HERMITIAN OPERATORS ON BANACH ALGEBRAS
OF LIPSCHITZ FUNCTIONS

FERNANDA BOTELHO, JAMES JAMISON, A. JIMÉNEZ-VARGAS,
AND MOISÉS VILLEGAS-VALLECILLOS

(Communicated by Thomas Schlumprecht)

Abstract. For compact metric spaces \((X,d)\), we show that the Lipschitz
spaces \(\text{Lip}(X,d)\) and the little Lipschitz spaces \(\text{lip}(X,d^\alpha)\) with \(0 < \alpha < 1\),
equipped with the sum norm, support only trivial hermitian operators, that
is, real multiples of the identity operator.

1. Introduction

Let \(A\) be a complex Banach algebra with unity \(I\) and let \(A^*\) be its dual space. Given \(a \in A\), recall that the algebraic numerical range \(V(a)\) is given by
\[
V(a) = \{ F(a) : F \in A^*, \|F\| = F(I) = 1 \}.
\]
An element \(a \in A\) is said to be hermitian if \(V(a) \subset \mathbb{R}\). It is known that \(a \in A\) is
hermitian if and only if \(\|\exp(ita)\| = 1\) for all \(t \in \mathbb{R}\); see [3].

Let \(E\) be a complex Banach space and \(B(E)\) the Banach algebra of all bounded linear operators on \(E\) equipped with the operator norm. It is well-known that
an operator \(T \in B(E)\) is hermitian if and only if \(\exp(itT)\) is an isometry for each \(t \in \mathbb{R}\); see [6, Theorem 5.2.6]. The set of hermitian operators on \(E\) is a real subspace
of \(B(E)\) which contains all operators of the form \(\lambda I\), where \(\lambda\) is a real number. A
hermitian operator is said to be trivial if it is a real multiple of the identity operator.

Some important Banach spaces only support trivial hermitian operators, as for
example, the Bergman spaces \(L^p_\alpha(\Delta)\) \((1 \leq p < \infty, p \neq 2)\) [9, Corollary 5.4] and
the Hardy spaces \(H^p(\Delta)\) \((1 \leq p < \infty, p \neq 2)\) [1]. Also, the hermitian operators
on several spaces of scalar-valued continuous functions defined on the interval \([0, 1]\)
are known to be just real scalar multiples of the identity. Such spaces include
the space of continuously differentiable functions \(C^1[0,1]\); the space of absolutely continuous functions \(AC[0,1]\); and the spaces of Lipschitz functions: \(\text{Lip}[0,1]\) and
\(\text{lip} \alpha, 0 < \alpha < 1\). We recall that \(\text{lip} \alpha\) consists of all period 1 functions on \(\mathbb{R}\)
satisfying \(|f(x) - f(y)| = o(|x - y|^{\alpha})\) uniformly as \(|x - y| \to 0\); cf. [2, Theorem 3.1].
In this paper we investigate the hermitian operators on spaces of Lipschitz functions defined on a compact metric space. More precisely, for a compact metric space \((X,d)\) and a positive real parameter \(\alpha \in (0,1]\), we consider the space of all \(\alpha\)-Lipschitz functions \(f : X \to \mathbb{C}\) such that
\[
p_{\alpha}(f) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)^{\alpha}} : x, y \in X, x \neq y \right\} < \infty,
\]
and also the subspace of all \(\alpha\)-Lipschitz functions \(f : X \to \mathbb{C}\) satisfying the additional local flatness condition:
\[
\lim_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x,y)^{\alpha}} = 0.
\]
These two spaces with the standard operations of addition, multiplication and scalar multiplication are complex algebras, and when equipped with the norm
\[
\|f\|_{\alpha} = p_{\alpha}(f) + \|f\|_{\infty}
\]
become Banach algebras. These two algebras are denoted by \(\text{Lip}(X,d^{\alpha})\) and \(\text{lip}(X,d^{\alpha})\), respectively.

It is important to observe that \(\text{Lip}(X,d^{\alpha})\) and \(\text{lip}(X,d^{\alpha})\) are unital semi-simple commutative complex Banach algebras, and \(\text{lip}(X,d^{\alpha})\) is a closed subalgebra of \(\text{Lip}(X,d^{\alpha})\). Notice that \(\text{lip}(X,d)\) may contain only constant functions, for example \(\text{lip}[0,1]\) with the usual metric. When \(X = [0,1]\) or \(X = \mathbb{T}\) with the usual metrics, \(\text{Lip}(X,d^{\alpha})\) and \(\text{lip}(X,d^{\alpha})\) are among the classical algebras considered by de Leeuw in [4]. These algebras were first studied by Sherbert in [14],[15].

In [2], it was shown that the hermitian operators on the Lipschitz spaces \(\text{Lip}[0,1]\) and \(\text{lip} \alpha, 0 < \alpha < 1\), are real multiples of the identity operator. In this paper we prove that the same property holds for the spaces \(\text{Lip}(X,d)\) and the spaces \(\text{lip}(X,d^{\alpha})\) with \(0 < \alpha < 1\), for \((X,d)\) a compact metric space. This generalizes the aforementioned result.

We also mention the natural connection between hermitian operators and the class of bi-circular projections. A projection \(P\) on a complex Banach space is bi-circular if \(e^{is}P + e^{it}(I-P)\) is an isometry for all \(s, t \in \mathbb{R}\). This type of projection was introduced by Stachó and Zalar in [17]. Jamison [10] showed that these projections are exactly the hermitian projections. Our result implies that the only bi-circular projections on \(\text{Lip}(X,d)\) and \(\text{lip}(X,d^{\alpha})\) with \(0 < \alpha < 1\) are the trivial projections, 0 and \(I\).

2. Preliminaries

In this section we give a representation for all surjective linear isometries on \(\text{Lip}(X,d)\) or \(\text{lip}(X,d^{\alpha})\) \((0 < \alpha < 1)\) that fix the constant function everywhere equal to 1. Then we characterize the hermitian elements of \(\text{Lip}(X,d)\) and \(\text{lip}(X,d^{\alpha})\) \((0 < \alpha < 1)\). The last result provides a useful description of the continuous linear functionals on both spaces.

Throughout this paper \((X,d)\) is a compact metric space, \(1_X\) denotes the constant function equal to 1 on \(X\), \(f_X\) represents the identity function on \(X\) and \(I\) is the identity operator on \(\text{Lip}(X,d)\) or \(\text{lip}(X,d^{\alpha})\), \(0 < \alpha < 1\). For each \(x \in X\), \(\delta_x\) stands for the evaluation functional at the point \(x\) defined on \(\text{Lip}(X,d)\) or \(\text{lip}(X,d^{\alpha})\), \(0 < \alpha < 1\).
Our approach requires that the surjective linear isometries on the spaces Lip\((X,d)\) and lip\((X,d^\alpha)\) \((0<\alpha<1)\) have a suitable representation. Rao and Roy [13] proved that any surjective linear isometry of Lip\([0,1]\) can be expressed as a weighted composition operator \(f \mapsto \tau f \circ \varphi\) \((f \in \text{Lip}[0,1])\), where \(\tau\) is a scalar of modulus 1 and \(\varphi\) is a surjective isometry of \([0,1]\). They asked whether every isometry on the Banach spaces Lip\((X,d)\) and lip\((X,d^\alpha)\) \((0<\alpha<1)\) are induced by the isometries of the metric space \(X\). Next we derive a characterization of surjective linear isometries on these spaces that fix \(1_X\). This characterization follows from a theorem due to Jarosz in [11], a theorem in [8] (page 144) and a result by Sherbert in [14].

**Theorem 2.1.** Let \(X\) be a compact metric space. Then \(T: \text{Lip}(X,d) \to \text{Lip}(X,d)\) is a surjective linear isometry such that \(T(1_X) = 1_X\) if and only if there exists a surjective isometry \(\varphi: X \to X\) such that \(T(f) = f \circ \varphi\) for all \(f \in \text{Lip}(X,d)\). The same characterization holds for a surjective linear isometry \(T\) on lip\((X,d^\alpha)\) \((0<\alpha<1)\) such that \(T(1_X) = 1_X\).

**Proof.** It is straightforward to check that an operator \(T\) of the form described in the theorem is a surjective isometry. Then we just prove the reversed implication.

Let \(A(X)\) represent either \(\text{Lip}(X,d)\) or lip\((X,d^\alpha)\) with \(0<\alpha<1\) and let \(C(X)\) be the algebra of continuous complex-valued functions on \(X\). We first observe that \(A(X)\) is a regular subspace of \(C(X)\) and the sum norm is a \(p\)-norm for the norm on \(\mathbb{R}^2\) given by \(p(s,t) = |s| + |t|\). Let us recall (see [11]) that given a compact Hausdorff space \(X\), a complex linear subspace \(A\) of \(C(X)\) that contains the function \(1_X\), is said to be regular if for any \(\varepsilon > 0\), any \(x_0 \in \text{Ch} A\) and any open neighborhood \(U\) of \(x_0\), there is an \(f \in A\) with \(\|f\|_\infty \leq 1 + \varepsilon\), \(f(x_0) = 1\), and \(| f(x) | < \varepsilon\) for \(x \in X \setminus U\). \(\text{Ch} A\) denotes the set of extreme points \(F\) of the unit ball of \((A, \|\cdot\|_\infty)^*\) such that \(F(1_X) = 1\), and we identify \(\text{Ch} A\) with a subset of \(X\). Suppose that \(T\) is a surjective linear isometry on \(A(X)\) such that \(T(1_X) = 1_X\). An application of the main theorem in [11] to \(A(X)\) yields that \(T\) is a surjective isometry on \((A(X), \|\cdot\|_\infty)\).

Next we quote a theorem from Hoffman’s book [8, p. 44]: Let \(X\) be a compact Hausdorff space and let \(B\) be a complex linear subalgebra of \(C(X)\) that contains the function \(1_X\). Suppose that \(S\) is a linear map of \(B\) onto \(B\) such that \(\|S(f)\|_\infty = \|f\|_\infty\) for all \(f \in B\). If \(S(1_X) = 1_X\), then \(S\) is multiplicative.

Therefore \(T\) is an automorphism of \(A(X)\). By Sherbert’s theorem [14] Corollary 5.2, every automorphism \(T\) of Lip\((X,d)\) that carries \(1_X\) into \(1_X\) is of the form \(T(f) = f \circ \varphi\), where \(\varphi: X \to X\) is a homeomorphism. Similarly, we can prove that this is also true for those automorphisms of lip\((X,d^\alpha)\) \((0<\alpha<1)\) that fix \(1_X\).

We now show that \(\varphi\) is an isometry of \(X\). Observe that given any \(\alpha \in (0,1]\), we have \(p_\alpha(T(f)) = p_\alpha(f)\) for all \(f \in A(X)\) since \(T\) is an isometry for both norms \(\|\cdot\|_\alpha\) and \(\|\cdot\|_\infty\).

For the case \(A(X) = \text{Lip}(X,d)\), fix \(y \in X\) and define \(f_y: X \to \mathbb{R}\) by \(f_y(z) = d(z, \varphi(y))\) for all \(z \in X\). Clearly, \(f_y \in \text{Lip}(X,d)\) and \(p_1(f_y) \leq 1\). For any \(x,y \in X\), we have

\[
d(\varphi(x), \varphi(y)) = |f_y(\varphi(x)) - f_y(\varphi(y))| = |T(f_y)(x) - T(f_y)(y)| \leq p_1(T(f_y))d(x,y) \leq d(x,y).
\]
For the case $A(X) = \text{lip}(X, d^\alpha)$ ($0 < \alpha < 1$), fix $x, y \in X$, $x \neq y$, choose $\beta \in (\alpha, 1)$ and define
\[
 f_{xy}(z) = \frac{d(z, \varphi(y))^\beta - d(z, \varphi(x))^\beta}{2d(\varphi(x), \varphi(y))^{\beta - \alpha}}, \quad \forall z \in X.
\]
It is not hard to check that $f_{xy} \in \text{lip}(X, d^\alpha)$ and $p_\alpha(f_{xy}) = 1$ (see, for example, [12, p. 62]). An easy calculation gives
\[
 d(\varphi(x), \varphi(y))^{\alpha} = |f_{xy}(\varphi(x)) - f_{xy}(\varphi(y))| = |T(f_{xy})(x) - T(f_{xy})(y)| \leq p_\alpha(T(f_{xy}))d(x, y)^\alpha = d(x, y)^\alpha.
\]
In either case we have $d(\varphi(x), \varphi(y)) \leq d(x, y)$ for all $x, y \in X$.

Since $T^{-1}$ is also a surjective linear isometry on $A(X)$ such that $T^{-1}(1_X) = 1_X$, the same argument used above implies the existence of a homeomorphism $\phi: X \to X$ such that $T^{-1}(f) = f \circ \phi$ for all $f \in A(X)$. Therefore $d(\phi(x), \phi(y)) \leq d(x, y)$ for all $x, y \in X$. Given $x \in X$, we have
\[
 f(\varphi^{-1}(x)) = T(T^{-1}(f))(\varphi^{-1}(x)) = T^{-1}(f)(x) = f(\phi(x))
\]
for all $f \in A(X)$. Since $A(X)$ separates the points of $X$, this implies that $\varphi^{-1} = \phi$. Consequently, $\varphi$ is a surjective isometry. This completes the proof of the theorem.

We will next characterize the hermitian elements of the spaces Lip$(X, d)$ and Lip$(X, d^\alpha)$, $0 < \alpha < 1$.

**Lemma 2.2.** Let $(X, d)$ be a compact metric space and $h \in \text{Lip}(X, d)$ (or Lip$(X, d^\alpha)$, $0 < \alpha < 1$). Then $h$ is a hermitian element in Lip$(X, d)$ (or Lip$(X, d^\alpha)$) if and only if $h$ is a real constant function.

**Proof.** Assume that $h$ is hermitian in Lip$(X, d)$. Then $F(h) \in V(h) \subset \mathbb{R}$ for all $F \in \text{Lip}(X, d)^*$ such that $\|F\| = F(1_X) = 1$. In particular, $h(x) = \delta_x(h) \in \mathbb{R}$ for all $x \in X$, and so $h$ is real-valued. Using that $|e^a - e^b| \leq |a - b| \exp(\max\{|a|, |b|\})$ for all $a, b \in \mathbb{C}$, we deduce that $\exp(ith)$ is a function in Lip$(X, d)$. We also have that, for each $t \in \mathbb{R}$, $\|\exp(ith)\|_1 = 1$. Since $\|\exp(ith)\|_\infty = 1$, it follows that $p_1(\exp(ith)) = 0$. Hence $\exp(ith)$ is a constant function on $X$ for all $t \in \mathbb{R}$ which implies that $h$ is constant.

Conversely, if $h$ is a real constant function, then $h$ is a real multiple of $1_X$. Therefore $h$ is hermitian in Lip$(X, d)$. The same proof works for lip$(X, d^\alpha)$, $0 < \alpha < 1$. \hfill $\Box$

Following an idea of de Leeuw [4], we embed the Banach spaces Lip$(X, d)$ and Lip$(X, d^\alpha)$ ($0 < \alpha < 1$) isometrically into some suitable spaces of complex-valued continuous functions.

Let $X$ be a compact metric space and let $\tilde{X}$ be the set $\{(x, y) \in X^2$: $x \neq y\}$. It is easy to check that $\tilde{X}$ is completely regular; we denote by $\beta\tilde{X}$ the Stone-Cech compactification of $\tilde{X}$. Let $C(X \cup \beta\tilde{X})$ denote the Banach space of all complex-valued continuous functions on $X \cup \beta\tilde{X}$, under the norm
\[
 \|f\| = \|f|_X\|_\infty + \|f|_{\beta\tilde{X}}\|_\infty \quad (f \in C(X \cup \beta\tilde{X})),
\]
and let $C_0(X \cup \tilde{X})$ denote the Banach space of all complex-valued continuous functions on $X \cup \tilde{X}$ vanishing at infinity, endowed with the norm
\[ \|f\| = \|f\|_X + \|f\|_{\tilde{X}} \]  
($f \in C_0(X \cup \tilde{X})$).

We now recall that the Riesz representation theorem states that the map $\mu \mapsto F_\mu$, given by
\[ F_\mu(f) = \int_{X \cup \beta \tilde{X}} f \, d\mu \quad (f \in C(X \cup \beta \tilde{X})) \]
defines an isometric isomorphism from the Banach space $\mathcal{M}(X \cup \beta \tilde{X})$ of all complex-valued regular Borel measures on $X \cup \beta \tilde{X}$ equipped with the norm of total variation:
\[ \|\mu\| = |\mu|(X \cup \beta \tilde{X}) \quad (\mu \in \mathcal{M}(X \cup \beta \tilde{X})) \]
ono{onto the dual space of $(C(X \cup \beta \tilde{X}), \|\cdot\|_\infty)$. Similarly, the map $\nu \mapsto G_\nu$ defined by
\[ G_\nu(f) = \int_{X \cup \tilde{X}} f \, d\nu \quad (f \in C_0(X \cup \tilde{X})) \]
is an isometric isomorphism from the Banach space $\mathcal{M}(X \cup \tilde{X})$ with the norm
\[ \|\nu\| = |\nu|(X \cup \tilde{X}) \quad (\nu \in \mathcal{M}(X \cup \tilde{X})) \]
ono{onto the dual space of $(C_0(X \cup \tilde{X}), \|\cdot\|_\infty)$. For each $f \in \text{Lip}(X, d)$ or $f \in \text{lip}(X, d^\alpha)$, $0 < \alpha < 1$, we set $\tilde{f} : \tilde{X} \to \mathbb{C}$ to be the map given by
\[ \tilde{f}(x, y) = \frac{f(x) - f(y)}{d(x, y)^\alpha}, \quad \forall (x, y) \in \tilde{X}, \]
where $\alpha = 1$ when $f \in \text{Lip}(X, d)$. It is easy to show that $\tilde{f}$ is continuous on $\tilde{X}$ and
\[ \|\tilde{f}\|_\infty = p_\alpha(f) \quad (0 < \alpha \leq 1). \]
Hence there exists a unique continuous function $\beta \tilde{f}$ on $\beta \tilde{X}$ such that $\beta \tilde{f}|_{\tilde{X}} = \tilde{f}$ and $\|\beta \tilde{f}\|_\infty = \|\tilde{f}\|_\infty$. Furthermore, if $f \in \text{lip}(X, d^\alpha)$, then $\tilde{f}$ vanishes at infinity on $\tilde{X}$. The maps $\Phi : \text{Lip}(X, d) \to C(X \cup \beta \tilde{X})$ and $\Psi : \text{lip}(X, d^\alpha) \to C_0(X \cup \tilde{X})$, defined by
\begin{align*}
(1) \quad \Phi(f)(w) &= \begin{cases} f(w) & \text{if } w \in X, \\ \beta \tilde{f}(w) & \text{if } w \in \beta \tilde{X}, \end{cases} \\
(2) \quad \Psi(f)(w) &= \begin{cases} f(w) & \text{if } w \in X, \\ \tilde{f}(w) & \text{if } w \in \tilde{X}, \end{cases}
\end{align*}
are isometric linear embeddings from $\text{Lip}(X, d)$ with the norm $\|\cdot\|_1$ into $C(X \cup \beta \tilde{X})$, and from $\text{lip}(X, d^\alpha)$ with the norm $\|\cdot\|_\alpha$ into $C_0(X \cup \tilde{X})$, respectively.

The Hahn–Banach theorem and the Riesz representation theorem yield the following lemma.

**Lemma 2.3.** Let $(X, d)$ be a compact metric space.
(1) For each $F \in \text{lip}(X,d)^*$, there exists $\mu \in M(X \cup \beta \bar{X})$ with $\|F\| \leq \|\mu\|$ satisfying

$$F(f) = \int_{X \cup \beta \bar{X}} \Phi(f)(w) \, d\mu(w), \quad \forall f \in \text{lip}(X,d).$$

(2) Let $\alpha \in (0,1)$. For each $G \in \text{lip}(X,d^\alpha)^*$, there exists $\nu \in M(X \cup \bar{X})$ with $\|G\| \leq \|\nu\|$ such that

$$G(f) = \int_{X \cup \bar{X}} \Psi(f)(w) \, d\nu(w), \quad \forall f \in \text{lip}(X,d^\alpha).$$

Proof. Let $F \in \text{lip}(X,d)^*$. The functional $T: \Phi(\text{lip}(X,d)) \to \mathbb{C}$, defined by $T(\Phi(f)) = F(f)$ for all $f \in \text{lip}(X,d)$, is linear, continuous and $\|T\| = \|F\|$. By the Hahn–Banach theorem, there exists a linear continuous functional $\tilde{T}: C(X \cup \beta \bar{X}) \to \mathbb{C}$ such that $\tilde{T}(\Phi(f)) = T(\Phi(f))$ for all $f \in \text{lip}(X,d)$ and $\|\tilde{T}\| = \|T\|$.

Since $\|g\| \leq 2 \|g\|_\infty$ for all $g \in C(X \cup \beta \bar{X})$, it follows that the linear functional $\tilde{T}$ is continuous on the space $C(X \cup \beta \bar{X})$ equipped with the norm $\|\|_\infty$. We denote by $\|\|_\infty^*$ the norm on the dual Banach space of $C(X \cup \beta \bar{X})$, $\|\|_\infty$. By the Riesz representation theorem, there exists $\mu \in M(X \cup \beta \bar{X})$ with $\|\tilde{T}\|_\infty^* = \|\mu\|$ satisfying

$$\tilde{T}(g) = \int_{X \cup \beta \bar{X}} g(w) \, d\mu(w), \quad \forall g \in C(X \cup \beta \bar{X}).$$

Since $\|\|_\infty \leq \|g\|$ for all $g \in C(X \cup \beta \bar{X})$, we have $\|\tilde{T}\| \leq \|\tilde{T}\|_\infty^*$, and so $\|F\| \leq \|\mu\|$.

Moreover,

$$F(f) = T(\Phi(f)) = \tilde{T}(\Phi(f)) = \int_{X \cup \beta \bar{X}} \Phi(f)(w) \, d\mu(w)$$

for all $f \in \text{lip}(X,d)$, as we wanted. Similarly, we prove statement (2). \hfill \Box

Such a $\mu$ is called a representing measure for $F$ (analogously, $\nu$ for $G$). We should note that a representing measure for $F$ or $G$ is not always determined uniquely.

3. The main result

In this section we describe all the hermitian operators on $\text{lip}(X,d)$ or $\text{lip}(X,d^\alpha)$ with $0 < \alpha < 1$. We proceed with the statement and proof of our main result.

**Theorem 3.1.** Let $(X,d)$ be a compact metric space. A bounded linear operator $T: \text{lip}(X,d) \to \text{lip}(X,d)$ is hermitian if and only if $T$ is a real multiple of the identity operator on $\text{lip}(X,d)$. An analogous assertion holds for $T: \text{lip}(X,d^\alpha) \to \text{lip}(X,d^\alpha)$ with $0 < \alpha < 1$.

Before proving this theorem we set notation and prove some preliminary lemmas.

Let $A(X)$ denote either $\text{lip}(X,d)$ or $\text{lip}(X,d^\alpha)$, $0 < \alpha < 1$. Recall that $\alpha = 1$ in the case $A(X) = \text{lip}(X,d)$.

**Lemma 3.2.** If $T: A(X) \to A(X)$ is a hermitian bounded linear operator, then the following statements hold:

(i) There exists $\lambda \in \mathbb{R}$ such that $T(1_X) = \lambda 1_X$.

(ii) For each $t \in \mathbb{R}$, $\exp(it(T - \lambda I))$ is a surjective linear isometry on $A(X)$ fixing $1_X$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
For each $t \in \mathbb{R}$, there exists a surjective isometry $\varphi_t$ on $X$ such that
\[ \exp(it(T - \lambda I))(f)(x) = f(\varphi_t(x)), \quad \forall f \in A(X), \forall x \in X. \]

(iii) For each $t \in \mathbb{R}$, there exists a surjective isometry $\varphi_t$ on $X$ such that
\[ \exp(it(T - \lambda I))(f)(x) = f(\varphi_t(x)), \quad \forall f \in A(X), \forall x \in X. \]

Proof. (i) For each $F \in A(X)^*$ with $\|F\| = F(1_X) = 1$, define $\Phi_F : B(A(X)) \to \mathbb{C}$ by
\[ \Phi_F(S) = F(S(1_X)), \quad \forall S \in B(A(X)). \]
It is easy to check that $\Phi_F$ is a linear functional on $B(A(X))$, and since
\[ |\Phi_F(S)| = |F(S(1_X))| \leq \|F\| \|S(1_X)\|_\alpha \leq \|S\| \|1_X\|_\alpha = \|S\|, \]
for all $S \in B(A(X))$, then $\Phi_F$ is continuous and $\|\Phi_F\| \leq 1$. Moreover, $\Phi_F(I) = F(1_X) = 1$; hence $\|\Phi_F\| \geq \|\Phi_F(I)\| = 1$ and thus $\|\Phi_F\| = \Phi_F(I) = 1$.

Since $T \in B(A(X))$ is hermitian, it follows that $F(T(1_X)) = \Phi_F(T) \in V(T) \subset \mathbb{R}$ for all $F \in A(X)^*$ such that $\|F\| = F(1_X) = 1$. This means that $T(1_X) = \lambda 1_X$.

(ii) By (i), we have $(T - \lambda I)(1_X) = 0$ and so $\exp(it(T - \lambda I))(1_X) = 1_X$ for all $t \in \mathbb{R}$. Indeed, since
\[ \exp(it(T - \lambda I)) = I + \sum_{n=1}^{\infty} i^n t^n (T - \lambda I)^n, \]
it follows that
\[ \exp(it(T - \lambda I))(1_X) = 1_X + \sum_{n=1}^{\infty} i^n t^n (T - \lambda I)^n(1_X) = 1_X. \]
Since $T$ and $\lambda I$ are hermitian operators in $B(A(X))$, it is easily seen that so is $T - \lambda I$. Indeed, using the fact that $\exp(it(T - \lambda I)) = \exp(itT)\exp(-it\lambda I)$ for all $t \in \mathbb{R}$, we have
\[ 1 = \|1_X\|_\alpha = \|\exp(it(T - \lambda I))(1_X)\|_\alpha \leq \|\exp(it(T - \lambda I))\| \leq \|\exp(itT)\| \|\exp(-it\lambda I)\| = 1. \]
Therefore, for each $t \in \mathbb{R}$, $\exp(it(T - \lambda I))$ is a linear isometry from $A(X)$ onto itself, fixing $1_X$.

(iii) In view of (ii), by applying Theorem 2.1 for each $t \in \mathbb{R}$ there exists a surjective isometry $\varphi_t$ on $X$ such that
\[ \exp(it(T - \lambda I))(f)(x) = f(\varphi_t(x)), \quad \forall f \in A(X), \forall x \in X. \]

\[ (\varphi_t)_{t \in \mathbb{R}} \text{ is a one-parameter group of surjective isometries on } X \text{ such that, for each } x \in X, \text{ the map } t \mapsto \varphi_t(x) \text{ from } \mathbb{R} \text{ to } X \text{ is continuous.} \]

(iv) For every $\varphi_t \in A(X)$,
\[ \lim_{t \to 0} (f \circ \varphi_t - f)(x) = 0, \quad \forall x \in X, \]
and
\[ \lim_{t \to 0} \frac{(f \circ \varphi_t - f)(x) - (f \circ \varphi_t - f)(y)}{d(x,y)^\alpha} = 0, \quad \forall (x,y) \in \overline{X}. \]
(iv) Using the fact that \( A(X) \) separates the points of \( X \), it is easily derived that \( \varphi_{s+t} = \varphi_s \circ \varphi_t \) for all \( s, t \in \mathbb{R} \) and \( \varphi_0 = I_X \). More precisely, given \( f \in A(X) \) and \( x \in X \), we have

\[
\begin{align*}
f(\varphi_{s+t}(x)) &= \exp(i(s + t)(T - \lambda I))(f)(x) \\
&= \exp(i(t + s)(T - \lambda I))(f)(x) \\
&= \exp(it(T - \lambda I))\exp(is(T - \lambda I))(f)(x) \\
&= \exp(is(T - \lambda I))(f)(\varphi_t(x)) \\
&= f(\varphi_s(\varphi_t(x))) \\
&= f(\varphi_s \circ \varphi_t(x))
\end{align*}
\]

and

\[
f(\varphi_0(x)) = \exp(i0(T - \lambda I))(f)(x) = \exp(0)(f)(x) = I(f)(x) = f(x).
\]

We next prove that for each \( x \in X \), the map \( t \mapsto \varphi_t(x) \) from \( \mathbb{R} \) to \( X \) is continuous. Note first that \( \delta: X \to A(X)^* \) defined by \( \delta(x) = \delta_x \) is a Lipschitz bijection from \((X, d^\alpha)\) onto \( \delta(X) \). Indeed, \( \delta \) is injective since \( A(X) \) separates points; and given \( x, y \in X \), we have

\[
|\delta(x) - \delta(y)|(f) = |f(x) - f(y)| \leq \|f\|_\alpha d(x, y)^\alpha
\]

for all \( f \in A(X) \). Hence \( \|\delta(x) - \delta(y)\| \leq d(x, y)^\alpha \). Since \( X \) is compact, we deduce that \( \delta^{-1}: \delta(X) \to X \) is continuous. Notice that \( \delta^{-1}(\delta_x) = x \) for all \( x \in X \).

Fix \( x \in X \). The maps \( t \mapsto \exp(it(T - \lambda I)) \) from \( \mathbb{R} \) to \( B(A(X)) \), \( U \mapsto U^* \) from \( B(A(X)) \) to \( B(A(X)^*) \) and \( S \mapsto S(\delta(x)) \) from \( B(A(X)^*) \) to \( A(X)^* \) are clearly continuous. From (5), we deduce that

\[
(\exp(it(T - \lambda I)))^*(\delta(x)) = \delta(\varphi_t(x)) \quad (t \in \mathbb{R}, x \in X).
\]

Since

\[
\varphi_t(x) = \delta^{-1}(\exp(it(T - \lambda I)))^*(\delta(x)) \quad (t \in \mathbb{R}, x \in X),
\]

we conclude that \( t \mapsto \varphi_t(x) \) from \( \mathbb{R} \) to \( X \) is continuous.

(v) Let \( f \in A(X) \). Given \( x \in X \), we have \( \lim_{t \to 0} (f \circ \varphi_t)(x) = (f \circ \varphi_0)(x) = f(x) \) by (iv), and thus \( \lim_{t \to 0} (f \circ \varphi_t - f)(x) = 0 \). Using this, for \( (x, y) \in \bar{X} \), we deduce that

\[
\lim_{t \to 0} \frac{(f \circ \varphi_t - f)(x) - (f \circ \varphi_t - f)(y)}{d(x, y)^\alpha} = 0.
\]

We recall that for \( f \in A(X) \) the map \( \bar{f}: \bar{X} \to \mathbb{C} \) is defined to be

\[
\bar{f}(x, y) = (f(x) - f(y))/d(x, y)^\alpha.
\]

We recall that \( \beta\bar{X} \) represents the Stone-Čech compactification of \( \bar{X} \). This entails that every bounded, continuous and scalar-valued map defined on \( \bar{X} \) has a unique continuous extension to \( \beta\bar{X} \).

**Lemma 3.3.** If \( f \in A(X) \), then

\[
\lim_{t \to 0} \beta f_t(w) = 0, \quad \forall w \in \beta\bar{X},
\]

where, for each \( t \in \mathbb{R} \), \( f_t \) denotes the function \( f \circ \varphi_t - f \).
Proof. We define \( g: [-1, 1] \times \tilde{X} \to \mathbb{C} \) by
\[
g(t, (x, y)) = \frac{(f \circ \varphi_t - f)(x) - (f \circ \varphi_t - f)(y)}{d(x, y)^\alpha}.
\]
The function \( g \) is continuous and bounded. In fact, we have
\[
|g(t, (x, y))| \leq p_\alpha(f) + p_\alpha(f) = 2p_\alpha(f)
\]
for all \( t \in [-1, 1] \) and \( (x, y) \in \tilde{X} \). For the continuity of \( g \), define \( \sigma: [-1, 1] \times \tilde{X} \to A(X)^* \) by
\[
\sigma(t, (x, y)) = \frac{\delta(\varphi_t(x)) - \delta(x) - (\delta(\varphi_t(y)) - \delta(y))}{d(x, y)^\alpha}
\]
and notice that
\[
g(t, (x, y)) = \sigma(t, (x, y))(f) \quad (t \in [-1, 1], (x, y) \in \tilde{X}).
\]
Taking into account the equality (6), for any \( t, s \in [-1, 1] \) and \( x, y \in X \), we have
\[
\|\delta(\varphi_t(x)) - \delta(\varphi_s(y))\| = \|(\exp(it(T - \lambda I)))^\ast(\delta(x)) - (\exp(is(T - \lambda I)))^\ast(\delta(y))\|
\leq \|(\exp(it(T - \lambda I)))^\ast\|\|\delta(x) - \delta(y)\| + \|(\exp(it(T - \lambda I)))^\ast - (\exp(is(T - \lambda I)))^\ast\|\|\delta(y)\|
\leq d(x, y) + \|\exp(it(T - \lambda I)) - \exp(is(T - \lambda I))\|\|\delta(y)\|.
\]
Let us recall now that if \( A \) and \( B \) are bounded commuting operators on a Banach algebra, then
\[
\|\exp(iA) - \exp(iB)\| \leq \|A - B\| \exp(\max \{|\|A\|, |\|B\||\}).
\]
Applying this formula to \( A = t(T - \lambda I) \) and \( B = s(T - \lambda I) \), we obtain
\[
\|\exp(it(T - \lambda I)) - \exp(is(T - \lambda I))\| \leq |t - s| k,
\]
where \( k = \|T - \lambda I\| \exp(\|T - \lambda I\|) \) is a constant, and so
\[
\|\delta(\varphi_t(x)) - \delta(\varphi_s(y))\| \leq d(x, y)^\alpha + k |t - s|.
\]
Therefore, for every \( t, s \in [-1, 1] \) and \( x, y \in X \), we have
\[
\|\delta(\varphi_t(x)) - \delta(x) - (\delta(\varphi_s(y)) - \delta(y))\| \leq 2d(x, y)^\alpha + k |t - s|.
\]
Hence the mapping \( (t, (x, y)) \mapsto \delta(\varphi_t(x)) - \delta(x) \), defined on \([ -1, 1 ] \times \tilde{X} \) and with values in \( A(X)^* \), is continuous. Since \((x, y) \mapsto d(x, y)^\alpha \) from \( \tilde{X} \) to \( \mathbb{R} \) is continuous, it follows that \( \sigma \) is continuous. Hence, given \( \varepsilon > 0 \) and \( (t_0, (x_0, y_0)) \in [-1, 1] \times \tilde{X} \), there is a neighborhood \( V \) of \( (t_0, (x_0, y_0)) \) such that if \((t, (x, y)) \in V\), then
\[
\|\sigma(t, (x, y)) - \sigma(t_0, (x_0, y_0))\| < \varepsilon/(1 + \|f\|_\alpha). \]
Therefore, for every \((t, (x, y)) \in V\), we have
\[
|g(t, (x, y)) - g(t_0, (x_0, y_0))| < \frac{\varepsilon}{1 + \|f\|_\alpha} \|f\|_\alpha < \varepsilon,
\]
and this proves that \( g \) is continuous.
We now check that the mapping \( t \mapsto g(t, \cdot) \) from \([-1, 1]\) into \( C(\tilde{X}) \) is continuous. Indeed, by using (5) and (6) we have

\[
|g(t, (x, y)) - g(s, (x, y))| = \left| \frac{\delta(\varphi_t(x)) - \delta(\varphi_s(x))}{d(x, y)} \right| \leq \frac{\left| \delta(\varphi_t(x)) - \delta(\varphi_s(x)) \right|}{d(x, y)} \leq k \|f\|_\alpha \frac{|t - s|}{d(x, y)}
\]

for all \( t, s \in [-1, 1] \) and \((x, y) \in \tilde{X}\), and so \( |g(t, \cdot) - g(s, \cdot)|_\infty \leq k \|f\|_\alpha |t - s|\).

Then \( g \) has a continuous extension \( h \) from \([-1, 1] \times \beta\tilde{X} \) to \( \mathbb{C} \) by [7] Lemma 2. Hence \( h(0, (x, y)) = g(0, (x, y)) = 0 \) for all \((x, y) \in \tilde{X}\). Now fix \( w \in \beta\tilde{X} \) and take a net \( \{(x_i, y_i)\} \) in \( \tilde{X} \) converging to \( w \). The continuity of \( h \) yields \( h(0, w) = \lim_{t \to 0} h(0, (x_i, y_i)) = 0 \). Using the continuity of \( \beta\tilde{f}_t \), with \( t \in [-1, 1] \), and the continuity of \( h \), we have

\[
\lim_{t \to 0} \beta\tilde{f}_t(w) = \lim_{t \to 0} \lim_{i \to 0} \beta\tilde{f}_t(x_i, y_i) = \lim_{i \to 0} \lim_{t \to 0} \tilde{f}_t(x_i, y_i) = \lim_{i \to 0} \lim_{t \to 0} g(t, (x_i, y_i)) = \lim_{i \to 0} h(t, (x_i, y_i)) = \lim_{i \to 0} h(t, w) = h(0, w) = 0.
\]

This completes the proof. \( \qed \)

We now proceed with the details for the proof of our main theorem.

**Proof of Theorem 3.1** Assume that \( T: A(X) \to A(X) \) is a hermitian bounded linear operator. In view of (3), (4) and (7), the maps defined in (1) and (2) satisfy

(9) \[
\lim_{t \to 0} \Phi(f \circ \varphi_t - f)(w) = 0, \quad \forall w \in X \cup \beta\tilde{X},
\]

and

(10) \[
\lim_{t \to 0} \Psi(f \circ \varphi_t - f)(w) = 0, \quad \forall w \in X \cup \tilde{X}.
\]

Note also that

(11) \[
|\Phi(f \circ \varphi_t - f)(w)| \leq \|\Phi(f \circ \varphi_t - f)\| = \|f \circ \varphi_t - f\|_1 \leq 2 \|f\|_1, \quad \forall t \in \mathbb{R}, \forall w \in X \cup \beta\tilde{X},
\]

and

(12) \[
|\Psi(f \circ \varphi_t - f)(w)| \leq \|\Psi(f \circ \varphi_t - f)\| = \|f \circ \varphi_t - f\|_\alpha \leq 2 \|f\|_\alpha, \quad \forall t \in \mathbb{R}, \forall w \in X \cup \tilde{X}.
\]

Now, for each \( t \in \mathbb{R} \), define \( T_t: A(X) \to A(X) \) by

(13) \[
T_t(f)(x) = f(\varphi_t(x)), \quad \forall f \in A(X), \forall x \in X.
\]

Each \( T_t \) is an isometric algebra automorphism of \( A(X) \). Moreover, it is immediate that \( T_{t+s} = T_s \circ T_t \) for all \( s, t \in \mathbb{R} \) and \( T_0 = I \). Indeed, for any \( f \in A(X) \), we have

\[
T_{t+s}(f) = f \circ \varphi_{t+s} = f \circ \varphi_t \circ \varphi_s = T_s(f \circ \varphi_t) = T_s(T_t(f)) = T_s \circ T_t(f)
\]

and

\[
T_0(f) = f \circ \varphi_0 = f \circ I_X = f.
\]
Thus \( \{T_t : t \in \mathbb{R}\} \) is a one-parameter group of isometric algebra automorphisms of \( A(X) \). Indeed, notice that \( T_{(-t)} = (T_t)^{-1} \) for every \( t \in \mathbb{R} \) since
\[
I = T_0 = T_{(t+(-t))} = T_t \circ T_{(-t)}, \quad I = T_0 = T_{(-t)+t} = T_{(-t)} \circ T_t \quad (t \in \mathbb{R}).
\]

Next we will prove that \( \{T_t : t \in \mathbb{R}\} \) is strongly continuous; that is, the maps \( t \mapsto T_t(f) \) from \( \mathbb{R} \) to \( A(X) \) are continuous for each \( f \in A(X) \) [5, Chapter I, 1.1 Definition]. According to [5, Chapter I, 1.6 Theorem], it is sufficient to show that \( \{T_t : t \in \mathbb{R}\} \) is weakly continuous; that is, the maps \( t \mapsto F(T_t(f)) \) from \( \mathbb{R} \) to \( \mathbb{C} \) are continuous for every \( f \in A(X) \) and \( F \in A(X)^* \). To prove this, notice that we only need to see that the aforementioned maps are continuous at 0, and it is easily checked.

Let \( f \in \text{Lip}(X,d) \) and \( F \in \text{Lip}(X,d)^* \). By Lemma 2.3 (1), there exists \( \mu \in \mathcal{M}(X \cup \beta X) \) such that
\[
F(g) = \int_{X \cup \beta X} \Phi(g)(w) \, d\mu(w) \quad (g \in \text{Lip}(X,d)).
\]

In particular, we have
\[
F(T_t(f) - f) = \int_{X \cup \beta X} \Phi(T_t(f) - f)(w) \, d\mu(w) \quad (t \in \mathbb{R}).
\]

Taking into account (9) and (11), we can apply Lebesgue’s bounded convergence theorem and get
\[
\lim_{t \to 0} F(T_t(f) - f) = \lim_{t \to 0} \int_{X \cup \beta X} \Phi(T_t(f) - f)(w) \, d\mu(w)
\]
\[
= \lim_{t \to 0} \int_{X \cup \beta X} \Phi(f \circ \varphi_t - f)(w) \, d\mu(w)
\]
\[
= \int_{X \cup \beta X} \lim_{t \to 0} \Phi(f \circ \varphi_t - f)(w) \, d\mu(w) = 0.
\]

Similarly, for any \( f \in \text{lip}(X,d^\alpha) \) and \( G \in \text{lip}(X,d^\alpha)^* \), we obtain
\[
\lim_{t \to 0} G(T_t(f) - f) = 0
\]
b by using Lemma 2.3 (2) and the equalities (10) and (12).

Let \( A : D(A) \subset A(X) \to A(X) \) be the generator of \( \{T_t : t \in \mathbb{R}\} \). That is, \( A \) is the linear operator
\[
A(f) = \lim_{t \to 0} \frac{T_t(f) - f}{t}
\]
defined for every \( f \) in its domain
\[
D(A) = \left\{ f \in A(X) : \exists \lim_{t \to 0} \frac{T_t(f) - f}{t} \right\}
\]
(see [5, Chapter II, 1.2 Definition]). By the equalities (5) and (13), we have
\[
T_t(f) = \exp(it(T - \lambda I))(f), \quad \forall f \in A(X).
\]

It follows that
\[
\lim_{t \to 0} \frac{T_t(f) - f}{t} = i(T - \lambda I)(f), \quad \forall f \in A(X)
\]
[5, Chapter I, 2.11 Proposition]; hence \( D(A) = A(X) \) and
\[
A(f) = i(T - \lambda I)(f), \quad \forall f \in A(X).
\]
Since \( \{T_t : t \in \mathbb{R} \} \) is a group of algebra homomorphisms on \( A(X) \), then \( A \) is a derivation on \( A(X) \); that is, \( A(fg) = A(f)g + fA(g) \) for all \( f, g \in A(X) \). Indeed,

\[
A(fg) = \lim_{t \to 0} \frac{T_t(fg) - fg}{t} \\
= \lim_{t \to 0} \frac{T_t(f)T_t(g) - fg}{t} \\
= \lim_{t \to 0} \frac{(T_t(f) - f)T_t(g) + f(T_t(g) - g)}{t} \\
= \lim_{t \to 0} \left( \frac{T_t(f) - f}{t} T_t(g) + \frac{f}{t} T_t(g) - g \right) \\
= A(f)T_0(g) + fA(g) \\
= A(f)g + fA(g).
\]

Since all linear derivations on a complex commutative semi-simple Banach algebra are trivial \([16,18]\), it follows that \( A \) is zero, and so \( T = \lambda I \), as desired. The converse implication follows immediately, and so the proof of the theorem is finished. \( \square \)

**Acknowledgement**

The authors wish to thank the referee for making several suggestions which improved this paper.

**References**


Department of Mathematical Sciences, The University of Memphis, Memphis, Tennessee 38152

E-mail address: mbotelho@memphis.edu

Department of Mathematical Sciences, The University of Memphis, Memphis, Tennessee 38152

E-mail address: jjamison@memphis.edu

Departamento de Álgebra y Análisis Matemático, Universidad de Almería, 04120 Almería, Spain

E-mail address: ajimenez@ual.es

Campus Universitario de Puerto Real, Facultad de Ciencias, Nuestra Universidad, 11510 Puerto Real, Cádiz, Spain

E-mail address: moises.villegas@uca.es