and then (using the continuity of the roots of a polynomial) by choosing a solution β to the equation $f(\alpha, Y) = 0$, with β near y_1 w.r.t. v_1. Then, the point $P = (\alpha, \beta)$ shall be near Q_1 w.r.t. v_1.

Now, by Theorem 1.1 we can also impose that α is near x_2 with respect to v_2. However, in the general case the constraint imposed by the previous choice of β (w.r.t. the first absolute value) completely determines β among the roots of $f(\alpha, Y) = 0$, which possibly may not fulfil the desired requirement for v_2.

Even by changing α there may be obstructions in the arithmetic of K which make it unclear whether this procedure may be successful for *some* choice of α (see the above Example 1.7 detailed in the Appendix).

Nevertheless, there is one situation where this obstacle disappears, namely when $f(x_2, Y) = 0$ happens to have only one solution. In this case, if α is near x_2 w.r.t. v_2, any β chosen in the first step shall be near the unique solution y_2 of $f(x_2, Y)$, and we would obtain that $P = (\alpha, \beta)$ is also near Q_2 w.r.t. v_2.

Geometrically, this condition on the roots of $f(x_2, Y)$ is ensured if we know that the map $x : \mathcal{C} \to \mathbb{P}_1$ is totally ramified above x_2. This is easy to achieve, for instance using Riemann-Roch.

Now, it turns out that this method does not merely work for two points and places, but for an arbitrary number n of places, provided however that $n - 1$ of the target points coincide. We detail this in the proof of the following lemma.

For the sake of notation, let S be enumerated as $v_1, \ldots, v_n$, with the v_i's pairwise distinct, and denote the target points as $Q_1, \ldots, Q_n$.

Lemma 2.1. *Let us suppose that $\mathcal{C}$ is an absolutely irreducible projective curve and that $Q_1 = \cdots = Q_{n-1} \neq Q_n$, with $Q_1 \in \mathcal{C}(K)$ a non-singular point and $Q_n \in \mathcal{C}(K_{v_n})$.*

Then for all $\varepsilon > 0$ there is a non-singular point $P \in \mathcal{C}(\overline{K})$ such that $d_{v_i}(Q_i, P) < \varepsilon$ for all $i = 1, \dots, n$, and moreover $[K(P) : K] \leq p_a + 1$.

Proof. Take a rational function x on $\mathcal{C}$ defined over K such that its only pole is Q_1. This is possible when Q_1 is non-singular, as in the hypothesis. In fact, if D is a divisor not containing singular points, then we can apply the Riemann-Roch theorem, in a suitable form adapted to singular curves. Denoting by $l(D)$ the dimension of the K-vector space of global sections of the sheaf associated to D, as in [11, Thm. IV.6.1, Eq. (IV.33)],[3] a generalization of the Riemann-Roch theorem in particular implies that

$$l(D) \geq \deg(D) + 1 - p_a.$$

Therefore, setting $D = (p_a + 1)Q_1$, we have $l(D) \geq 2$, so we may indeed choose x as a non-constant function defined over K and regular on $\mathcal{C}$ outside of Q_1.

It is then a standard (and not difficult) result that a value $x(R)$ can be large w.r.t. an absolute value v only at points $R \in \mathcal{C}$ v-adically near Q_1. In other words, there is a number C such that $|x(R)|_v > C$ implies $d_v(R, Q_1) < \varepsilon$ for any v in the set $v_1, \dots, v_{n-1}$ (for the reader's convenience we shall recall a proof of this in Proposition 3.1 in the Appendix).

Now, for any $\delta > 0$ let us fix, using Theorem 1.1, $\alpha_\delta \in K$ such that $|\alpha_\delta - x(Q_n)|_{v_n} < \delta$ and $|\alpha_\delta|_{v_i} > C$ for $i = 1, \dots, n-1$.

Consider the fiber $x^{-1}(\alpha_\delta)$. The map x is locally open on the set $\mathcal{C}(\overline{K})$ w.r.t. the topology induced by v_n on $\overline{K}$ (see Proposition 3.2 in the Appendix); in particular, if δ is small enough we can find a point $P \in x^{-1}(\alpha_\delta)$ such that $d_{v_n}(P, Q_n) < \varepsilon$. Clearly, we can also choose α_δ such that P is non-singular, as there are only finitely many singular points in $\mathcal{C}(\overline{K})$.

Note that we have achieved the approximation conclusion of the lemma. Indeed, we have $d_{v_n}(P, Q_n) < \varepsilon$. Also, $|x(P)|_{v_i} = |\alpha_\delta|_{v_i} > C$; then, by the above remark, $d_{v_i}(P, Q_1) = d_{v_i}(P, Q_i) < \varepsilon$ for $i = 1, \dots, n-1$.

As to the conclusion about the degree, since the function x is defined over K, and its pole divisor (on a smooth model of $\mathcal{C}$) is at most $(p_a + 1)Q_1$, its degree in $K(\mathcal{C})$ is at most $(p_a + 1)$; in particular, the fact that $x(P) = \alpha_\delta$ is in K implies that $[K(P) : K] \leq (p_a + 1)$. □

2.2. **The general case.** As we have mentioned, the special case of Theorem 1.4 represented by Lemma 2.1 rapidly leads to a proof of Theorem 1.4 in general, and then in turn to that of Theorem 1.2.

First of all, let us prove Theorem 1.4, i.e., the case of curves.

As above, let S be enumerated as $v_1, \dots, v_n$, with the v_i's pairwise distinct, and denote the target points as $Q_1, \dots, Q_n$.

Proof of Theorem 1.4. Essentially, we have to remove in Lemma 2.1 both assumptions that $n - 1$ of the given points coincide and that they lie in $\mathcal{C}(K)$ rather than in $\mathcal{C}(K_v)$. We argue by induction on the number n of absolute values, starting at

[3]We have $l(D) = \dim_K\{f \in K(\mathcal{C}) : \operatorname{div}(f) \geq -D\}$ when $\mathcal{C}$ is non-singular. Similarly in general, where we restrict f to be well defined and regular everywhere out of D on the singular curve.

$n = 1$. In our inductive hypothesis, we also require that the approximating point we find is non-singular.

When $n = 1$ we have only one point, and we just need to move it to a nearby non-singular algebraic point. Let x be a non-constant function $x \in K(\mathcal{C})$ of minimal degree and regular at all the singular points of $\mathcal{C}$; by definition its degree is $G(K, \mathcal{C})$. We may assume that x has no pole at Q_1 (just replace x with $(x + c)^{-1}$ if needed). As in the proof of Lemma 2.1, for a given $\delta > 0$, consider an $\alpha_\delta \in K$ such that $|\alpha_\delta - x(Q_1)|_{v_1} < \delta$ (this exists since $x(Q_1) \in K_{v_1}$). If δ is small enough, we can find a point P in the fiber $x^{-1}(\alpha_\delta)$ such that $d_{v_1}(P, Q_1) < \varepsilon$.[4] Since $x(P) = \alpha_\delta \in K$, and since the degree of x is $G(K, \mathcal{C})$, we have $[K(P) : K] \leq G(K, \mathcal{C})$, which proves the conclusion of the theorem for $n = 1$. Moreover, avoiding finitely many values of α we can be sure that P is non-singular.

Now suppose $n > 1$ and the conclusion true up to $n - 1$. We apply first the inductive hypothesis to $Q_1, \ldots, Q_{n-1}$: we can find a non-singular algebraic point $R \in \mathcal{C}(\overline{K})$ such that $d_{v_i}(R, Q_i) < \varepsilon/2$ for $i = 1, \ldots, n - 1$, and $[K(R) : K] \leq G(K, \mathcal{C}) \cdot (p_a + 1)^{n-2}$. Then we apply Lemma 2.1 to the sequence $R, \ldots, R, Q_n$ and to the field $K(R)$ in place of K: there is a point P such that $d_{v_i}(P, R) < \varepsilon/2$ for $i = 1, \ldots, n - 1$ and $d_{v_n}(P, Q_n) < \varepsilon/2$, and moreover $[K(R, P) : K(R)] \leq (p_a + 1)$. By possibly reducing ε, we can also make sure that P is non-singular.

In particular, the point P is non-singular and such that $d_{v_i}(P, Q_i) < \varepsilon$ for $i = 1, \ldots, n$, and $[K(P) : K] \leq G(K, \mathcal{C}) \cdot (p_a + 1)^{n-1}$, proving the desired conclusion. $\square$

Theorem 1.4 now easily yields Theorem 1.2, as in the proof below. The idea is to reduce the dimension by intersecting the variety with a subspace passing near the original points in such a way that we get an absolutely irreducible curve.

Proof of Theorem 1.2. In the case $\dim V = 1$, we may estimate the arithmetic genus of V by projecting the curve onto a plane so that its arithmetic genus does not decrease and its degree remains the same; by the genus formula for plane curves, we obtain the bound $p_a(\mathcal{C}) \leq (d - 1)(d - 2)/2 \leq d^2$. Moreover, $G(K, \mathcal{C})$ is bounded by d. The desired conclusion can then be deduced by Theorem 1.4.

Let us assume that $\dim V > 1$. We may pick n linear subspaces $L_1, \ldots, L_n$ of $\mathbb{P}_N$ such that, for $i = 1, \ldots, n$, L_i is defined over K_{v_i}, passes through Q_i, and moreover $\dim L_i = \operatorname{codim} V$. We may assume that the intersection $L_i \cap V$ is finite for $i = 1, \ldots, n$.

In order to simplify the following discussion, we assume that the subspaces L_i belong to a one-parameter family $L(t)$, with $t \in \mathbb{P}_1$, defined over K; for instance, we can choose them so that their intersection is a prescribed subspace of dimension $(\operatorname{codim} V - 1)$ defined over K. Let us call $t_i \in K_{v_i}$ the parameter such that $L_i = L(t_i)$.

Applying Theorem 1.1 to the sequence $(t_1, \ldots, t_n)$, we can find an approximating $t_0 \in K$ such that the subspace $L(t_0)$ is 'near' L_i w.r.t. v_i for $i = 1, \ldots, n$. We can choose t_0 such that moreover $L(t_0) \cap V$ is finite. By continuity of the roots of a polynomial (see the Appendix), the set $L(t_0) \cap V$ contains, for each i, a point Q'_i near Q_i w.r.t. v_i.

Since the cardinality of $L(t_0) \cap V$ is at most d, we have that $[K(Q'_i) : K] \leq d$ for $i = 1, \ldots, n$. Letting $K' = K(Q'_1, \ldots, Q'_n)$, we clearly have $[K' : K] \leq d^n$.

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[4] Again, this well-known fact is recalled in Proposition 3.2 in the Appendix.

Now let us choose a subspace Λ containing $L(t_0)$ such that $\dim \Lambda = (\mathrm{codim} V + 1)$. By Bertini's Theorem, for a suitable choice of the initial family $L(t)$, of the parameter t_0 and of Λ, we may assume that $\Lambda \cap V$ is an absolutely irreducible curve of degree at most d.

We can now apply Theorem 1.4 to the curve $\mathcal{C} := \Lambda \cap V$, to the points $Q'_1, \ldots, Q'_n$ on it and to the field K'. Observe that since $\mathcal{C}$ is embedded in $\mathbb{P}_N$ as a curve of degree $\leq d$, $G(K', \mathcal{C})$ is at most d; also, as before, $p_a(\mathcal{C}) \leq (d-1)(d-2)/2 \leq d^2$. Then, by Theorem 1.4, we can find an approximating point $P \in \mathcal{C}$ such that $[K'(P) : K'] \leq d \cdot d^{2(n-1)} = d^{2n-1}$.

In particular, the point P is an approximating point on V and its degree is such that $[K(P) : K] \leq d^{3n-1}$. $\square$

Remark 2.2. (i) If the points Q_i are non-singular, it is seemingly possible to find the points Q'_i defined over K_{v_i}, in which case we save a factor d^n. We do not know whether such an improvement is possible in general.

(ii) For a proof of a result similar to Theorem 1.2 one can also find an irreducible curve passing *through* $Q_1, \ldots, Q_n$, rather than near them (the trick is to blow up the variety V above $Q_1, \ldots, Q_n$ as in [9, Lemma at p. 56]).

2.3. **Elliptic curves, groups and Jacobians.** In the case of algebraic groups, e.g., for abelian varieties, we can exploit the group structure to give an alternative proof of a (slightly weaker) form of Theorem 1.2, starting from Lemma 2.1.

Using such lemma, we can take points P_v such that P_v is near the identity with respect to v' for all $v' \neq v$, but also near Q_v with respect to v. (Here, as in the final argument of Theorem 1.2, we have to take some suitable curves passing near the identity and near Q_v.) Then the continuity of the translation maps $x \mapsto x + P$ easily implies that $P := \sum_{v \in S} P_v$ is an approximating point.

If we know that the group is embedded into a projective space with degree d, this construction again leads to a bound on $[K(P) : K]$. Indeed, the degree of each P_v can be bounded by d^3, hence $[K(P) : K]$ is no more than d^{3n}.

Moreover, by embedding a (non-singular) curve of genus > 0 into its Jacobian, this leads to an alternative proof of the qualitative part of Theorem 1.4. We sketch the argument.

Let $\mathcal{C}$ be a non-singular curve of genus g. If $g = 0$, the theorem is equivalent to Theorem 1.1, so we may assume that $g > 0$. In this case, let $\mathcal{J}$ be the Jacobian and $\phi : \mathcal{C}^g / S_g \to \mathcal{J}$ be the usual birational isomorphism with the g-th symmetric power of $\mathcal{C}$. Let $U \subset \mathcal{J}$ be an open dense subset where ϕ is one-to-one.

Using Lemma 2.1, we can find points $P_1, \ldots, P_n \in U$ such that P_i is near the identity w.r.t. v_j for $j \neq i$, but also near the point $\phi(Q_i, \ldots, Q_i)$ w.r.t. v_i.

Now, we proceed as above: the sum $R := P_1 + \cdots + P_n$ lies near P_i w.r.t. v_i. If the points P_i are taken sufficiently near the identity, we have that R lies on U as well. In particular, there is a unique point $(R_1, \ldots, R_g) \in \mathcal{C}^g / S_g$ such that $\phi(R_1, \ldots, R_g) = R$. Clearly, the R_i are suitable approximation points.

3. Appendix

Since one of the motivations for the present paper is to give a simplified treatment of some statements, for the reader's convenience we include in this appendix a few details on rather standard facts and also make unambiguous our terminology. (All of this is well known but complete references seem not to be easily accessible.)

3.1. **Distances.** Given a projective variety $V \subset \mathbb{P}_N$ defined over K, and a place v of K, there are several ways to define a distance on $V(K)$ from v, and all these distances induce the same topology.

Since we are working with projective varieties, we may define a distance on the ambient space $\mathbb{P}_N(K)$ and take the restriction to $V(K)$. For instance, we could define the distance using the sup norm on the standard affine charts, as usual, and then patch the distances on $\mathbb{P}_N(K)$.

Using this definition it is easy to prove the following proposition.

Proposition 3.1. *Let $\mathcal{C}$ be an absolutely irreducible projective curve, Q a non-singular point in $\mathcal{C}(K)$, and x a function in $K(\mathcal{C})$ with Q as its only pole viewed into some normalization of $\mathcal{C}$. Let v be an absolute value of K and $d_v(\cdot,\cdot)$ be a distance induced by v on $\mathcal{C}$.*

Then for all $\varepsilon > 0$ there is a number C such that if $|x(P)|_v > C$, then $d_v(P,Q) < \varepsilon$.

Proof. Without loss of generality, we may assume that the curve $\mathcal{C}$ is non-singular. If this is not the case, it is sufficient to work on a normalization $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$. The function x, by hypothesis, lifts to a function $\tilde{x} : \tilde{\mathcal{C}} \to \mathbb{P}_1$ with a unique pole in Q, and it is therefore sufficient to study what happens on $\tilde{\mathcal{C}}$.

Let $y_0, \dots, y_N$ be the homogeneous coordinates on the projective space $\mathbb{P}_N$ containing $\mathcal{C}$. Without loss of generality, we can assume that $|y_0(Q)|_v$ is maximal among the $y_i(Q)$. Letting x_i be the function induced on $\mathcal{C}$ by y_i/y_0, we have $x_0(Q) = 1$ and $|x_i(Q)|_v \leq 1$ for all $i = 1, \dots, N$.

Let us consider the functions $w_i := (x_i - x_i(Q))^n x$, where n is the order of Q as a pole of x. They are regular at Q. Let us also define

$$z_i := \prod_{R\,:\,w_i(R)=\infty} \frac{1}{(x - x(R))^{[K(R):K]_i}},$$

where the product is over all the poles of w_i and $[K(R) : K]_i$ is the inseparable degree of R over K. Then z_i is defined over K (this is the reason for inserting the inseparable degree).

Note that z_i has at least the same poles as w_i. Hence, the function w_i is integral over $K[z_i]$. Thus, there is a monic polynomial relation

$$w_i^d = a_{d-1}(z_i) w_i^{d-1} + \cdots + a_0(z_i),$$

with $a_i(z) \in K[z]$. As is well known, on dividing the equation by w_i^{d-1}, this produces a bound of the form $|w_i(P)|_v \leq B \max\{1, |z_i(P)|_v^e\}$ for some numbers B, e depending only on the coefficients and the degrees of the $a_j(z)$'s.

Now, by construction, if we take a point P such that $|x(P)|_v$ is large enough, then $|z_i(P)|_v \leq 1$, whence $|w_i(P)|_v \leq B$.

In turn, recalling the definition of w_i, this implies $|(x_i(P) - x_i(Q))^n x(P)|_v \leq B$. Thus, when $|x(P)|_v$ is large, $|x_i(P) - x_i(Q)|_v$ is small. In particular, they fall in the same affine chart (the one with $y_0(Q) \neq 0$) and they are near w.r.t. the distance induced by v on this affine chart. Now, by our construction of the distance, there is a number C such that $|x(P)|_v > C$ implies $d_v(P,Q) < \varepsilon$. $\square$

Recall that this proposition was used in Lemma 2.1. Our purpose there was to find points v-adically near a given Q, and this shows that it suffices to look at large values of a function with Q as its only pole.

3.2. **Locally open maps.** In the proof of Lemma 2.1 we used the fact that if x is a non-constant rational function on a curve $\mathcal{C}$, then it is locally open as a map $\mathcal{C}(\overline{K}) \to \mathbb{P}_1(\overline{K})$ in the topology induced by any place of $\overline{K}$. Note that this is not true in general on the set $\mathcal{C}(K)$, when K is not algebraically closed.

We formally state this well-known fact in the following.

Proposition 3.2. *Let $\mathcal{C}$ be an absolutely irreducible projective curve over K, $Q \in \mathcal{C}(\overline{K})$ and $x \in \overline{K}(\mathcal{C})$ defined at Q. Let v be a place of $\overline{K}$. Then there is an open neighborhood U of Q in the topology induced by v on $\mathcal{C}(\overline{K})$ such that $x_{\restriction U}$ is an open map.*

Proof. Without loss of generality, we shall assume that $\mathcal{C}$ is non-singular. Indeed, if $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$ is a desingularization, the map x lifts to $\tilde{\mathcal{C}}$, and it is sufficient to verify that the lifted $\tilde{x} : \tilde{\mathcal{C}} \to \mathbb{P}_1$ is locally open. We may also assume that $x(Q) \neq \infty$.

We can choose a local parameter $t \in \overline{K}(\mathcal{C})$ at Q. Then, as is well known, $\overline{K}(\mathcal{C})$ may be embedded in $\overline{K}((t))$; also, each Laurent series expressing an element of $\overline{K}(\mathcal{C})$ is v-convergent in a suitable disc. In particular, we may express x as a power series (convergent in a disc around 0):

$$x = a_0 + a_1 t + a_2 t^2 + \cdots , \qquad a_i \in \overline{K}.$$

Then, to obtain the desired conclusion we have just to prove that the image of such a convergent series contains a disc around a_0. In characteristic 0 this follows from the fact that up to invertible analytic functions, each such map, and in particular x, is equivalent to $u \mapsto u^k$ for some k, which is locally open. In positive characteristic, this last claim does not generally hold, but still the desired conclusion is true, as can be proved, for instance, using Newton's polygons. See [4, Thm. II.2.1]. $\square$

3.3. **Details of Example 1.7.** Here we give the details of Example 1.7 mentioned in Remark 1.5 and also alluded to in the introduction. We show that if we strengthen the conclusion of Theorem 1.2 by requiring that $x(P)$ is in the ground field K, for a fixed non-constant function x in $K(\mathcal{C})$, the resulting statement may well be false for suitable fields K and in any case leads to very intriguing Diophantine questions in K.

To start the construction, let us suppose that our curve is defined by the equation $y^2 = p(x)$, where $p(x) \in K[x]$ is such that the values $p(\alpha)$ for $\alpha \in K$ are infinitely many, non-zero, and mutually multiplicatively independent modulo squares in K. (As remarked earlier, the question of the existence of such a polynomial seems extremely hard to decide. We will hint later at a more explicit and unconditional example using a polynomial $p(X,Y)$ over $K = \overline{\mathbb{Q}}(t)$ whose Galois group over $K(X)$ is S_5.)

In order to simplify the discussion, we will also assume that K is countable. (As observed by the referee, for the general case it is sufficient to replace the inductive part of the construction with a corresponding transfinite induction up to the cardinality of K.)

Given these data, we shall now construct two places v_1 and v_2 on $\overline{K}$ and two points $Q_1 \in \mathcal{C}(K_{v_1})$, $Q_2 \in \mathcal{C}(K_{v_2})$ such that $x(Q_1), x(Q_2) \in K$ and such that if $P \in \mathcal{C}(\overline{K})$ is sufficiently near Q_i w.r.t. v_i, for $i = 1, 2$, then $x(P) \notin K$.

As distances on K^2 we take the sup norm: we fix two absolute values $|\cdot|_1, |\cdot|_2$ corresponding to v_1, v_2, and we set $d_i((x,y),(x',y')) := \sup\{|x - x'|_i, |y - y'|_i\}$.

Let $Q_1 = (1, y_1)$, $Q_2 = (2, y_2)$, where $y_1, y_2 \in \overline{K}$ satisfy $y_i^2 = p(i)$ for $i = 1, 2$. Note that indeed $x(Q_i) = i \in K$, so $Q_1, Q_2 \in \mathcal{C}(\overline{K})$; we also suppose that $Q_i \in \mathcal{C}(K_{v_i})$.

Let us enumerate the distinct elements of K as $(1, 2, \alpha_3, \alpha_4, \dots)$. Let us consider the increasing sequence $K_n := K(\sqrt{p(1)}, \sqrt{p(2)}, \sqrt{p(\alpha_3)}, \dots, \sqrt{p(\alpha_n)})$. Note that by the fundamental property of our polynomial $p(x)$, if $p(\alpha_n)$ is a new value, different from $p(\alpha_i)$ for $i < n$, then K_n is a proper extension of degree 2 of the field K_{n-1} for all n. The principle shall be to construct the v_1, v_2 by extending them inductively to the sequence of fields (K_n) in such a way as to obstruct the approximation condition.

Suppose that v_1, v_2 have been extended to $K(y_1, y_2)$. Let us fix $\varepsilon < \min_i |y_i|_i/2$ and $\delta > 0$ such that if $|\xi - i|_i < \delta$ for $i = 1, 2$, then $\min_\pm |\sqrt{p(\xi)} \pm y_i|_i < \varepsilon$ for $i = 1, 2$. We may assume that $\delta \leq \varepsilon$. We shall work by induction, the induction hypothesis being that if $p(x(P)) \in \{p(1), p(2), p(\alpha_3), p(\alpha_4), \dots, p(\alpha_n)\}$, then $\max_{i=1,2} d_i(P, Q_i) \geq \delta$. In this process, we assume that the v_i have been extended to K_{n-1}, and we will extend them to K_n in a suitable way during the induction.

This works by definition for $n = 2$. Suppose that $n > 2$ and that we have achieved this up to $n - 1$. If $p(\alpha_n)$ is equal to $p(\alpha_i)$ for some $i < n$, then the inductive hypothesis carries on trivially to n. We may assume then that $p(\alpha_n)$ is a new value; by the assumption on the values of $p(x)$, this implies that K_n is a proper extension of K_{n-1} of degree 2.

If $|\alpha_n - i|_i \geq \delta$ for $i = 1$ or $i = 2$, then for any point $P \in \mathcal{C}$ with $x(P) = \alpha_n$ we obtain that $d_i(P, Q_i) \geq \delta$, and we are done. Hence, let us suppose that $|\alpha_n - i|_i < \delta$ for $i = 1, 2$.

We now extend v_1 in an arbitrary way to K_n. By the above choice of δ, there is a point $P = (\alpha_n, y_n)$ with $y_n^2 = p(\alpha_n)$ and $|y_n - y_1|_1 < \varepsilon$. By the definition of ε, this uniquely determines y_n among the set $\{y_n, -y_n\}$. Now, v_2 is by assumption determined on K_{n-1}, and we have at most two possible extensions of v_2 to K_n, say $|\cdot|$ and $|\cdot|'$, related by the equation $|\xi| = |\xi^\sigma|'$, where $\xi \in K_n$ and σ is the unique non-trivial automorphism of K_n/K_{n-1} (these extensions $|\cdot|$ and $|\cdot|'$ may or may not coincide). In particular, $|y_n - y_2| = |y_n + y_2|'$.

At this stage, we can prove that if $x(P) = \alpha_n$, then $\max_i d_i(P, Q_i) \geq \delta$. Indeed, if $|y_n - y_2| < \varepsilon$, then $|y_n - y_2|' \geq |2y_2|_2 - |y_n + y_2|' \geq 2\varepsilon - \varepsilon \geq \delta$. We may choose the extension v_2 to K_n so that $|y_n - y_2|_2 \geq \delta$, and we are done.

This example is conditional on the existence of the said suitable polynomial p. However, we have unconditional examples, although more complicated. If we replace the curve $y^2 = p(x)$ with the curve defined by

$$y^5 + y^4 + (x^5 - tx)^{16} - \frac{64}{3125} = 0$$

over $K = \overline{\mathbb{Q}}(t)$, we find that an analogous construction is possible.

The crucial property of this polynomial is that for each non-zero specialization of x, the splitting fields over K of the corresponding specialized polynomials have Galois group S_5 and they are mutually linearly disjoint (namely, any of these fields is linearly disjoint from the compositum of the other ones). The Galois groups and the linear disjointness can be determined by a careful inspection of the ramification points. Moreover, there are exactly five rational points given by the specialization $x \mapsto 0$.

Now, taking Q_1, Q_2 as two of the five rational points, it is possible as above to define two absolute values v_1, v_2 on $\overline{K} = \overline{\mathbb{Q}(t)}$ such that if $x(P) \in K$, then $d_{v_i}(P, Q_i)$ is larger than a fixed constant for either $i = 1$ or $i = 2$.

Acknowledgements

The authors would like to thank Jean-Louis Colliot-Thélène, Pierre Dèbes, Peter Roquette and Lucien Szpiro for their comments and for references. They also thank an anonymous referee for a very careful reading of the paper and helpful comments.

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