

MAXIMAL COVARIANCE GROUP OF WIGNER TRANSFORMS AND PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. We show that the linear symplectic and antisymplectic transformations form the maximal covariance group for both the Wigner transform and Weyl operators. The proof is based on a new result from symplectic geometry which characterizes symplectic and antisymplectic matrices and which allows us, in addition, to refine a classical result on the preservation of symplectic capacities of ellipsoids.

INTRODUCTION

It is well known [3–5, 19] that the Wigner transform

$$(0.1) \quad W\psi(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p \cdot y} \psi\left(x + \frac{1}{2}y\right) \overline{\psi}\left(x - \frac{1}{2}y\right) dy$$

of a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ has the following *symplectic covariance* property: let S be a linear symplectic automorphism of \mathbb{R}^{2n} (equipped with its standard symplectic structure) and \widehat{S} one of the two metaplectic operators covering S ; then

$$(0.2) \quad W\psi \circ S = W(\widehat{S}^{-1}\psi).$$

It has been a long-standing question whether this property can be generalized in some way to arbitrary non-linear symplectomorphisms (the question actually harks back to the early days of quantum mechanics, following a question of Dirac [1, 2]). In a recent paper [6] one of us has shown that one cannot expect to find an operator \widehat{F} (unitary or not) such that $W\psi \circ F^{-1} = W(\widehat{F}\psi)$ for all $\psi \in \mathcal{S}'(\mathbb{R}^n)$ when $F \in \text{Symp}(n)$ (the group of all symplectomorphisms of the standard symplectic space) unless F is linear (or affine). In this paper we show that one cannot expect to have covariance for arbitrary linear automorphisms of \mathbb{R}^{2n} . More specifically, fix $M \in GL(2n, \mathbb{R})$ and suppose that for any $\psi \in \mathcal{S}'(\mathbb{R}^n)$ there exists $\psi' \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$W\psi \circ M = W\psi';$$

then M is either symplectic or antisymplectic (*i.e.* MC is symplectic, where C is the reflection $(x, p) \mapsto (x, -p)$) (Theorem 1).

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The covariance property (0.2) is intimately related to the following property of Weyl operators: assume that \widehat{A} is a continuous linear operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ with Weyl symbol a ; writing this correspondence $\widehat{A} \xleftrightarrow{\text{Weyl}} a$ we then have

$$(0.3) \quad \widehat{S}^{-1} \widehat{A} \widehat{S} \xleftrightarrow{\text{Weyl}} a \circ S$$

for every symplectic automorphism S . Properties (0.2) and (0.3) are in fact easily deduced from one another. One shows [16, 19] that property (0.3) is really characteristic of Weyl calculus: it is the only pseudo-differential calculus enjoying this symplectic covariance property (however, see [7] for partial covariance results for Shubin operators). We will see, as a consequence of our study of the Wigner function, that one cannot extend property (0.3) to non-symplectic automorphisms. More precisely, if S is not a symplectic or antisymplectic matrix, then there exists no unitary operator \widehat{S} such that (0.3) holds for all $\widehat{A} \xleftrightarrow{\text{Weyl}} a$. In other words, the group of linear symplectic and antisymplectic transformations is the maximal covariance group for the Weyl–Wigner calculus.

It turns out that the methods we use allow us in addition to substantially improve a result from symplectic topology. Recall that a symplectic capacity on \mathbb{R}^{2n} is a mapping c associating to every subset $\Omega \subset \mathbb{R}^{2n}$ a non-negative number, or $+\infty$, and satisfying the following properties:

- *Symplectic invariance:* $c(f(\Omega)) = c(\Omega)$ if $f \in \text{Symp}(n)$.
- *Monotonicity:* $\Omega \subset \Omega' \implies c(\Omega) \leq c(\Omega')$.
- *Conformality:* $c(\lambda\Omega) = \lambda^2 c(\Omega)$ for every $\lambda \in \mathbb{R}$.
- *Non-triviality and normalization:*

$$c(B^{2n}(R)) = \pi R^2 = c(Z_j^{2n}(R)),$$

where $B^{2n}(R)$ is the ball $|z| \leq R$ and $Z_j^{2n}(R)$ is the cylinder $x_j^2 + p_j^2 \leq R^2$.

An important property is that all symplectic capacities agree on ellipsoids in \mathbb{R}^{2n} . Now, a well known result is that if $f \in GL(2n, \mathbb{R})$ preserves the symplectic capacity of *all* ellipsoids in \mathbb{R}^{2n} , then f is either symplectic or antisymplectic (the notion will be defined below). It turns out that our Lemma 1, which we use to prove our main results about covariance, yields the following sharper result (Proposition 1): if f preserves the symplectic capacity of all symplectic balls, then f is either symplectic or antisymplectic (a symplectic ball is an ellipsoid which is the image of $B^{2n}(R)$ by a linear symplectic automorphism; see section 1.2).

Notation and Terminology. The standard symplectic form on $\mathbb{R}^{2n} \equiv T^*\mathbb{R}^n$ is defined by $\sigma(z, z') = p \cdot x' - p' \cdot x$ if $z = (x, p)$, $z' = (x', p')$. An automorphism S of \mathbb{R}^{2n} is symplectic if $\sigma(Sz, Sz') = \sigma(z, z')$ for all $z, z' \in \mathbb{R}^{2n}$. These automorphisms form a group $\text{Sp}(n)$ (the standard symplectic group). The metaplectic group $\text{Mp}(n)$ is a group of unitary operators on $L^2(\mathbb{R}^n)$ isomorphic to the double cover $\text{Sp}_2(n)$ of the symplectic group. The standard symplectic matrix is $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I (resp. 0) is the $n \times n$ identity (resp. zero) matrix. We have $\sigma(z, z') = Jz \cdot z' = (z')^T Jz$ and $S \in \text{Sp}(n)$ if and only if $S^T J S = J$ (or, equivalently, $S J S^T = J$).

1. A RESULT ABOUT SYMPLECTIC MATRICES

1.1. Two symplectic diagonalization results. We denote by $\text{Sp}^+(n)$ the subset of $\text{Sp}(n)$ consisting of symmetric positive definite symplectic matrices. We recall that if $G \in \text{Sp}^+(n)$, then $G^\alpha \in \text{Sp}^+(n)$ for every $\alpha \in \mathbb{R}$. We also recall that the unitary group $U(n, \mathbb{C})$ is identified with the subgroup

$$(1.1) \quad U(n) = \text{Sp}(n) \cap O(2n, \mathbb{R})$$

of $\text{Sp}(n)$ by the embedding

$$A + iB \longrightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Recall [3, 4, 12] that if $G \in \text{Sp}^+(n)$, then there exists $U \in U(n)$ such that

$$(1.2) \quad G = U^T \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} U,$$

where Λ is the diagonal matrix whose diagonal elements are the n eigenvalues ≥ 1 of G (counting the multiplicities).

For further use we also recall the following classical result: let N be a (real) symmetric positive definite $2n \times 2n$ matrix; then there exists $S \in \text{Sp}(n)$ such that

$$(1.3) \quad S^T N S = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix},$$

where Σ is the diagonal matrix whose diagonal entries are the symplectic eigenvalues of N , *i.e.* the moduli of the eigenvalues $\pm i\lambda$ ($\lambda > 0$) of the product JN (“Williamson diagonalization theorem” [18]; see [3–5, 12] for proofs).

1.2. A lemma and its consequence. Recall that an automorphism M of \mathbb{R}^{2n} is antisymplectic if $\sigma(Mz, Mz') = -\sigma(z, z')$ for all $z, z' \in \mathbb{R}^{2n}$; in matrix notation $M^T J M = -J$. Equivalently $CM \in \text{Sp}(n)$, where $C = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

Lemma 1. *Let $M \in GL(2n, \mathbb{R})$ and assume that $M^T G M \in \text{Sp}(n)$ for every*

$$(1.4) \quad G = \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} \in \text{Sp}^+(n).$$

Then M is either symplectic or antisymplectic.

Proof. We first remark that, taking $G = I$ in the condition $M^T G M \in \text{Sp}(n)$, we have $M^T M \in \text{Sp}(n)$. Next, we can write $M = HP$, where $H = M(M^T M)^{-1/2}$ is orthogonal and $P = (M^T M)^{1/2} \in \text{Sp}^+(n)$ (polar decomposition theorem). It follows that the condition $M^T G M \in \text{Sp}(n)$ is equivalent to $P(H^T G H)P \in \text{Sp}(n)$; since P is symplectic so is P^{-1} , and hence $H^T G H \in \text{Sp}(n)$ for all G of the form (1.4).

Let us now make the following particular choice for G : it is any diagonal matrix

$$G = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}, \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n),$$

with $\lambda_j > 0$ for $1 \leq j \leq n$. We thus have

$$H^T \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} H \in \text{Sp}^+(n)$$

for every Λ of this form. Let $U \in U(n)$ be such that

$$H^T \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} H = U^T \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} U$$

($H^TGH \in \text{Sp}^+(n)$ and the eigenvalues of H^TGH are those of G since H is orthogonal) and set $R = HU^T$; the equality above is equivalent to

$$(1.5) \quad \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} R = R \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}.$$

Writing $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we get the conditions

$$\begin{aligned} \Lambda A &= A\Lambda, & \Lambda B &= B\Lambda^{-1}, \\ \Lambda^{-1}C &= C\Lambda, & \Lambda^{-1}D &= D\Lambda^{-1} \end{aligned}$$

for all Λ . It follows from these conditions that A and D must themselves be diagonal: $A = \text{diag}(a_1, \dots, a_n)$, $D = \text{diag}(d_1, \dots, d_n)$. On the other hand, choosing $\Lambda = \lambda I$, $\lambda \neq 1$, we get $B = C = 0$. Hence, taking into account the fact that $R \in O(2n, \mathbb{R})$ we must have

$$(1.6) \quad R = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad A^2 = D^2 = I.$$

Conversely, if R is of the form (1.6), then (1.5) holds for any positive-definite diagonal Λ . We conclude that M has to be of the form $M = RUP$, where R is of the form (1.6). Since $UP \in \text{Sp}(n)$, $M^TGM \in \text{Sp}(n)$ implies $RGR \in \text{Sp}(n)$.

To proceed, for each pair i, j with $1 \leq i < j \leq n$ we choose the following matrix X in (1.4):

$$(1.7) \quad X^{(ij)} = I + \frac{1}{2}E^{(ij)},$$

where $E^{(ij)}$ is the symmetric matrix whose entries are all zero except the ones on the i -th row and j -th column and on the j -th row and i -th column which are equal to one. For instance, if $n = 4$, we have

$$(1.8) \quad X^{(13)} = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A simple calculation then reveals that

$$(1.9) \quad RGR = \begin{pmatrix} AX^{(ij)}A & 0 \\ 0 & D(X^{(ij)})^{-1}D \end{pmatrix}.$$

If we impose $RGR \in \text{Sp}(n)$, we obtain

$$(1.10) \quad AX^{(ij)}AD(X^{(ij)})^{-1}D = I \iff X^{(ij)}AD = ADX^{(ij)}.$$

In other words the matrix AD commutes with every real positive-definite $n \times n$ matrix $X^{(ij)}$ of the form (1.7).

Let us write $AD = \text{diag}(c_1, \dots, c_n)$ with $c_j = a_jd_j$ for $1 \leq j \leq n$. Applying (1.10) to (1.7) for $i < j$, we conclude that

$$(1.11) \quad c_i = c_j.$$

This means that the entries of the matrix AD are all equal, that is, either $AD = I$ or $AD = -I$, or equivalently $A = D$ or $A = -D$. In the first case, R is symplectic and so is M . In the second case R is antisymplectic, but then M is also antisymplectic. \square

There is a very interesting link between Lemma 1 and symplectic topology (in particular the notion of symplectic capacities of ellipsoids). In fact it is proven in [12] that the only linear mappings that preserve the symplectic capacities [5, 12, 14] of ellipsoids in the symplectic space $(\mathbb{R}^{2n}, \sigma)$ are either symplectic or antisymplectic. Recall that the symplectic capacity of an ellipsoid $\Omega_G = \{z : Gz \cdot z \leq 1\}$ (G a real symmetric positive-definite $2n \times 2n$ matrix) can be defined in terms of Gromov's width by

$$c(\Omega_G) = \sup_{f \in \text{Symp}(n)} \{\pi r^2 : f(B^{2n}(r)) \subset \Omega_G\},$$

where $B^{2n}(r) = \{z : |z| \leq r\}$ is the closed ball of radius r . The number $c(\Omega_G)$ is in practice calculated as follows: let λ_{\max} be the largest symplectic eigenvalue of G ; then $c(\Omega_G) = \pi/\lambda_{\max}$. Now ([12], Theorem 5, p. 61, and its Corollary, p. 64) assume that f is a linear map $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $c(f(\Omega_G)) = c(\Omega_G)$ for all G . Then f is symplectic or antisymplectic. It turns out that our Lemma 1 yields a sharper result: let us call a symplectic ball the image of $B^{2n}(r)$ by an element $S \in \text{Sp}(n)$. A symplectic ball $S(B^{2n}(r))$ is an ellipsoid having symplectic capacity $c(S(B^{2n}(r))) = c(B^{2n}(r)) = \pi r^2$. Then:

Proposition 1. *Let $k : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a linear automorphism taking any symplectic ball to a symplectic ball. Then k is either symplectic or antisymplectic.*

Proof. The symplectic ball $S(B^{2n}(r))$ is defined by the inequality $Gz \cdot z \leq 1$, where $G = (1/r^2)(S^T)^{-1}S^{-1} \in \text{Sp}^+(n)$. Let K be the matrix of k in the canonical basis. We have

$$k(S(B^{2n}(r))) = \{z : (K^{-1})^T G K^{-1} z \cdot z \leq 1\};$$

hence $k(S(B^{2n}(r)))$ is a symplectic ball if and only if $(K^{-1})^T G K^{-1} \in \text{Sp}^+(n)$. The proof now follows from Lemma 1 with $M = K^{-1}$. \square

2. THE MAIN RESULT

Let us now prove our main result:

Theorem 1. *Let $M \in GL(2n, \mathbb{R})$.*

(i) *Assume that M is antisymplectic: $S = CM \in \text{Sp}(n)$, where $C = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.*

Then for every $\psi \in \mathcal{S}'(\mathbb{R}^n)$

$$(2.1) \quad W\psi(Mz) = W(\widehat{S}^{-1}\overline{\psi})(z),$$

where \widehat{S} is any of the two elements of $\text{Mp}(n)$ covering S .

(ii) *Conversely, assume that for any $\psi \in \mathcal{S}'(\mathbb{R}^n)$ there exists $\psi' \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$(2.2) \quad W\psi(Mz) = W\psi'(z).$$

Then M is either symplectic or antisymplectic.

Proof. (i) It is sufficient to assume that $\psi \in \mathcal{S}(\mathbb{R}^n)$. We have

$$\begin{aligned} W\psi(Cz) &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}p \cdot y} \psi\left(x + \frac{1}{2}y\right) \overline{\psi}\left(x - \frac{1}{2}y\right) dy \\ &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p \cdot y} \psi\left(x - \frac{1}{2}y\right) \overline{\psi}\left(x + \frac{1}{2}y\right) dy \\ &= W\overline{\psi}(z). \end{aligned}$$

It follows that

$$W\psi(Mz) = W\psi(CSz) = W\overline{\psi}(Sz),$$

hence formula (2.1).

(ii) Choosing for ψ a Gaussian of the form

$$(2.3) \quad \psi_X(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} (\det X)^{1/4} e^{-\frac{1}{2\hbar}Xx \cdot x}$$

(X is real symmetric and positive definite) we have

$$(2.4) \quad W\psi_X(z) = \left(\frac{1}{\pi\hbar}\right)^n e^{-\frac{1}{\hbar}Gz \cdot z}$$

where

$$(2.5) \quad G = \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix}$$

is positive definite and belongs to $\text{Sp}(n)$. Condition (2.2) implies that we must have

$$W\psi'(z) = \left(\frac{1}{\pi\hbar}\right)^n e^{-\frac{1}{\hbar}M^TGMz \cdot z},$$

the Wigner transform of a function being a Gaussian if and only if the function itself is a Gaussian (see [4,5]). This can be seen in the following way. The matrix M^TGM being symmetric and positive definite, we can use a Williamson diagonalization (1.3): there exists $S \in \text{Sp}(n)$ such that

$$(2.6) \quad S^T(M^TGM)S = \Delta = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}$$

and hence

$$W\psi'(Sz) = \left(\frac{1}{\pi\hbar}\right)^n e^{-\frac{1}{\hbar}\Delta z \cdot z}.$$

In view of the symplectic covariance of the Wigner transform, we have

$$W\psi'(Sz) = W\psi''(z) \quad , \quad \psi'' = \widehat{S}^{-1}\psi',$$

where $\widehat{S} \in \text{Mp}(n)$ is one of the two elements of the metaplectic group covering S . We now show that the equality

$$(2.7) \quad W\psi''(z) = \left(\frac{1}{\pi\hbar}\right)^n e^{-\frac{1}{\hbar}\Delta z \cdot z}$$

implies that ψ'' must be a Gaussian of the form (2.3) and hence $W\psi''$ must be of the type (2.4), (2.5). That ψ'' must be a Gaussian follows from $W\psi'' \geq 0$ and Hudson's theorem (see e.g. [3]). If ψ'' were of the more general type,

$$(2.8) \quad \psi_{X,Y}(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} (\det X)^{1/4} e^{-\frac{1}{2\hbar}(X+iY)x \cdot x}$$

(X, Y are real and symmetric and X is positive definite), the matrix G in (2.4) would be

$$(2.9) \quad G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix},$$

which is compatible with (2.7) only if $Y = 0$. In addition, due to the parity of $W\psi''$, ψ'' must be even; hence Gaussians more general than $\psi_{X,Y}$ are excluded. It follows from these considerations that we have

$$\Delta = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix}$$

so that $\Sigma = \Sigma^{-1}$. Since $\Sigma > 0$ this implies that we must have $\Sigma = I$, and hence, using formula (2.6), $S^T(M^TGM)S = I$. It follows that we must have $M^TGM \in \text{Sp}(n)$ for every $G = \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} \in \text{Sp}^+(n)$. In view of Lemma 1 the matrix M must then be either symplectic or antisymplectic. □

Remark 1. An alternative way of proving that (2.7) implies that $\Sigma = I$ is to use the formulation of Hardy’s uncertainty principle [11] for Wigner transforms introduced in [8] (see [5], Theorem 105, for a detailed study).

3. APPLICATION TO WEYL OPERATORS

3.1. The Weyl correspondence. Let $a \in \mathcal{S}(\mathbb{R}^{2n})$; the operator \widehat{A} defined for all $\psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$(3.1) \quad \widehat{A}\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^n \iint_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}p \cdot (x-y)} a\left(\frac{1}{2}(x+y), p\right) \psi(y) dy dp$$

is called the Weyl operator with symbol a . For more general symbols $a \in \mathcal{S}'(\mathbb{R}^{2n})$ one can define $\widehat{A}\psi$ in a variety of ways [3, 4]; we will see one below. The Weyl correspondence $a \xleftrightarrow{\text{Weyl}} \widehat{A}$ is linear and one-to-one: If $a \xleftrightarrow{\text{Weyl}} \widehat{A}$ and $a' \xleftrightarrow{\text{Weyl}} \widehat{A}$, then $a = a'$, and we have $1 \xleftrightarrow{\text{Weyl}} I$, where I is the identity operator on $\mathcal{S}'(\mathbb{R}^n)$.

There is a fundamental relation between Weyl operators and the cross-Wigner transform, which is a straightforward generalization of the Wigner transform [17]: it is defined, for $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$, by

$$W(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p \cdot y} \psi\left(x + \frac{1}{2}y\right) \overline{\phi}\left(x - \frac{1}{2}y\right) dy$$

(in particular $W(\psi, \psi) = W\psi$). In fact, if $\widehat{A} \xleftrightarrow{\text{Weyl}} a$, then

$$(3.2) \quad \langle \widehat{A}\psi, \overline{\phi} \rangle = \langle \langle a, W(\psi, \phi) \rangle \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the distributional bracket on \mathbb{R}^n and $\langle \langle \cdot, \cdot \rangle \rangle$ is that on \mathbb{R}^{2n} ; the latter pairs distributions $\Psi \in \mathcal{S}'(\mathbb{R}^{2n})$ and Schwartz functions $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$. When $\Psi \in L^2(\mathbb{R}^{2n})$ we thus have

$$\langle \langle \Psi, \Phi \rangle \rangle = \int_{\mathbb{R}^{2n}} \Psi(z) \Phi(z) dz.$$

This relation can actually be taken as a concise definition of an arbitrary Weyl operator $\widehat{A} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$; for $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ we have $W(\psi, \phi) \in \mathcal{S}(\mathbb{R}^{2n})$ where the right-hand side is defined for arbitrary $a \in \mathcal{S}'(\mathbb{R}^{2n})$, and this defines unambiguously $\widehat{A}\psi$ since ϕ is arbitrary. The symplectic covariance property for Weyl operators

$$(3.3) \quad \widehat{S}^{-1} \widehat{A} \widehat{S} \xleftrightarrow{\text{Weyl}} a \circ S$$

easily follows: since

$$(3.4) \quad W(\widehat{S}\psi, \widehat{S}\phi)(z) = W(\psi, \phi)(S^{-1}z)$$

we have

$$\begin{aligned} \langle \langle a \circ S, W(\psi, \phi) \rangle \rangle &= \langle \langle a, W(\psi, \phi) \circ S^{-1} \rangle \rangle \\ &= \langle \langle a, W(\widehat{S}\psi, \widehat{S}\phi) \rangle \rangle \\ &= \langle \widehat{A}\widehat{S}\psi, \overline{\widehat{S}\phi} \rangle \\ &= \langle \widehat{S}^{-1}\widehat{A}\widehat{S}\psi, \overline{\phi} \rangle, \end{aligned}$$

which proves the covariance relation (3.3).

3.2. Maximal covariance of Weyl operators. Theorem 1 implies the following maximal covariance result for Weyl operators:

Corollary 1. *Let $M \in GL(2n, \mathbb{R})$. Assume that there exists a unitary operator $\widehat{M} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ such that $\widehat{M}\widehat{A}\widehat{M}^{-1} \overset{\text{Weyl}}{\longleftrightarrow} a \circ M^{-1}$ for all $\widehat{A} \overset{\text{Weyl}}{\longleftrightarrow} a \in \mathcal{S}(\mathbb{R}^{2n})$. Then M is symplectic or antisymplectic.*

Proof. Suppose that $\widehat{M}\widehat{A}\widehat{M}^{-1} \overset{\text{Weyl}}{\longleftrightarrow} a \circ M^{-1}$; then, by (3.2),

$$\begin{aligned} (\widehat{M}\widehat{A}\widehat{M}^{-1}\psi|\phi)_{L^2} &= \langle \langle a \circ M^{-1}, W(\psi, \phi) \rangle \rangle \\ &= \langle \langle a, W(\psi, \phi) \circ M \rangle \rangle. \end{aligned}$$

On the other hand, using the unitarity of \widehat{M} and (3.2),

$$\begin{aligned} (\widehat{M}\widehat{A}\widehat{M}^{-1}\psi|\phi)_{L^2} &= (\widehat{A}\widehat{M}^{-1}\psi|\widehat{M}^{-1}\phi)_{L^2} \\ &= \langle \langle a, W(\widehat{M}^{-1}\psi, \widehat{M}^{-1}\phi) \rangle \rangle. \end{aligned}$$

It follows that we must have

$$\langle \langle a, W(\psi, \phi) \circ M \rangle \rangle = \langle \langle a, W(\widehat{M}^{-1}\psi, \widehat{M}^{-1}\phi) \rangle \rangle$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$, and hence, in particular, taking $\psi = \phi$,

$$\langle \langle a, W\psi \circ M \rangle \rangle = \langle \langle a, W(\widehat{M}^{-1}\psi) \rangle \rangle$$

for all $\psi \in \mathcal{S}(\mathbb{R}^n)$. Since a is arbitrary this implies that we must have $W\psi \circ M = W(\widehat{M}^{-1}\psi)$. In view of Theorem 1 the automorphism M must be symplectic or antisymplectic. □

4. DISCUSSION AND CONCLUDING REMARKS

The results above, together with those in [6], where it was proved that one cannot expect a covariance formula for non-linear symplectomorphisms, show that the symplectic group indeed is a maximal linear covariance group for both Wigner transforms and general Weyl pseudo-differential operators. As briefly mentioned in the Introduction, one can prove [6, 7] partial symplectic covariance results for other classes of pseudo-differential operators (Shubin or Born–Jordan operators). Corollary 1 proves that one cannot expect to extend these results to more general linear non-symplectic automorphisms.

The link between Lemma 1, its consequence, Proposition 1, and the notion of symplectic capacity of ellipsoids in the symplectic space $(\mathbb{R}^{2n}, \sigma)$ is not after all so surprising: as has been shown in [5, 8, 9] there is a deep and certainly essential

interplay between Weyl calculus, the theory of Wigner transforms, the uncertainty principle, and Gromov's non-squeezing theorem [10]. For instance, the methods used in this paper can be used to show that the uncertainty principle in its strong Robertson–Schrödinger form [9] is invariant only under symplectic or antisymplectic transforms.

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