A CHARACTERIZATION OF SUBMODULES VIA
THE BEURLING-LAX-HALMOS THEOREM

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Abstract. Shift invariant subspaces in the vector-valued Hardy space \( H^2(E) \) play important roles in Nagy-Foias operator model theory. A theorem by Beurling, Lax and Halmos characterizes such invariant subspaces by operator-valued inner functions \( \Theta(z) \). When \( E = H^2(D) \), \( H^2(E) \) is the Hardy space over the bidisk \( H^2(D^2) \). This paper shows that for some well-known examples of invariant subspaces in \( H^2(D^2) \), the function \( \Theta(z) \) turns out to be strikingly simple.

1. Introduction

Let \( E \) be a separable Hilbert space of infinite dimension and \( H^2(E) \) be the \( E \)-valued Hardy space, i.e.
\[
H^2(E) := \{ u(z) = \sum_{j=0}^{\infty} z^j x_j : \sum_{j=0}^{\infty} \| x_j \|^2_E < \infty, z \in D \}.
\]

Let \( T_z \) be the Toeplitz operator on \( H^2(E) \) such that for any \( f \in H^2(E), T_z f(z) = z f(z). \) \( T_z \) is often called a shift operator. A closed subspace \( M \subset H^2(E) \) is said to be invariant if \( T_z M \subset M \). Denote the set of bounded linear operators on \( E \) by \( B(E) \). Then a \( B(E) \)-valued analytic function \( \Theta(z) \) on \( D \) is said to be inner if \( \| \Theta(z) \| \leq 1 \) for each \( z \in D \), and on the boundary \( T \), \( \Theta(z) \) is almost everywhere an isometry (with respect to the Lebesgue measure on \( T \)). In [1], Beurling gave a representation of invariant subspaces of \( T_z \) on the classical Hardy space \( H^2(D) \) in terms of inner functions. This was generalized by Lax [2] and Halmos [3], and is called the Beurling-Lax-Halmos Theorem: \( M \subset H^2(E) \) is invariant if and only if there exists a \( B(E) \)-valued inner function \( \Theta \) such that \( M = \Theta(z) H^2(E) \). Compression of \( T_z \) to the quotient space \( H^2(E) \oplus M \) shall be denoted by \( S_z \). It is a classical theorem (cf. [5]) that every bounded linear operator on a separable Hilbert space is of the form \( S_z \) up to a scalar multiple and unitary equivalence. Moreover, there exists a spectral connection. Let \( G_1 = \{ \lambda \in \mathbb{D} : \Theta(\lambda) \) is not invertible \( \} \), and let \( G_2 \) be the collection of \( \lambda \in \partial \mathbb{D} \) such that \( \Theta \) has no analytic extension to a neighborhood \( U \) of \( \lambda \) and \( \Theta \) is unitary-valued on \( U \cap \partial \mathbb{D} \). Then the spectrum of \( \Theta \) is defined as \( \sigma(\Theta) = G_1 \cup G_2 \). Then we have the following lemma (cf. [8]).

Lemma 1.1. \( \sigma(S_z) = \sigma(\Theta). \)
When $E = H^2(\mathbb{D})$, $H^2(E)$ is the Hardy space over the bidisk $H^2(\mathbb{D}^2)$. A closed subspace $M$ that is invariant for both $T_z$ and $T_w$ is often called a submodule. As opposed to invariant subspaces in $H^2(\mathbb{D})$, submodules in $H^2(\mathbb{D}^2)$ are far more complicated, and a complete characterization of them seems beyond reach at this point. However, many promising studies have been made in recent years, and two notable examples emerge in this study. One is given by a sequence of inner functions, and the other is given by two inner functions.

**Example 1.** An inner sequence is a sequence of inner functions $\{\phi_l(w) : l = 0, 1, 2, \ldots\}$ with $\phi_{l+1}(w)\phi_l(w)$. Now consider the inner-sequence-based submodule

$$
M = \bigoplus_{l=0}^{\infty} \phi_l(w)H^2_w z^l.
$$

It is not hard to check that $M$ is a submodule. Rudin considered an example of this kind in [6], where he showed that this kind of submodule can have infinite rank. This form of definition, as well as a general study, was made in [9] by Seto and the second author. Among other things, it is shown that $S_w$ is, in this case, a model of the so-called $C_0$ class contractions.

**Example 2.** Suppose $\phi_1(z)$ and $\phi_2(w)$ are two inner functions (or 0) with variables $z$ and $w$ respectively. Define

$$
M = \phi_1(z)H^2(\mathbb{D}^2) + \phi_2(w)H^2(\mathbb{D}^2).
$$

Then it can be verified that $M$ is a submodule. This submodule emerges in the work of Izuchi, Nakazi and Seto in [4] as they study the commutator $[S^*_z, S_w]$, where $S_w$ is the compression of $T_w$ to $H^2(\mathbb{D}^2) \ominus M$. They showed that a submodule of this form if and only if $[S^*_z, S_w] = 0$.

This paper determines the operator inner function $\Theta(z)$ for the two examples above.

## 2. INNER-SEQUENCE-BASED INVARIANT SUBSPACES

Notice that $\Theta(z)$ is analytic on $\mathbb{D}$. Assume its power series representation is $\Theta(z) = \sum_{l=0}^{\infty} z^l P_l$, where $z \in \mathbb{D}$ and $P_l$ are operators on $H^2_w$. For an inner sequence $\{\phi_l(w) : l \geq 0\}$, we let $M = \bigoplus_{l=0}^{\infty} \phi_l(w)H^2_z z^l$. One checks easily that $M$ is a submodule in $H^2(\mathbb{D}^2)$ (cf. [9]). One note needs to be made here. First, if we let $\phi_\infty$ be the greatest common divisor of $\{\phi_l(w) : l \geq 0\}$, then up to some scalar normalization, the sequence $\{\phi_l(w) : l \geq 0\}$ converges to $\phi_\infty$ in $H^2_w$. If $\phi_\infty$ is nontrivial, then we can write $M = \phi_\infty(w)M'$, where $M'$ is an inner-sequence-based submodule with the inner sequence having greatest common divisor equal to 1. Since $M$ and $M'$ are unitarily equivalent, we assume in this paper, without loss of generality, that $\phi_\infty = 1$. The following theorem completely characterizes inner-sequence-based submodules in terms of $\Theta(z)$.

**Theorem 2.1.** Let $\Theta(z) = \sum_{l=0}^{\infty} z^l P_l$ be the operator inner function for a submodule $M$. Then $M$ is inner-sequence-based if and only if $P_l, l = 0, 1, 2, \ldots,$ are orthogonal projections on $H^2_w$ with perpendicular ranges.
Proof. \(\Rightarrow\): Since \(M = \bigoplus_{i=1}^{\infty} \phi_i(w) H^2_w z^i\), it’s easy to compute that
\[
M \ominus zM = \phi_0 H^2_w \ominus \bigoplus_{i=1}^{\infty} z^i (\phi_i H^2_w \ominus \phi_{i-1} H^2_w).
\]
For simplicity, we let \(N_l = \phi_l H^2_w \ominus \phi_{l-1} H^2_w, l \geq 1\). Let \(P_l\) be the orthogonal projection from \(H^2_w\) onto \(N_l\), \(l \geq 1\), \(P_0\) be the orthogonal projection from \(H^2_w\) onto \(\phi_0 H^2_w\), and set \(\Theta(z) = \sum_{l=0}^{\infty} z^l P_l\). Two observations need to be made here. First, if there is a nonnegative integer \(k\) such that \(\phi_j(w) H^2_w = \phi_k(w) H^2_w\) for any \(j \geq k\), then \(P_j = 0\) for \(j \geq k\), and in this case \(\Theta(z)\) is a polynomial. Second, by the assumption that \(\phi_\infty = 1\),
\[
H^2_w = \phi_0 H^2_w \oplus \bigoplus_{l=1}^{\infty} N_l;
\]
hence \(\sum_{l=0}^{\infty} P_l\) converges strongly to the identity operator \(I\). We now show that \(\Theta(z)\) is an isometry a.e. on \(\mathbb{T}\). Check that for any \(g(w) \in H^2_w\), \(\Theta(z)g = \sum_{l=0}^{\infty} z^l P_l g\) and
\[
\|\Theta(z)g(w)\| = \sum_{l=0}^{\infty} \|P_l g(w)\| \leq \sum_{l=0}^{\infty} \|P_l g(w)\| = \|g(w)\|.
\]
We conclude that \(\Theta(z)\) is an operator inner function. Further, since \(M \ominus zM = \Theta(z) H^2_w\), \(M = \Theta(z) H^2(\mathbb{D}^2)\).

\(\Leftarrow\): Suppose \(P_l\), \(l \geq 0\), are orthogonal projections on \(H^2_w\) with perpendicular ranges, and \(\Theta(z) = \sum_{l=0}^{\infty} z^l P_l\). Then
\[
M = \Theta(z) H^2(\mathbb{D}^2) = \sum_{l=0}^{\infty} z^l P_l \left( \bigoplus_{k=0}^{\infty} H^2_w \right) = \sum_{n=0}^{\infty} z^n \left( \bigoplus_{l=0}^{n} P_l H^2_w \right) = \bigoplus_{l=0}^{n} P_l H^2_w = \phi_n(w) H^2_w.
\]
(2.1)

Since \(T_w M \subset M\), each closed subspace \(\bigoplus_{l=0}^{n} P_l H^2_w, n \geq 0\), is an invariant subspace for \(T_w\). By Beurling’s Theorem, there exists an inner function \(\phi_n(w)\) such that
\[
\bigoplus_{l=0}^{n} P_l H^2_w = \phi_n(w) H^2_w.
\]
Clearly, \(\phi_n | \phi_{n-1}, n \geq 1\); hence \(\{\phi_n\}\) is an inner sequence, and by (2.1)
\[
M = \bigoplus_{n=0}^{\infty} z^n \phi_n H^2_w.
\]
\(\square\)

Since \(M\) is also invariant under \(T_w\), there is an operator inner function \(\Gamma(w)\) such that \(M = \Gamma H^2(\mathbb{D}^2)\), where \(\Gamma(\lambda)\) is an operator on \(H^2_z\) for each \(\lambda \in \mathbb{D}\). Thus \(\Gamma\) can also be determined.
Proposition 2.2. Let $M = \bigoplus_{l=0}^{\infty} \phi_l(w)H^2_wz^l$. If $\Gamma(w)$ is the operator inner function such that $M = \Gamma(w)H^2(\mathbb{D}^2)$, then $\Gamma(w) = \sum_{l=0}^{\infty} \phi_l(w)P_l$, where $P_l$ is the projection from $H^2_w$ to $\mathbb{C}z^l$.

Proof. It is not hard to verify that $M \ominus wM = \bigoplus_{l=0}^{\infty} \phi_l(w)Cz^l$. So if one defines

$$\Gamma(w) = \sum_{l=0}^{\infty} \phi_l(w)P_l,$$

then $\Gamma(w)H^2_w = M \ominus wM$, and hence $M = \Gamma(w)H^2(\mathbb{D}^2)$. Further, for every $g \in H^2_w$,

$$\|\Gamma(w)g\|^2 = \sum_{l=0}^{\infty} |\phi_l(w)|^2\|P_lg\|^2$$

$$= \sum_{l=0}^{\infty} \|P_lg\|^2$$

$$= \|g\|^2$$

for almost every $w \in \partial\mathbb{D}$. This verifies that $\Gamma$ is inner. \qed

3. Submodule generated by two inner functions

Suppose $\phi_1(z)$ and $\phi_2(w)$ are two inner functions with variables $z$ and $w$ respectively. In this section we consider the submodule of the form (1.2).

Theorem 3.1. Let $\Theta(z)$ be the operator inner function for a submodule $M$. Then $M$ is of the form (1.2) if and only if $\Theta(z) = \phi_1(z)P_0 + P_1$, where $P_0$ and $P_1$ are complementary projections on $H^2_w$, e.g., $P_0P_1 = 0$ and $P_0 + P_1 = I$.

Proof. Suppose $M$ is of the form (1.2). In [10], it was shown that

$$M \ominus zM = \phi_1(z)(H^2_w \ominus \phi_2(w)H^2_w) \ominus \phi_2(w)H^2_w.$$

Set $\Theta(z) = \phi_1(z)P_0 + P_1$, where $P_0 : H^2_w \rightarrow H^2_w \ominus \phi_2(w)H^2_w$ is the orthogonal projection and $P_1 = I - P_0$. Then for every $g \in H^2_w$, by the Pythagorean theorem,

$$\|\Theta(z)g\|^2 = |\phi_1(z)|^2\|P_0g\|^2 + \|P_1g\|^2$$

$$= \|P_0g\|^2 + \|P_1g\|^2$$

$$= \|g\|^2$$

a.e. on $\mathbb{T}$. This shows that $\Theta$ is an operator inner function. Further, it is not hard to check $\Theta(z)H^2_w = M \ominus zM$, and hence

$$M = \bigoplus_{n=0}^{\infty} z^n(M \ominus zM) = \Theta(z)H^2(\mathbb{D}^2).$$

On the other hand, suppose $\Theta(z) = \phi_1(z)P_0 + P_1$, where $\phi_1$ is inner and $P_0$ and $P_1$ are complementary projections on $H^2_w$. Then

$$M = \Theta(z)H^2(\mathbb{D}^2)$$

$$= \phi_1(H^2_z \otimes P_0H^2_w) \oplus (H^2_z \otimes P_1H^2_w).$$

(3.1)

First, we will show that $T_w$ and $P_1$ commute on $M$. Denote $M_0 = H^2_z \otimes P_0H^2_w$ and $M_1 = H^2_z \otimes P_1H^2_w$, and let $P_{M_0}$ and $P_{M_1}$ stand for the projections from $H^2(\mathbb{D}^2)$ to
$M_0$ and $M_1$ respectively. Then, with respect to the decomposition (3.1), we rewrite $T_w$ on $M$ as

$$T_w = \begin{pmatrix} P_{M_0} T_w P_{M_0} & P_{M_0} T_w P_{M_1} \\ P_{M_1} T_w P_{M_0} & P_{M_1} T_w P_{M_1} \end{pmatrix}.$$ 

Since $M$ is invariant under $T_w$, we have

$$\begin{pmatrix} P_{M_0} T_w P_{M_0} & P_{M_0} T_w P_{M_1} \\ P_{M_1} T_w P_{M_0} & P_{M_1} T_w P_{M_1} \end{pmatrix} \begin{pmatrix} \phi_1 M_0 \\ M_1 \end{pmatrix} \subset \begin{pmatrix} \phi_1 M_0 \\ M_1 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} \phi_1 P_{M_0} T_w M_0 + P_{M_0} T_w M_1 \\ \phi_1 P_{M_1} T_w M_0 + P_{M_1} T_w M_1 \end{pmatrix} \subset \begin{pmatrix} \phi_1 M_0 \\ M_1 \end{pmatrix}.$$ (3.2)

Consider the first line in (3.2). It is clear that $\phi_1 P_{M_0} T_w M_0 \subset \phi_1 M_0$, and hence $P_{M_0} T_w M_1 \subset \phi_1 M_0$. Check that $P_{M_0} T_w M_1 = H_\omega^2 \otimes P_0 w P_1 H_\omega^2$ and $\phi_1 M_0 = \phi_1 H_\omega^2 \otimes P_0 H_\omega^2$. Therefore, since $\phi_1$ is nontrivial, the first inclusion in (3.2) holds only if $P_0 w P_1 H_\omega^2 = \{0\}$. This implies $T_w P_1 H_\omega^2 \subset P_1 H_\omega^2$. By Beurling’s Theorem, there exists an inner function $\phi_2(w)$ such that $P_1 H_\omega^2 = \phi_2(w) H_\omega^2$. Thus

$$M = \phi_1(H_\omega^2 \otimes P_0 H_\omega^2) \oplus (H_\omega^2 \otimes \phi_2 H_\omega^2) = \phi_1 H^2(\mathbb{D}^2) + \phi_2 H^2(\mathbb{D}^2).$$

Because of symmetry, the operator inner function $\Gamma(w)$ for $M$ in the variable $w$ is $P_0 + \phi_2(w) P_1$.

4. Some applications

For a submodule $M$, the quotient module $N = H^2 \ominus M$ is defined as the orthogonal complement of $M$ in $H^2(\mathbb{D}^2)$. We let $S_z$ (respectively $S_w$) denote the compression of $T_z$ (respectively $T_w$) to $N$, i.e. $S_z = P_N T_z|N$ (respectively $S_w = P_N T_w|N$), where $P_N$ denotes the orthogonal projection from $H^2(\mathbb{D}^2)$ onto $N$. Lemma 1.1 indicates that the spectra of $S_z$ and $S_w$ can be calculated through the corresponding operator inner functions $\Theta(z)$ and $\Gamma(w)$. This section will make use of the results in Sections 2 and 3 to this end. All the results in this section are known, but the work here will show how elegantly these results can be obtained through operator inner functions.

For the inner-sequence-based submodule, spectra of both $S_z$ and $S_w$ are determined in [7]. The proof here is new.

**Proposition 4.1.** Let $M$ be an inner-sequence-based invariant subspace defined in (1.1). Then $\sigma(S_z)$ is either $\{0\}$ or $\mathbb{D}$, and $\sigma(S_w) = \sigma(\phi_0)$.

**Proof.** For $\lambda \in \mathbb{D}$, $\lambda$ is in $\sigma(S_z)$ if and only if $\Theta$ is not invertible at $\lambda$. We know by Theorem 2.1 that $\Theta(z) = \sum_{l=0}^{\infty} z^l P_l$ with $P_l$ orthogonal. There are two cases. First, if there is a nonnegative integer $k$ such that $P_l = 0$ for all $l \geq k$, then $\Theta(z)$ is a polynomial, and by (2.1),

$$M = \sum_{n=0}^{k-1} z^n (\bigoplus_{l=0}^{n} P_l H_\omega^2) \oplus z^k H^2(\mathbb{D}^2).$$

This implies that $(S_z)^k = 0$ and hence $\sigma(S_z) = \{0\}$. Second, if $\Theta(z)$ is not a polynomial, then $\Theta(\lambda)$ is not invertible for all $\lambda \in \mathbb{D}$, because we can select $\{e_n\}_{n=0}^{\infty}$ with $e_n \in Range(P_n)$ and $\|e_n\| = 1$ (or 0 if $P_n = 0$); then

$$\|\Theta(\lambda)e_n\| = |\lambda^n| \rightarrow 0.$$
Now since $\mathbb{D}$ is dense in $\overline{\mathbb{D}}$ and $\sigma(S_z)$ is closed, we have $\overline{\mathbb{D}} \subseteq \sigma(S_z)$. Also, since $S_z$ is a contraction, $\sigma(S_z) \subseteq \mathbb{D}$. Therefore $\sigma(S_z) = \mathbb{D}$.

For $\sigma(S_w)$, we recall that $\Gamma(w) = \sum_{m=0}^{\infty} \phi_m(w)P_m$, where $P_m$ is the orthogonal projection from $H^2_z$ to $\mathbb{C}z^m$. So for a point $w$ in $\mathbb{D}$, $\Gamma(w)$ is invertible if and only if $|\phi_m(w)|$ is bounded below for all $m \geq 0$. Because $|\phi_m(w)| \geq |\phi_0(w)|$ for each $m$, it is equivalent to $\phi_0(w)$ being bounded below at $w$, i.e. $\phi_0(w) \neq 0$. For $w \in \partial\mathbb{D}$, $w \notin G_2$ (defined above in Lemma 1.1) if and only if $\phi_m$ has an analytic continuation to a neighborhood of $w$ for each $m$. Clearly, this is so if and only if $\phi_0$ has an analytic continuation to a neighborhood of $w$. In conclusion, $\sigma(S_w) = \sigma(\Gamma) = \sigma(\phi_0)$. $\square$

Next, we look at the submodule generated by two inner functions. In this case since $\Theta(z) = \phi_1P_0 + P_1$ and $\Gamma(w) = P_0 + \phi_2(w)P_1$ for some complementary projections $P_0$ and $P_1$, the next proposition is immediate.

**Proposition 4.2.** Let $M$ be a submodule of the form (1.2). Then $\sigma(S_z) = \sigma(\phi_1(z))$ and $\sigma(S_w) = \sigma(\phi_2(w))$.

**References**


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