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A CHARACTERIZATION OF SUBMODULES VIA THE BEURLING-LAX-HALMOS THEOREM

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Abstract. Shift invariant subspaces in the vector-valued Hardy space $H^2(E)$ play important roles in Nagy-Foias operator model theory. A theorem by Beurling, Lax and Halmos characterizes such invariant subspaces by operator-valued inner functions $\Theta(z)$. When $E = H^2(\mathbb{D})$, $H^2(E)$ is the Hardy space over the bidisk $H^2(\mathbb{D}^2)$. This paper shows that for some well-known examples of invariant subspaces in $H^2(\mathbb{D}^2)$, the function $\Theta(z)$ turns out to be strikingly simple.

1. Introduction

Let E be a separable Hilbert space of infinite dimension and $H^2(E)$ be the E-valued Hardy space, i.e.

$$H^2(E) := \{u(z) = \sum_{j=0}^{\infty} z^j x_j : \sum_{j=0}^{\infty} \|x_j\|_E^2 < \infty, z \in \mathbb{D}\}.$$

Let T_z be the Toeplitz operator on $H^2(E)$ such that for any $f \in H^2(E)$, $T_z f(z) = zf(z)$. T_z is often called a shift operator. A closed subspace $M \subset H^2(E)$ is said to be invariant if $T_z M \subset M$. Denote the set of bounded linear operators on E by $B(E)$. Then a $B(E)$-valued analytic function $\Theta(z)$ on $\mathbb{D}$ is said to be inner if $\|\Theta(z)\| \leq 1$ for each $z \in \mathbb{D}$, and on the boundary $\mathbb{T}$, $\Theta(z)$ is almost everywhere an isometry (with respect to the Lebesgue measure on $\mathbb{T}$). In [1], Beurling gave a representation of invariant subspaces of T_z on the classical Hardy space $H^2(\mathbb{D})$ in terms of inner functions. This was generalized by Lax [2] and Halmos [3], and is called the Beurling-Lax-Halmos Theorem: $M \subset H^2(E)$ is invariant if and only if there exists a $B(E)$-valued inner function Θ such that $M = \Theta(z)H^2(E)$. Compression of T_z to the quotient space $H^2(E) \ominus M$ shall be denoted by S_z. It is a classical theorem (cf. [5,8]) that every bounded linear operator on a separable Hilbert space is of the form S_z up to a scalar multiple and unitary equivalence. Moreover, there exists a spectral connection. Let $G_1 = \{\lambda \in \mathbb{D} : \Theta(\lambda) \text{ is not invertible}\}$, and let G_2 be the collection of $\lambda \in \partial\mathbb{D}$ such that Θ has no analytic extension to a neighborhood U of λ and Θ is unitary-valued on $U \cap \partial\mathbb{D}$. Then the spectrum of Θ is defined as $\sigma(\Theta) = G_1 \cup G_2$. Then we have the following lemma (cf. [8]).

Lemma 1.1. $\sigma(S_z) = \sigma(\Theta)$.

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When $E = H^2(\mathbb{D})$, $H^2(E)$ is the Hardy space over the bidisk $H^2(\mathbb{D}^2)$. A closed subspace M that is invariant for both T_z and T_w is often called a submodule. As opposed to invariant subspaces in $H^2(\mathbb{D})$, submodules in $H^2(\mathbb{D}^2)$ are far more complicated, and a complete characterization of them seems beyond reach at this point. However, many promising studies have been made in recent years, and two notable examples emerge in this study. One is given by a sequence of inner functions, and the other is given by two inner functions.

Example 1. An inner sequence is a sequence of inner functions $\{\phi_l(w) : l = 0, 1, 2, \ldots\}$ with $\phi_{l+1}(w)|\phi_l(w)$. Now consider the inner-sequence-based submodule

$$M = \bigoplus_{l=0}^{\infty} \phi_l(w) H_w^2 z^l. \tag{1.1}$$

It is not hard to check that M is a submodule. Rudin considered an example of this kind in [6], where he showed that this kind of submodule can have infinite rank. This form of definition, as well as a general study, was made in [9] by Seto and the second author. Among other things, it is shown that S_w is, in this case, a model of the so-called C_0 class contractions.

Example 2. Suppose $\phi_1(z)$ and $\phi_2(w)$ are two inner functions (or 0) with variables z and w respectively. Define

$$M = \phi_1(z)H^2(\mathbb{D}^2) + \phi_2(w)H^2(\mathbb{D}^2). \tag{1.2}$$

Then it can be verified that M is a submodule. This submodule emerges in the work of Izuchi, Nakazi and Seto in [4] as they study the commutator $[S_z^*,\ S_w]$, where S_w is the compression of T_w to $H^2(\mathbb{D}^2) \ominus M$. They showed that a submodule is of this form if and only if $[S_z^*,\ S_w] = 0$.

This paper determines the operator inner function $\Theta(z)$ for the two examples above.

2. Inner-sequence-based invariant subspaces

Notice that $\Theta(z)$ is analytic on $\mathbb{D}$. Assume its power series representation is $\Theta(z) = \sum_{l=0}^{\infty} z^l P_l$, where $z \in \mathbb{D}$ and P_l are operators on H_w^2. For an inner sequence $\{\phi_l(w) : l \geq 0\}$, we let $M = \bigoplus_{l=0}^{\infty} \phi_l(w) H_w^2 z^l$. One checks easily that M is a submodule in $H^2(\mathbb{D}^2)$ (cf. [9]). One note needs to be made here. First, if we let ϕ_∞ be the greatest common divisor of $\{\phi_l(w) : l \geq 0\}$, then up to some scalar normalization, the sequence $\{\phi_l(w) : l \geq 0\}$ converges to ϕ_∞ in H_w^2. If ϕ_∞ is nontrivial, then we can write $M = \phi_\infty(w)M'$, where M' is an inner-sequence-based submodule with the inner sequence having greatest common divisor equal to 1. Since M and M' are unitarily equivalent, we assume in this paper, without loss of generality, that $\phi_\infty = 1$. The following theorem completely characterizes inner-sequence-based submodules in terms of $\Theta(z)$.

Theorem 2.1. *Let $\Theta(z) = \sum_{l=0}^{\infty} z^l P_l$ be the operator inner function for a submodule M. Then M is inner-sequence-based if and only if P_l, $l = 0, 1, 2 \ldots$, are orthogonal projections on H_w^2 with perpendicular ranges.*

Proof. $\Rightarrow$: Since $M = \bigoplus_{l=1}^{\infty} \phi_l(w) H_w^2 z^l$, it's easy to compute that

$$M \ominus zM = \phi_0 H_w^2 \oplus \bigoplus_{l=1}^{\infty} z^l (\phi_l H_w^2 \ominus \phi_{l-1} H_w^2).$$

For simplicity, we let $N_l = \phi_l H_w^2 \ominus \phi_{l-1} H_w^2,\ l \geq 1$. Let P_l be the orthogonal projection from H_w^2 onto N_l, $l \geq 1$, P_0 be the orthogonal projection from H_w^2 onto $\phi_0 H_w^2$, and set $\Theta(z) = \sum_{l=0}^{\infty} z^l P_l$. Two observations need to be made here. First, if there is a nonnegative integer k such that $\phi_j(w) H_w^2 = \phi_k(w) H_w^2$ for any $j \geq k$, then $P_j = 0$ for $j \geq k$, and in this case $\Theta(z)$ is a polynomial. Second, by the assumption that $\phi_\infty = 1$,

$$H_w^2 = \phi_0 H_w^2 \oplus \bigoplus_{l=1}^{\infty} N_l;$$

hence $\sum_{l=0}^{\infty} P_l$ converges strongly to the identity operator I. We now show that $\Theta(z)$ is an isometry a.e. on $\mathbb{T}$. Check that for any $g(w) \in H_w^2$, $\Theta(z) g = \sum_{l=0}^{\infty} z^l P_l g$ and

$$\begin{aligned} \|\Theta(z) g(w)\|^2 &= \sum_{l=0}^{\infty} \|P_l g(w)\|^2 \\ &= \|\sum_{l=0}^{\infty} P_l g(w)\|^2 \\ &= \|g(w)\|^2. \end{aligned}$$

We conclude that $\Theta(z)$ is an operator inner function. Further, since $M \ominus zM = \Theta(z) H_w^2$, $M = \Theta(z) H^2(\mathbb{D}^2)$.

$\Leftarrow$: Suppose $P_l,\ l \geq 0$, are orthogonal projections on H_w^2 with perpendicular ranges, and $\Theta(z) = \sum_{l=0}^{\infty} z^l P_l$. Then

$$\begin{aligned} M &= \Theta(z) H^2(\mathbb{D}^2) \\ &= \sum_{l=0}^{\infty} z^l P_l (\bigoplus_{k=0}^{\infty} z^k H_w^2) \\ &= \sum_{n=0}^{\infty} z^n (\bigoplus_{l=0}^{n} P_l H_w^2). \end{aligned} \tag{2.1}$$

Since $T_w M \subset M$, each closed subspace $\bigoplus_{l=0}^{n} P_l H_w^2,\ n \geq 0$, is an invariant subspace for T_w. By Beurling's Theorem, there exists an inner function $\phi_n(w)$ such that

$$\bigoplus_{l=0}^{n} P_l H_w^2 = \phi_n(w) H_w^2.$$

Clearly, $\phi_n | \phi_{n-1},\ n \geq 1$; hence $\{\phi_n\}$ is an inner sequence, and by (2.1)

$$M = \bigoplus_{n=0}^{\infty} z^n \phi_n H_w^2.$$

□

Since M is also invariant under T_w, there is an operator inner function $\Gamma(w)$ such that $M = \Gamma H^2(\mathbb{D}^2)$, where $\Gamma(\lambda)$ is an operator on H_z^2 for each $\lambda \in \mathbb{D}$. Thus Γ can also be determined.

Proposition 2.2. *Let $M = \bigoplus_{l=0}^{\infty} \phi_l(w) H_w^2 z^l$. If $\Gamma(w)$ is the operator inner function such that $M = \Gamma(w) H^2(\mathbb{D}^2)$, then $\Gamma(w) = \sum_{l=0}^{\infty} \phi_l(w) P_l$, where P_l is the projection from H_z^2 to $\mathbb{C}z^l$.*

Proof. It is not hard to verify that $M \ominus wM = \bigoplus_{l=0}^{\infty} \phi_l(w) \mathbb{C} z^l$. So if one defines

$$\Gamma(w) = \sum_{l=0}^{\infty} \phi_l(w) P_l,$$

then $\Gamma(w) H_z^2 = M \ominus wM$, and hence $M = \Gamma(w) H^2(\mathbb{D}^2)$. Further, for every $g \in H_z^2$,

$$\begin{aligned} \|\Gamma(w) g\|^2 &= \sum_{l=0}^{\infty} |\phi_l(w)|^2 \|P_l g\|^2 \\ &= \sum_{l=0}^{\infty} \|P_l g\|^2 \\ &= \|g\|^2 \end{aligned}$$

for almost every $w \in \partial\mathbb{D}$. This verifies that Γ is inner. □

3. Submodule generated by two inner functions

Suppose $\phi_1(z)$ and $\phi_2(w)$ are two inner functions with variables z and w respectively. In this section we consider the submodule of the form (1.2).

Theorem 3.1. *Let $\Theta(z)$ be the operator inner function for a submodule M. Then M is of the form (1.2) if and only if $\Theta(z) = \phi_1(z) P_0 + P_1$, where P_0 and P_1 are complementary projections on H_w^2, e.g., $P_0 P_1 = 0$ and $P_0 + P_1 = I$.*

Proof. Suppose M is of the form (1.2). In [10], it was shown that

$$M \ominus zM = \phi_1(z)(H_w^2 \ominus \phi_2(w) H_w^2) \oplus \phi_2(w) H_w^2.$$

Set $\Theta(z) = \phi_1(z) P_0 + P_1$, where $P_0 : H_w^2 \to H_w^2 \ominus \phi_2(w) H_w^2$ is the orthogonal projection and $P_1 = I - P_0$. Then for every $g \in H_w^2$, by the Pythagorean theorem,

$$\begin{aligned} \|\Theta(z) g\|^2 &= |\phi_1(z)|^2 \|P_0 g\|^2 + \|P_1 g\|^2 \\ &= \|P_0 g\|^2 + \|P_1 g\|^2 \\ &= \|g\|^2 \end{aligned}$$

a.e. on $\mathbb{T}$. This shows that Θ is an operator inner function. Further, it is not hard to check $\Theta(z) H_w^2 = M \ominus zM$, and hence

$$M = \bigoplus_{n=0}^{\infty} z^n (M \ominus zM) = \Theta(z) H^2(\mathbb{D}^2).$$

On the other hand, suppose $\Theta(z) = \phi_1(z) P_0 + P_1$, where ϕ_1 is inner and P_0 and P_1 are complementary projections on H_w^2. Then

$$\begin{aligned} M &= \Theta(z) H^2(\mathbb{D}^2) \\ &= \phi_1(H_z^2 \otimes P_0 H_w^2) \oplus (H_z^2 \otimes P_1 H_w^2). \end{aligned} \tag{3.1}$$

First, we will show that T_w and P_1 commute on M. Denote $M_0 = H_z^2 \otimes P_0 H_w^2$ and $M_1 = H_z^2 \otimes P_1 H_w^2$, and let P_{M_0} and P_{M_1} stand for the projections from $H^2(\mathbb{D}^2)$ to

M_0 and M_1 respectively. Then, with respect to the decomposition (3.1), we rewrite T_w on M as

$$T_w = \begin{pmatrix} P_{M_0}T_wP_{M_0} & P_{M_0}T_wP_{M_1} \\ P_{M_1}T_wP_{M_0} & P_{M_1}T_wP_{M_1} \end{pmatrix}.$$

Since M is invariant under T_w, we have

$$\begin{pmatrix} P_{M_0}T_wP_{M_0} & P_{M_0}T_wP_{M_1} \\ P_{M_1}T_wP_{M_0} & P_{M_1}T_wP_{M_1} \end{pmatrix} \begin{pmatrix} \phi_1 M_0 \\ M_1 \end{pmatrix} \subset \begin{pmatrix} \phi_1 M_0 \\ M_1 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} \phi_1 P_{M_0}T_wM_0 + P_{M_0}T_wM_1 \\ \phi_1 P_{M_1}T_wM_0 + P_{M_1}T_wM_1 \end{pmatrix} \subset \begin{pmatrix} \phi_1 M_0 \\ M_1 \end{pmatrix}. \tag{3.2}$$

Consider the first line in (3.2). It is clear that $\phi_1 P_{M_0}T_wM_0 \subset \phi_1 M_0$, and hence $P_{M_0}T_wM_1 \subset \phi_1 M_0$. Check that $P_{M_0}T_wM_1 = H_z^2 \otimes P_0 wP_1 H_w^2$ and $\phi_1 M_0 = \phi_1 H_z^2 \otimes P_0 H_w^2$. Therefore, since ϕ_1 is nontrivial, the first inclusion in (3.2) holds only if $P_0 wP_1 H_w^2 = \{0\}$. This implies $T_w P_1 H_w^2 \subset P_1 H_w^2$. By Beurling's Theorem, there exists an inner function $\phi_2(w)$ such that $P_1 H_w^2 = \phi_2(w) H_w^2$. Thus

$$M = \phi_1(H_z^2 \otimes P_0 H_w^2) \oplus (H_z^2 \otimes \phi_2 H_w^2) = \phi_1 H^2(\mathbb{D}^2) + \phi_2 H^2(\mathbb{D}^2). \qquad \square$$

Because of symmetry, the operator inner function $\Gamma(w)$ for M in the variable w is $P_0 + \phi_2(w)P_1$.

4. Some applications

For a submodule M, the quotient module $N = H^2 \ominus M$ is defined as the orthogonal complement of M in $H^2(\mathbb{D}^2)$. We let S_z (respectively S_w) denote the compression of T_z (respectively T_w) to N, i.e. $S_z = P_N T_z|N$ (respectively $S_w = P_N T_w|N$), where P_N denotes the orthogonal projection from $H^2(\mathbb{D}^2)$ onto N. Lemma 1.1 indicates that the spectra of S_z and S_w can be calculated through the corresponding operator inner functions $\Theta(z)$ and $\Gamma(w)$. This section will make use of the results in Sections 2 and 3 to this end. All the results in this section are known, but the work here will show how elegantly these results can be obtained through operator inner functions.

For the inner-sequence-based submodule, spectra of both S_z and S_w are determined in [7]. The proof here is new.

Proposition 4.1. *Let M be an inner-sequence-based invariant subspace defined in (1.1). Then $\sigma(S_z)$ is either $\{0\}$ or $\bar{\mathbb{D}}$, and $\sigma(S_w) = \sigma(\phi_0)$.*

Proof. For $\lambda \in \mathbb{D}$, λ is in $\sigma(S_z)$ if and only if Θ is not invertible at λ. We know by Theorem 2.1 that $\Theta(z) = \sum_{l=0}^{\infty} z^l P_l$ with P_l orthogonal. There are two cases. First, if there is a nonnegative integer k such that $P_l = 0$ for all $l \geq k$, then $\Theta(z)$ is a polynomial, and by (2.1),

$$M = \sum_{n=0}^{k-1} z^n \Big(\bigoplus_{l=0}^{n} P_l H_w^2\Big) \oplus z^k H^2(\mathbb{D}^2).$$

This implies that $(S_z)^k = 0$ and hence $\sigma(S_z) = \{0\}$. Second, if $\Theta(z)$ is not a polynomial, then $\Theta(\lambda)$ is not invertible for all $\lambda \in \mathbb{D}$, because we can select $\{e_n\}_{n=0}^{\infty}$ with $e_n \in Range(P_n)$ and $\|e_n\| = 1$ (or 0 if $P_n = 0$); then

$$\|\Theta(\lambda)e_n\| = |\lambda^n| \to 0.$$

Now since $\mathbb{D}$ is dense in $\bar{\mathbb{D}}$ and $\sigma(S_z)$ is closed, we have $\bar{\mathbb{D}} \subseteq \sigma(S_z)$. Also, since S_z is a contraction, $\sigma(S_z) \subseteq \bar{\mathbb{D}}$. Therefore $\sigma(S_z) = \bar{\mathbb{D}}$.

For $\sigma(S_w)$, we recall that $\Gamma(w) = \sum_{m=0}^{\infty} \phi_m(w) P_m$, where P_m is the orthogonal projection from H_z^2 to $\mathbb{C}z^m$. So for a point w in $\mathbb{D}$, $\Gamma(w)$ is invertible if and only if $|\phi_m(w)|$ is bounded below for all $m \geq 0$. Because $|\phi_m(w)| \geq |\phi_0(w)|$ for each m, it is equivalent to $\phi_0(w)$ being bounded below at w, i.e. $\phi_0(w) \neq 0$. For $w \in \partial\mathbb{D}$, $w \notin G_2$ (defined above in Lemma 1.1) if and only if ϕ_m has an analytic continuation to a neighborhood of w for each m. Clearly, this is so if and only if ϕ_0 has an analytic continuation to a neighborhood of w. In conclusion, $\sigma(S_w) = \sigma(\Gamma) = \sigma(\phi_0)$. □

Next, we look at the submodule generated by two inner functions. In this case since $\Theta(z) = \phi_1 P_0 + P_1$ and $\Gamma(w) = P_0 + \phi_2(w) P_1$ for some complementary projections P_0 and P_1, the the next proposition is immediate.

Proposition 4.2. *Let M be a submodule of the form* (1.2)*. Then $\sigma(S_z) = \sigma(\phi_1(z))$ and $\sigma(S_w) = \sigma(\phi_2(w))$.*

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