MONOTONICITY THEOREMS FOR ANALYTIC FUNCTIONS CENTERED AT INFINITY

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ABSTRACT. We consider the family of analytic functions centered at infinity with Laurent expansion $f(z) = cz + c_0 + \sum_{j=1}^{\infty} c_j z^{-j}$. We prove some monotonicity theorems involving geometric quantities such as diameter, radius and length.

1. INTRODUCTION

Let us consider the analytic functions f on $\mathbb{C} \setminus \overline{\mathbb{D}}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, with Laurent expansion at infinity

$$(1.1) \quad f(z) = cz + c_0 + \sum_{j=1}^{\infty} c_j z^{-j},$$

where $c \neq 0$. We denote by $\mathcal{M}$ the family of all the functions of the above form with the usual normalization $c = 1$. Let also Σ be the subclass of $\mathcal{M}$ which contains the univalent functions in $\mathcal{M}$. The class Σ is a classical topic of research; see [4], [7].

For the class Σ , a well-known result of Pólya and Szegő [11, Problem 141, Page 23] asserts that the diameter D_f of the complement of the image of $\mathbb{C} \setminus \overline{\mathbb{D}}$, under f , satisfies the inequalities $2 \leq D_f \leq 4$. The equality $D_f = 4$ is attained if and only if f is of the form $f(z) = z + c + e^{it}/z$, $c \in \mathbb{C}$, $t \in \mathbb{R}$. The uniqueness statement of the equality $D_f = 2$ is that $D_f = 2$ if and only if f is a translation (i.e. $f(z) = z + c_0$, for some constant $c_0 \in \mathbb{C}$). This assertion appeared in [11] but was not proved there. It was proved in [5] by Jenkins. Alternative proofs appeared in [6], [8] and [9].

For every $f \in \mathcal{M}$ and $r > 1$ we set $C_r = \{z \in \mathbb{C} : |z| = r\}$, $\Gamma_f(r) = f(C_r)$. We denote by $D_f(r)$ the diameter of $\Gamma_f(r)$ and we prove the following monotonicity theorem:

Theorem 1 (Diameter Theorem). (a) *If $f \in \mathcal{M}$, then the function*

$$\varphi_D(r) = \frac{D_f(r)}{2r}, \quad 1 < r < \infty,$$

is strictly decreasing, except when f is a translation, in which case φ_D is constant.

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- (b) If $f \in \mathcal{M}$, then $\lim_{r \rightarrow \infty} \varphi_D(r) = 1$.
 (c) If $f \in \Sigma$, then $\lim_{r \rightarrow 1} \varphi_D(r) = D_f/2$.
 (d) If $f \in \Sigma$, then $D_f \geq 2$. Moreover, $D_f = 2$ if and only if f is a translation.

Similar monotonicity theorems for holomorphic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ have been a subject of interest of several authors. Aulaskari and Chen [1, Theorem 6] prove that the function $\phi_{\text{Area}}(r) = (\pi r^2)^{-1} \text{Area} f(r\mathbb{D})$ is increasing for $0 < r < 1$. Here $\text{Area} A$ is the area of the set A and $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$. In [3], Burckel, Marshall, Minda, Poggi-Corradini and Ransford consider the functions

$$\phi_{\text{Diam}}(r) = \frac{\text{Diam} f(r\mathbb{D})}{2r}, \quad \phi_{\text{Cap}}(r) = \frac{\text{Cap} f(r\mathbb{D})}{r},$$

where $\text{Diam} A$, $\text{Cap} A$ are the diameter and the logarithmic capacity of the set A respectively, and they prove that ϕ_{Diam} , ϕ_{Cap} are strictly increasing for $0 < r < 1$, except when f is linear (i.e. $f(z) = az + b$, for some $a, b \in \mathbb{C}$), in which case ϕ_{Diam} , ϕ_{Cap} are constant. Moreover, they prove that

$$\lim_{r \downarrow 0} \phi_{\text{Diam}}(r) = \lim_{r \downarrow 0} \phi_{\text{Cap}}(r) = |f'(0)| \quad \text{and} \quad \lim_{r \downarrow 0} \phi_{\text{Area}}(r) = |f'(0)|^2.$$

In fact, the classical Schwarz Lemma itself can be viewed as a monotonicity theorem (see [3]): We set

$$(1.2) \quad \text{Rad} f(r\mathbb{D}) = \sup_{z \in r\mathbb{D}} |f(z) - f(0)|, \quad 0 < r < 1.$$

Then the function $\phi_{\text{Rad}}(r) = r^{-1} \text{Rad} f(r\mathbb{D})$ is strictly increasing for $0 < r < 1$, except when f is linear, in which case ϕ_{Rad} is constant. Moreover, $\lim_{r \downarrow 0} \phi_{\text{Rad}}(r) = |f'(0)|$.

In [2] Betsakos and Pouliasis prove that the function

$$\Phi_I(r) = \frac{R(f(r\mathbb{D}), f(0))}{r},$$

where $R(D, z)$ is the inner radius of D at z , is increasing. For some other related monotonicity theorems, we refer to [2] and the references therein.

For functions $f \in \Sigma$, with Laurent expansion

$$(1.3) \quad f(z) = z + c_0 + \sum_{j=1}^{\infty} c_j z^{-j}.$$

Pólya and Szegő [10, p. 129, Problem 126] (see also [4, p. 29]) prove that the area $A_f(r)$ of the domain which is bounded by the Jordan curve $\Gamma_f(r) = f(C_r)$, $r > 1$, is given by the formula

$$(1.4) \quad A_f(r) = \pi \left(r^2 - \sum_{j=1}^{\infty} j |c_j|^2 r^{-2j} \right),$$

and so the function

$$(1.5) \quad \varphi_A(r) = (\pi r^2)^{-1} A_f(r)$$

equals

$$(1.6) \quad \varphi_A(r) = 1 - \sum_{j=1}^{\infty} j |c_j|^2 r^{-2j-2}.$$

Thus φ_A is strictly increasing unless $c_j = 0$, for every $j = 1, 2, \dots$, in which case φ_A is constant and f is a translation. Also, by (1.6) we get

$$(1.7) \quad \lim_{r \rightarrow \infty} \varphi_A(r) = 1.$$

The proof of Theorem 1 is in section 2. As we mentioned above, the proof of (d) can also be found in [5, 6, 8, 9]. Some other monotonicity results for functions in $\mathcal{M}$ and Σ appear in section 3. These results involve geometric quantities such as radius, length and capacity.

2. PROOF OF THEOREM 1

(a) Let $f \in \mathcal{M}$ have the Laurent expansion (1.3). Let $1 < r < s$ and $z_1, z_2 \in C_s$ be such that $D_f(s) = |f(z_1) - f(z_2)|$. We set $w = z_1/z_2 = e^{i\alpha}$, $\alpha \in \mathbb{R}$, and we consider the function

$$(2.1) \quad \Phi_s(z) = f(wz) - f(z), \text{ for every } z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

The function $(2z)^{-1}\Phi_s(z)$ is holomorphic in $\mathbb{C} \setminus \overline{\mathbb{D}}$ (including the point at infinity). Also,

$$\max_{z \in C_r} \left| \frac{\Phi_s(z)}{2z} \right| \leq \frac{D_f(r)}{2r}.$$

Thus, by the maximum principle, for every $z \in \mathbb{C} \setminus r\mathbb{D}$,

$$(2.2) \quad \left| \frac{\Phi_s(z)}{2z} \right| \leq \frac{D_f(r)}{2r}.$$

We set $z = z_2$ in (2.2) and by (2.1) we conclude that

$$\varphi_D(s) = \left| \frac{\Phi_s(z_2)}{2z_2} \right| \leq \frac{D_f(r)}{2r} = \varphi_D(r),$$

and thus φ_D is decreasing.

Let us now assume that φ_D is not strictly decreasing. So there are $1 < r < s$ such that $\varphi_D(r) = \varphi_D(s)$. Then, since φ_D is decreasing,

$$\varphi_D(r) = \varphi_D(\rho) = \varphi_D(s), \text{ for every } \rho \in [r, s].$$

By the maximum principle, there is a constant $C \in \mathbb{C}$ such that

$$\frac{\Phi_s(z)}{2z} = C, \text{ for every } z \in \mathbb{C} \setminus r\mathbb{D},$$

or equivalently, using (1.3) and (2.1), we have identically in $\mathbb{C} \setminus r\mathbb{D}$

$$(w-1)z + \sum_{j=1}^{\infty} c_j(w^{-j}-1)z^{-j} = 2Cz,$$

which means that $w-1 = 2C$ and

$$(2.3) \quad c_j(w^{-j}-1) = 0, \text{ for every } j = 1, 2, \dots$$

Since the function $(2z)^{-1}\Phi_s(z)$ is constant, for every $\rho \in [r, s]$ and every $z_\rho \in C_\rho$,

$$\left| \frac{\Phi_s(z_\rho)}{2z_\rho} \right| = |C| = \varphi_D(s) = \varphi_D(\rho).$$

Thus $D_f(\rho) = |\Phi_s(z_\rho)| = |f(wz_\rho) - f(z_\rho)|$. So the modulus

$$|f(\rho e^{it}) - f(\rho e^{i\theta})|^2$$

is maximized for $t = \alpha + \theta$, for every $\rho \in [r, s]$ and every $\theta \in \mathbb{R}$. Therefore, its partial derivative with respect to t vanishes when $t = \alpha + \theta$, for every $\rho \in [r, s]$. Using the expansion (1.3), we find an expression for the partial derivative for $t = a + \theta$. The coefficients of all the powers of ρ in this expression should be zero. The coefficient of ρ^2 is

$$i(w^{-1} - w) = 2\operatorname{Im}(w).$$

Hence $w = \pm 1$. But for $w = 1$, we get $z_1 = z_2$, and thus $D_f(s) = 0$; then f should be constant on the circle C_s , which gives a contradiction. So $w = -1$. We go back to (2.3) and substitute $w = -1$. Then

$$(2.4) \quad c_{2n+1} = 0, \text{ for every } n = 0, 1, \dots$$

Using (2.4) we find that the coefficients of the powers ρ^{-2n+1} are

$$-8n\operatorname{Im}(c_{2n}e^{-i(2n+1)\theta}),$$

which implies that

$$(2.5) \quad c_{2n} = 0, \text{ for every } n = 1, 2, \dots$$

We conclude that $c_j = 0$ for every $j = 1, 2, \dots$, and so f is a translation and the proof of (a) is complete.

(b) Let $f \in \mathcal{M}$ have the expansion (1.3). Then the function

$$g(z) = \frac{1}{f(1/z)} = \frac{z}{1 + c_0z + \sum_{j=1}^{\infty} c_j z^{1+j}}$$

is holomorphic in a neighbourhood of $z_0 = 0$. We have that $g(0) = 0$ and $g'(0) = 1 \neq 0$, and so $z_0 = 0$ is a first order zero of g . This implies that there exists an $\epsilon > 0$ such that g is univalent in the disc $\epsilon\mathbb{D}$. So the function $f(z) = 1/g(1/z)$ is univalent in $\mathbb{C} \setminus (1/\epsilon)\overline{\mathbb{D}}$. Let now $r > 1/\epsilon$. Since f is univalent in $\mathbb{C} \setminus (1/\epsilon)\overline{\mathbb{D}}$, $\Gamma_f(r)$ is a Jordan curve. We denote by $I_f(r)$ the interior of $\Gamma_f(r)$ and we set $A_f(r) = \operatorname{Area} I_f(r)$. The well-known isodiametric inequality asserts that $A_f(r) \leq \pi \operatorname{Diam}^2 I_f(r)/4$, and hence, by (1.5),

$$(2.6) \quad \sqrt{\varphi_A(r)} \leq \varphi_D(r).$$

By (1.3), one easily obtains that

$$(2.7) \quad \varphi_D(r) \leq 1 + \sum_{j=1}^{\infty} |c_j| r^{-j-1}.$$

By (2.6), (2.7) and (1.7), we get

$$\lim_{r \rightarrow \infty} \varphi_D(r) = 1.$$

(c) The limit $\lim_{r \rightarrow 1} \varphi_D(r) = D_f/2$ is evident since f is univalent.

(d) Let $f \in \Sigma$ and $r > 1$. By (a), (b) and (c) we have that

$$1 = \lim_{\rho \rightarrow \infty} \varphi_D(\rho) \leq \varphi_D(r) \leq \lim_{\rho \rightarrow 1} \varphi_D(\rho) = \frac{D_f}{2},$$

and so $D_f \geq 2$. The equality $D_f = 2$ now holds if and only if φ_D is constant and so if and only if f is a translation.

3. RADIUS, LENGTH AND CAPACITY THEOREMS

Suppose $f \in \mathcal{M}$ has the expansion (1.3). We introduce a kind of radius. We cannot use (1.2) since $f(0)$ is not at our disposal. We substitute it with the constant term c_0 of the Laurent expansion, which is called the conformal center of the complement of $f(\mathbb{C} \setminus \overline{\mathbb{D}})$; see [11, p. 23], [7, p. 12]. We thus set

$$R_f(r) = \sup_{z \in C_r} |f(z) - c_0|, \quad r > 1.$$

Theorem 2 (Radius Theorem). *Let $f \in \mathcal{M}$. The function $\varphi_R(r) = r^{-1}R_f(r)$ is strictly decreasing for $r > 1$, except when f is a translation, in which case φ_R is constant. Moreover, $\lim_{r \rightarrow \infty} \varphi_R(r) = 1$.*

Proof. Let $f \in \mathcal{M}$ with the expansion (1.3). Then $f(z) - c_0 = z + \sum_{j=1}^{\infty} c_j z^{-j}$. Let $1 < r < s$. We have that

$$\begin{aligned} (3.1) \quad \varphi_R(r) &= \frac{\sup_{z \in C_r} |z + \sum_{j=1}^{\infty} c_j z^{-j}|}{r} \\ &= \sup_{z \in C_r} \left| 1 + \sum_{j=1}^{\infty} c_j z^{-j-1} \right|. \end{aligned}$$

We apply the maximum principle to the function

$$h(\zeta) = 1 + \sum_{j=1}^{\infty} c_j \zeta^{j+1}, \quad \zeta \in \mathbb{D},$$

and we conclude that φ_R is strictly decreasing unless f is a translation. Moreover, it follows from (3.1) that $\lim_{r \rightarrow \infty} \varphi_R(r) = 1$. $\square$

We now present an analogous monotonicity result involving length. For $f \in \Sigma$, $\Gamma_f(r)$ is a smooth Jordan curve. Let $L_f(r)$ be its length.

Theorem 3 (Length Theorem). *Let $f \in \Sigma$. The function $\varphi_L(r) = (2\pi r)^{-1}L_f(r)$ is strictly decreasing for $r > 1$, except when f is a translation, in which case φ_L is constant. Moreover, $\lim_{r \rightarrow \infty} \varphi_L(r) = 1$.*

Proof. Let $f \in \Sigma$. The function $g(z) = (f'(z))^{1/2}$ has the Laurent expansion at the point of infinity

$$(3.2) \quad g(z) = 1 + \sum_{j=2}^{\infty} b_j z^{-j}.$$

Let $r > 1$. As we mentioned above, the curve $\Gamma_f(r)$ is a smooth Jordan curve and has the parametrization $\gamma_r(t) = f(re^{it})$, $t \in [0, 2\pi]$. So its length is

$$(3.3) \quad L_f(r) = \int_0^{2\pi} |\gamma_r'(t)| dt = r \int_0^{2\pi} |g(re^{it})|^2 dt.$$

After standard calculations involving the power series (3.2), we find that

$$(3.4) \quad \int_0^{2\pi} |g(re^{it})|^2 dt = 2\pi + 2\pi \sum_{j=2}^{\infty} |b_j|^2 r^{-2j}.$$

Combining the relations (3.3) and (3.4), we obtain

$$(3.5) \quad \varphi_L(r) = 1 + \sum_{j=2}^{\infty} |b_j|^2 r^{-2j},$$

and so φ_L is strictly decreasing unless $b_j = 0$, for every $j = 2, 3, \dots$, in which case φ_L is constant. In this case, because of (3.2), we have $f'(z) = 1$, which means exactly that f is a translation. Also, by (3.5), we obtain that $\lim_{r \rightarrow \infty} \varphi_L(r) = 1$. $\square$

We give via conformal mapping [7] the definition of logarithmic capacity of a compact set $E \subset \mathbb{C}$, and we denote it by $\text{Cap}E$. For other definitions (via transfinite diameter and equilibrium measure) we refer the reader to [12].

Let $E \subset \mathbb{C}$, be a simply connected compact set. Let

$$f(z) = a_1 z + a_0 + \sum_{j=1}^{\infty} a_{-j} z^{-j}$$

be the unique conformal mapping from $\mathbb{C} \setminus \overline{\mathbb{D}}$ onto $\mathbb{C} \setminus E$ with $a_1 > 0$, which maps ∞ to ∞ . Then the logarithmic capacity of the set E is defined as $\text{Cap}E = a_1$.

Now for every $f \in \Sigma$ and every $r > 1$ we set $C_f(r) = \text{Cap}I_f(r)$. The following theorem follows directly from the above definition of logarithmic capacity.

Theorem 4 (Capacity Theorem). *Let $f \in \Sigma$. The function $\varphi_C(r) = r^{-1}C_f(r)$ is constant and equals 1 for every $r > 1$.*

4. FINAL REMARKS

In [3] the authors study the so-called higher-order diameters, which are defined for a set $E \subset \mathbb{C}$ as

$$d_n(E) = \sup \left(\prod_{j < k} |z_j - z_k| \right)^{2/(n(n-1))},$$

where the supremum is taken over all n -tuples of points from E .

There are two limit cases here:

(a) When $n = 2$, we have that $d_2(E) = \text{Diam}(E)$ and

(b) when $n = \infty$, we have that $d_{\infty}(E) := \lim_{n \rightarrow \infty} d_n(E) = \text{Cap}(E)$.

For the last see also [12, p. 153]. Also it is well-known that $d_n(\mathbb{D}) = n^{1/(n-1)}$, and so $\text{Cap}(\mathbb{D}) = d_{\infty}(\mathbb{D}) = 1$. In [3] the authors proved that the functions

$$\phi_{\text{n-Diam}}(r) = \frac{d_n(f(r\mathbb{D}))}{d_n(r\mathbb{D})} \quad \text{and} \quad \phi_{\text{Cap}}(r) = \frac{\text{Cap}(f(r\mathbb{D}))}{\text{Cap}(r\mathbb{D})} = \frac{d_{\infty}(f(r\mathbb{D}))}{d_{\infty}(r\mathbb{D})}$$

are increasing. In our situation we give the following open problem:

Conjecture 1. *If $f \in \mathcal{M}$ and $n = 2, 3, \dots$, then the function*

$$\varphi_{\text{n-Diam}}(r) := \frac{d_n(f(C_r))}{d_n(C_r)}, \quad 1 < r < \infty,$$

is strictly decreasing, except when f is a translation, in which case $\varphi_{\text{n-Diam}}(r)$ is constant. Moreover, $\lim_{r \rightarrow \infty} \varphi_{\text{n-Diam}}(r) = 1$.

As we already mentioned for $n = 2$ we have the case of diameter, and so Theorem 1 covers this conjecture. Note also that, in the class Σ , the limit-case function $\varphi_C(r)$ is constant, as one may see in Theorem 4.

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