LOGIC FOR METRIC STRUCTURES AND THE NUMBER OF UNIVERSAL SOFIC AND HYPERLINEAR GROUPS

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ABSTRACT. Using the model theory of metric structures, the author gave an alternative proof of the following result by Thomas: If the Continuum Hypothesis fails, then there are $2^{2^{\aleph_0}}$ universal sofic groups up to isomorphism. This method is also applicable to universal hyperlinear groups, giving a positive answer to a question posed by Thomas.

1. INTRODUCTION

Sofic and hyperlinear groups are two classes of discrete groups that have received the attention of many mathematicians in different areas in the last ten years. It is known that the class of sofic groups is contained in the class of hyperlinear groups, but it is not known if this containment is proper or whether the class of hyperlinear groups is equal to the class of all discrete groups. For a complete presentation of this topic, the reader is referred to [11]. In [3], Elek and Szabó proved that a countable group Γ is sofic if and only if it can be embedded in some (or, equivalently, every) ultraproduct of the symmetric groups, regarded as a bi-invariant metric group with respect to the normalized Hamming distance. (See [1] for the definition of metric ultraproducts and an introduction to the logic for metric structures.) An analogous characterization holds for hyperlinear groups, where the symmetric groups are replaced with the finite rank unitary groups, endowed with the normalized distance induced by the Hilbert-Schmidt norm. In view of this characterization, metric ultraproducts of symmetric groups are said to be universal sofic groups, and metric ultraproducts of unitary groups are said to be universal hyperlinear groups. In [12], Thomas proved that if the Continuum Hypothesis fails, then there are $2^{\mathfrak{c}}$ many metric ultraproducts of the symmetric groups up to (algebraic) isomorphism, where $\mathfrak{c}$ denotes the cardinality of the continuum, and asked if the same statement holds for ultraproducts of the unitary groups. In this paper, we give a proof of Thomas' result, by means of the logic for metric structures, which also applies in the case of the unitary groups. From this, we deduce also the existence, under the failure of the Continuum Hypothesis, of $2^{\mathfrak{c}}$

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many metric ultraproducts of the matrix algebras regarded as ranked regular rings, up to algebraic isomorphism. This problem was raised by Gábor Elek, in view of Proposition 8.3 in [6], asserting the existence, under the failure of the Continuum hypothesis, of 2^c isomorphism classes of metric ultraproducts of the complex matrix algebras, regarded as tracial von Neumann algebras.

Under the Continuum Hypothesis, the number of metric ultraproducts of symmetric and unitary groups up to isomorphism is still unknown. In this case, the statement that they are all isomorphic is equivalent to the assertion that they are all elementarily equivalent as metric structures. At the end of this article, we prove a partial result in this direction, showing that they have the same Σ_2 -theories as metric structures.

This article is organized as follows. In Section 2, we show that the sequences of symmetric and unitary groups have the order property. This implies that, under the failure of the Continuum Hypothesis, there are 2^c many ultraproducts of these sequences up to isometric isomorphism. In Section 3, we introduce some results and terminology from [6] in order to deduce the existence, under the failure of the Continuum Hypothesis, of 2^c many ultraproducts up to algebraic isomorphism. In Section 4, we infer from this result the existence, under the failure of the Continuum Hypothesis, of 2^c many ultraproducts of the sequence of matrix algebras up to algebraic isomorphism. In Section 5, we prove that all universal sofic groups are elementarily equivalent as metric structures with respect to Σ_2 formulas and that the same holds for universal hyperlinear groups.

If $n \in \mathbb{N}$, the symmetric group acting on $\{1, \dots, n\}$ on the left is denoted by S_n and its identity by e_n . The unitary group of $n \times n$ matrices over $\mathbb{C}$ is denoted by U_n and its identity by I_n . The symmetric group S_n is regarded as metric group with respect to the bi-invariant metric defined by

$$d_{S_n}(\sigma, \tau) = \frac{1}{n} |\{i \in \{1, \dots, n\} \mid \sigma(i) \neq \tau(i)\}|,$$

called the normalized Hamming distance. The unitary group U_n is endowed with the metric

$$d_{U_n}(A, B) = \frac{\|A - B\|_2}{2\sqrt{n}},$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. Usually the factor $\frac{1}{2}$ is omitted. It is introduced here only because, in the logic for bounded metric structures, for convenience all the metric spaces are supposed to have diameter at most 1. By the universal sofic and, respectively, hyperlinear groups, we will mean in the following the metric ultraproducts of the sequences of the symmetric and, respectively, unitary groups. Consider, for $n \in \mathbb{N}$, the injective homomorphism $\sigma \mapsto A_\sigma$ from S_n to U_n defined by

$$(1.1) \quad A_\sigma(b_i) = b_{\sigma(i)}$$

for $i \in \{1, 2, \dots, n\}$, where $b_1, \dots, b_n$ is the canonical basis of $\mathbb{C}^n$, and observe that

$$(1.2) \quad d_{U_n}(A_\sigma, A_\tau) = \sqrt{\frac{d_{S_n}(\sigma, \tau)}{2}}$$

(see [11]).

In the rest of the paper, we will use the following notational conventions: If a, b are elements of a group G , then $[a, b]$ denotes the element $aba^{-1}b^{-1}$ of G . Uppercase calligraphic letters such as $\mathcal{U}, \mathcal{V}$ are reserved for ultrafilters over $\mathbb{N}$. If

$(M_n)_{n \in \mathbb{N}}$ is a sequence of metric structures and $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$, the corresponding metric ultraproduct is denoted by $\prod_n^{\mathcal{U}} M_n$, while the ultrapower of a metric structure M by $\mathcal{U}$ is denoted by $M^{\mathcal{U}}$. We will denote by $\bar{x}$ and $\bar{y}$ m -tuples of variables $x_1, \dots, x_m$ and $y_1, \dots, y_m$. Every metric will be denoted by d . The context will make clear which metric we are referring to each time. The set of natural numbers $\mathbb{N}$ is not supposed to contain 0, and if r is a real number, then $\lceil r \rceil$ denotes the smallest integer greater than or equal to r . For convenience, we suppose 0 to be a multiple of any natural number. Finally we will write, as usual, the acronym CH to stand for “Continuum Hypothesis”.

2. THE ORDER PROPERTY FOR SYMMETRIC AND UNITARY GROUPS

In [6], Theorem 6.1, aiming to count the number of ultrapowers of a C^* -algebra or of a von Neumann algebra, Farah and Shelah isolate a condition ensuring that a sequence of metric structures has 2^c many ultraproducts up to isometric isomorphism, under the failure of CH.

In this section, we will consider a particular case of [6], Theorem 6.1, for bi-invariant metric groups, and will infer from that the following:

Proposition 2.1. *If CH fails and $(k_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers, then the sequences $(S_{k_n})_{n \in \mathbb{N}}$ and $(U_{k_n})_{n \in \mathbb{N}}$ have 2^c many ultraproducts up to isometric isomorphism.*

In the following section, after introducing notation and definitions from [6], we will refine this result, showing that in this case there are in fact 2^c many ultraproducts up to algebraic isomorphism, under the failure of CH.

The following proposition is a particular case of [6], Theorem 6.1, obtained by considering the language of bi-invariant metric groups and the formula

$$\varphi(x_1, x_2, y_1, y_2)$$

defined by

$$d([x_1, y_2], e).$$

Proposition 2.2. *Let $(k_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers and $(G_n)_{n \in \mathbb{N}}$ be a sequence of bi-invariant metric groups with uniformly bounded diameter. Suppose that, for some constant $\gamma > 0$ and every $l \in \mathbb{N}$, for all but finitely many $n \in \mathbb{N}$, G_n contains sequences $(g_{n,i})_{i=1}^l$ and $(h_{n,i})_{i=1}^l$ such that, for every $1 \leq i < j \leq l$, $g_{n,i}$ and $h_{n,j}$ commute, while if $1 \leq j \leq i \leq l$,*

$$d([g_{n,i}, h_{n,j}], e_{G_n}) \geq \gamma.$$

Then, under the failure of CH, there are 2^c many metric ultraproducts of the sequence $(G_{k_n})_{n \in \mathbb{N}}$ up to isometric isomorphism.

Thus, in order to prove Proposition 2.1, it is enough to show that the sequences of symmetric and unitary groups satisfy the hypothesis of Proposition 2.2.

Lemma 2.3. *For every $l \in \mathbb{N}$, there are sequences $(\sigma_{l,i})_{i=1}^l, (\tau_{l,i})_{i=1}^l$ of elements of S_{3^l} such that if $1 \leq i < j \leq l$, $\sigma_{l,i}$ and $\tau_{l,j}$ commute, while if $1 \leq j \leq i \leq l$, $[\sigma_{l,i}, \tau_{l,j}]$ is the product of 3^{l-1} disjoint cycles of length 3.*

Proof. If $l \in \mathbb{N}$, consider the action of $\overbrace{S_3 \times \dots \times S_3}^{l \text{ times}}$ on $\{1, 2, 3\}^l$ defined by

$$(\sigma_1, \dots, \sigma_l)(i_1, \dots, i_l) = (\sigma_1(i_1), \dots, \sigma_l(i_l)),$$

where $\sigma_1, \dots, \sigma_l \in S_3$ and $i_1, \dots, i_l \in \{1, 2, 3\}$. This action defines an isometric embedding

$$(\sigma_1, \dots, \sigma_l) \rightarrow \alpha(\sigma_1, \dots, \sigma_l)$$

of $\overbrace{S_3 \times \dots \times S_3}^{l \text{ times}}$ into the group of permutations of $\{1, 2, 3\}^l$, which can be identified with S_{3^l} . Define also, for $i = 1, \dots, l$,

$$\sigma_{l,i} = \alpha \left(\overbrace{(12), \dots, (12)}^{i \text{ times}}, \overbrace{e_3, \dots, e_3}^{l-i \text{ times}} \right)$$

and

$$\tau_{l,i} = \alpha \left(\overbrace{e_3, \dots, e_3}^{i-1 \text{ times}}, (23), \overbrace{e_3, \dots, e_3}^{l-i \text{ times}} \right).$$

Observe that, for $i < j$,

$$[\sigma_{l,i}, \tau_{l,j}] = e_{3^l},$$

while, for $i \geq j$,

$$[\sigma_{l,i}, \tau_{l,j}] = \alpha \left(\overbrace{e_3, \dots, e_3}^{j-1 \text{ times}}, (132), \overbrace{e_3, \dots, e_3}^{l-j \text{ times}} \right). \quad \square$$

Lemma 2.4. *If $n, k, l \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ are such that $n = 3^l k + r$ and $0 \leq r < 3^l$, then there are sequences $(\Sigma_{n,i})_{i=1}^l$ and $(T_{n,i})_{i=1}^l$ in S_n such that, for $1 \leq i, j \leq l$, $\Sigma_{n,i}$ and $T_{n,i}$ commute if $i < j$, while $[\Sigma_{n,i}, T_{n,j}]$ is the product of $3^{l-1}k$ disjoint cycles of length 3 and, in particular,*

$$d([\Sigma_{n,i}, T_{n,j}], e) = \frac{3^l k}{3^l k + r} \geq \frac{k}{k+1} \geq \frac{1}{2},$$

if $i \geq j$.

Proof. Consider the action of S_{3^l} on $\{1, \dots, 3^l\} \times \{1, \dots, k\}$ defined by

$$\sigma(i, j) = (\sigma(i), j)$$

for every $i \in \{1, \dots, 3^l\}$ and $j \in \{1, \dots, k\}$. This defines an isometric embedding of S_{3^l} into $S_{3^l k}$. Moreover, letting $S_{3^l k}$ act on the first $3^l k$ elements of $\{1, \dots, n\}$ defines an algebraic embedding of $S_{3^l k}$ into S_n . The composition Φ of these two embeddings is an algebraic embedding of S_{3^l} into S_n . For $1 \leq i \leq l$, define

$$\Sigma_{n,i} = \Phi(\sigma_{l,i})$$

and

$$T_{n,i} = \Phi(\tau_{l,i}),$$

where $\sigma_{l,i}$ and $\tau_{l,i}$ are the elements of S_{3^l} defined in Lemma 2.3. Then, if $1 \leq i, j \leq l$,

$$[\Sigma_{n,i}, T_{n,j}] = [\Phi(\sigma_{l,i}), \Phi(\tau_{l,j})] = \Phi([\sigma_{l,i}, \tau_{l,j}]).$$

If $i < j$, $[\sigma_{l,i}, \tau_{l,j}]$ is the identity and, hence, $[\Sigma_{n,i}, T_{n,j}]$ is the identity. If $i \geq j$, then $[\sigma_{l,i}, \tau_{l,j}]$ is a product of 3^{l-1} disjoint 3-cycles and, hence, $[\Sigma_{n,i}, T_{n,j}]$ is the product of $3^{l-1}k$ disjoint 3-cycles. $\square$

Lemma 2.5. *If $n, k, l \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ are as in the statement of Lemma 2.4, then there are sequences $(b_{n,i})_{i=1}^l$ and $(c_{n,i})_{i=1}^l$ in U_n such that $b_{n,i}$ and $c_{n,i}$ commute if $i < j$, while $d([b_{n,i}, c_{n,j}]) \geq \frac{1}{2}$ if $i \geq j$.*

Proof. Let $\sigma \mapsto A_\sigma$ be the embedding of S_n into U_n given by (1.1) and let

$$b_{n,i} = A_{\Sigma_{n,i}} \quad \text{and} \quad c_{n,i} = A_{T_{n,i}},$$

where $\Sigma_{n,i}, T_{n,i} \in S_n$ are the permutations given by Lemma 2.4. Observe that if $i < j$, then

$$[\Sigma_{n,i}, \Sigma_{n,j}] = e_n,$$

and hence

$$[A_{\Sigma_{n,i}}, A_{T_{n,j}}] = A_{[\Sigma_{n,i}, T_{n,j}]} = A_{e_n} = I_n,$$

while, if $i \geq j$,

$$d([\Sigma_{n,i}, T_{n,i}], e_n) \geq \frac{1}{2}$$

and, by (1.2),

$$d([A_{\Sigma_{n,i}}, A_{T_{n,j}}], I_n) = \sqrt{\frac{d([\Sigma_{n,i}, T_{n,j}], e_n)}{2}} \geq \frac{1}{2}. \quad \square$$

The proposition is now an immediate consequence of Lemma 2.4 and Lemma 2.5, together with Proposition 2.2.

3. NON-ISOMORPHIC UNIVERSAL SOFIC AND HYPERLINEAR GROUPS

In this section, we will prove the following strengthening of Proposition 2.1.

Theorem 3.1. *If CH fails and $(k_n)_{n \in \mathbb{N}}$ is an increasing sequence of natural numbers, then, up to algebraic isomorphism, there are $2^{\mathfrak{c}}$ many ultraproducts of both the sequence $(S_{k_n})_{n \in \mathbb{N}}$ and the sequence $(U_{k_n})_{n \in \mathbb{N}}$.*

This result has already been proved by Thomas in [12] for permutation groups. Łukasz Grabowski pointed out that the case of unitary groups can be deduced from Proposition 8.3 in [6], using the fact that non-isomorphic type II von Neumann algebras have non-isomorphic unitary groups ([7], Theorem 4) and that the unitary group of an ultraproduct of finite von Neumann algebras is the ultraproduct of the unitary groups ([8], Proposition 2.1). In the following, we will give a direct proof of this result by means of the logic for metric structures. This generalization of the usual discrete logic is suitable to deal with structures endowed with a metric. For an introduction to this topic, the reader is referred to [1]. Structures and formulas in the usual discrete logic can be considered particular cases of metric structures and formulas, where the distance is interpreted as the trivial discrete distance defined by $d(x, y) = 1$ iff $x \neq y$. Thus, definitions and theorems stated in the setting of the logic for metric structures subsume the analogous definitions and theorems for the usual discrete logic as a particular case. For the sake of simplicity, all the languages are henceforth supposed without relation symbols, apart from the metric. If M is a structure in such a language $\mathcal{L}$, denote by M_{alg} the $\mathcal{L}$ -structure obtained from M by replacing the metric on M by the trivial discrete metric.

We now have to introduce some notation and recall some results from [6]. If $\mathcal{L}$ is a language, $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}$ -formula, $\varepsilon \geq 0$ and M is an $\mathcal{L}$ -structure, then the relation $\prec_{\psi, \varepsilon}$ on M^k is defined by

$$\bar{a} \prec_{\psi, \varepsilon} \bar{b} \Leftrightarrow (\psi^M(\bar{a}, \bar{b}) \leq \varepsilon \wedge \psi^M(\bar{b}, \bar{a}) \geq 1 - \varepsilon).$$

A chain in M^k with respect to the relation $\prec_{\psi, \varepsilon}$ will be called a (ψ, ε) -chain in M . The relation $\prec_{\psi, 0}$ will be denoted by $\prec_\psi$ and a $(\psi, 0)$ -chain will be called a ψ -chain. If M is an $\mathcal{L}$ -structure and $\varphi(\bar{x}, \bar{y})$ an $\mathcal{L}$ -formula, a ψ -chain $\mathcal{C}$ is called an $(\aleph_1, \psi)$ -skeleton like if, for every $\bar{a} \in M^k$ there exists a countable $\mathcal{C}_{\bar{a}} \subset \mathcal{C}$ such that, for every $\bar{b}, \bar{c} \in \mathcal{C}$ such that

$$\{x \in \mathcal{C}_{\bar{a}} \mid \bar{b} \preceq_\psi x \preceq_\psi \bar{c}\} = \emptyset,$$

one has

$$\psi^M(\bar{a}, \bar{b}) = \psi^M(\bar{a}, \bar{c}) \quad \text{and} \quad \psi^M(\bar{b}, \bar{a}) = \psi^M(\bar{c}, \bar{a}).$$

The notion of a ψ -chain and an $(\aleph_1, \psi)$ -skeleton like ψ -chain in a discrete structure for a discrete formula ψ are obtained from the previous ones, as a particular case.

The following statement is proved in [6] (Proposition 6.6).

Lemma 3.2. *If $\varphi(\bar{x}, \bar{y})$ is an $\mathcal{L}$ -formula, I is a linear order of cardinality $\mathfrak{c}$ and $(M_n)_{n \in \mathbb{N}}$ is a sequence of $\mathcal{L}$ -structures such that, $\forall n \in \mathbb{N}$, M_n contains a φ -chain of length n , then there is an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that $\prod_n^{\mathcal{U}} M_n$ contains an $(\aleph_1, \varphi)$ -skeleton like φ -chain of order type I .*

The same fact for discrete structures and formulas can be inferred from this lemma as a particular case. The following definition, taken from [4], is of key importance for the proof of the main result.

Definition 3.3. A sequence $(M_n)_{n \in \mathbb{N}}$ of $\mathcal{L}$ -structures has the *order property* witnessed by the $\mathcal{L}$ -formula $\varphi(\bar{x}, \bar{y})$ if, for every $\varepsilon > 0$ and $l \in \mathbb{N}$, M_n contains a (φ, ε) -chain of length l for all but finitely many $n \in \mathbb{N}$.

The assumption that a sequence $(M_n)_{n \in \mathbb{N}}$ of $\mathcal{L}$ -structures has the order property is slightly weaker than the assumption on $(M_n)_{n \in \mathbb{N}}$ in Lemma 3.2. Nonetheless, if a sequence $(M_n)_{n \in \mathbb{N}}$ has the order property, then the same conclusion as in Lemma 3.2 holds, namely, for every linear order I of cardinality $\mathfrak{c}$, there is an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that $\prod_n^{\mathcal{U}} M_n$ contains an $(\aleph_1, \varphi)$ -skeleton like φ -chain of order type I . This is easily seen via a suitable modification of the proof of Proposition 6.6 in [6].

The connection between the number of non-isomorphic ultraproducts and the $(\aleph_1, \varphi)$ -skeleton like φ -chains is given by the following lemma. It is proved in [6] (Proposition 3.14) in the setting of usual first order logic. As pointed out in Section 6.5 of the same paper, the proof can be easily adapted to the metric case.

Lemma 3.4. *If CH fails, $\varphi(\bar{x}, \bar{y})$ is an $\mathcal{L}$ -formula and $\mathcal{K}$ is a class of $\mathcal{L}$ -structures such that, for every linear order I of cardinality $\mathfrak{c}$, there is an element M of $\mathcal{K}$ such that M contains an $(\aleph_1, \varphi)$ -skeleton like φ -chain, then there are $2^{\mathfrak{c}}$ many pairwise non-isometrically isomorphic $\mathcal{L}$ -structures in $\mathcal{K}$.*

The following lemma is useful when, as in our case, one is interested in counting the number of metric ultraproducts up to algebraic isomorphism.

Lemma 3.5. *Suppose that $\varphi(\bar{x}, \bar{y})$ is an $\mathcal{L}$ -formula and $\psi(\bar{x}, \bar{y})$ is a discrete $\mathcal{L}$ -formula such that, for every $\mathcal{L}$ -structure M , for every $\bar{a}, \bar{b} \in M^k$, $\psi(\bar{a}, \bar{b})$ holds in*

M_{alg} if and only if $\varphi^M(\bar{a}, \bar{b}) = 0$. If M is an $\mathcal{L}$ -structure and $\mathcal{C}$ is an $(\aleph_1, \varphi)$ -skeleton like φ -chain in M , then $\mathcal{C}$ is an $(\aleph_1, \psi)$ -skeleton like ψ -chain of the same order type in M_{alg} .

Proof. The hypothesis implies that $\prec_\psi$ in M_{alg}^k refines $\prec_\varphi$ in M^k . Thus, a φ -chain in M is also a ψ -chain in M_{alg} of the same order type. Moreover, suppose $\bar{a} \in M^k$ and $\mathcal{C}_{\bar{a}}$ is as in the definition of $(\aleph_1, \varphi)$ -skeleton like. If $\bar{b}, \bar{c} \in \mathcal{C}$ are such that

$$\{x \in \mathcal{C}_{\bar{a}} \mid \bar{b} \preceq_\psi x \preceq_\psi \bar{c}\} = \emptyset,$$

then also

$$\{x \in \mathcal{C}_{\bar{a}} \mid \bar{b} \preceq_\varphi x \preceq_\varphi \bar{c}\} = \emptyset.$$

Hence,

$$\varphi^M(\bar{a}, \bar{b}) = \varphi^M(\bar{a}, \bar{c}) \quad \text{and} \quad \varphi^M(\bar{b}, \bar{a}) = \varphi^M(\bar{c}, \bar{a}).$$

Since, by hypothesis, $M_{alg} \models \psi^M(\bar{a}, \bar{b})$ is equivalent to $\varphi^M(\bar{a}, \bar{b}) = 0$ and $M_{alg} \models \psi^M(\bar{a}, \bar{c})$ is equivalent to $\varphi^M(\bar{a}, \bar{c}) = 0$, one gets

$$M_{alg} \models (\psi^M(\bar{a}, \bar{b}) \leftrightarrow \psi^M(\bar{a}, \bar{c})).$$

In the same way,

$$M_{alg} \models (\psi^M(\bar{b}, \bar{a}) \leftrightarrow \psi^M(\bar{c}, \bar{a}))$$

is deduced. Thus, $\mathcal{C}$ is an $(\aleph_1, \psi)$ -skeleton like ψ -chain in M_{alg} . $\square$

Remark 3.6. If s, t are $\mathcal{L}$ -terms and $q : [0, 1] \rightarrow [0, 1]$ is a continuous function such that $q(x) = 0$ iff $x = 0$, the formulas

$$\varphi(s, t) = "q(d(s, t))" \quad \text{and} \quad \psi(s, t) = "s = t"$$

satisfy the hypothesis of the previous lemma.

Proposition 3.7. Assume that CH fails. If $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ are as in Lemma 3.5, $(M_n)_{n \in \mathbb{N}}$ is a sequence of $\mathcal{L}$ -structures with the order property witnessed by φ , and $(k_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers, then the family of $\mathcal{L}$ -structures

$$\left\{ \left(\prod_n^{\mathcal{U}} M_{k_n} \right)_{alg} \mid \mathcal{U} \text{ is an ultrafilter over } \mathbb{N} \right\},$$

where $\prod_n^{\mathcal{U}} M_{k_n}$ denotes the metric ultraproduct, contains $2^{\mathfrak{c}}$ many pairwise non-isomorphic elements. In other words, there are $2^{\mathfrak{c}}$ many metric ultraproducts of the sequence $(M_{k_n})_{n \in \mathbb{N}}$ up to algebraic isomorphism, i.e., up to bijections preserving all the function symbols but not necessarily the distance.

Proof. Since every subsequence of $(M_n)_{n \in \mathbb{N}}$ has the order property witnessed by φ , there is no loss of generality to assume $k_n = n$ for every $n \in \mathbb{N}$. By Lemma 3.2, for every linear order I of cardinality $\mathfrak{c}$, there is an ultrafilter $\mathcal{U}$ such that $\prod_n^{\mathcal{U}} M_n$ has an $(\aleph_1, \varphi)$ -skeleton like φ -chain $\mathcal{C}$ of order type I . By Lemma 3.5, $\mathcal{C}$ is also an $(\aleph_1, \psi)$ -skeleton like ψ -chain of the same order type in $\left(\prod_n^{\mathcal{U}} M_n \right)_{alg}$. Since this is true for every linear order I of cardinality $\mathfrak{c}$, the conclusion follows from Lemma 3.4. $\square$

Observe that in this result $2^{\mathfrak{c}}$ is the maximum number possible, because it is the number of ultrafilters over $\mathbb{N}$. The following result is an immediate consequence of Proposition 3.7 and Remark 3.6.

Corollary 3.8. *If $(M_n)_{n \in \mathbb{N}}$ is a sequence of $\mathcal{L}$ -structures with the order property witnessed by the $\mathcal{L}$ -formula $q(d(s, t))$, where s and t are terms and $q: [0, 1] \rightarrow [0, 1]$ is a continuous function such that $q(x) = 0$ iff $x = 0$, then the conclusion of Proposition 3.7 holds.*

Now, in order to prove Theorem 3.1 it is enough to show that the sequences $(S_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$ have the order property, witnessed by a formula φ as in Corollary 3.8. Let $\eta(x_1, x_2, y_1, y_2)$ be the metric formula defined by

$$\min \{2d([x_1, y_2], e), 1\}.$$

Fix $l \in \mathbb{N}$ and $n \geq 3^l$, and consider the sequences $(\Sigma_{n,i})_{i=1}^l$ and $(T_{n,i})_{i=1}^l$ in S_n defined in Lemma 2.4 and the sequences $(b_{n,i})_{i=1}^l$ and $(c_{n,i})_{i=1}^l$ in U_n defined in Lemma 2.5. It is not difficult to infer from Lemma 2.4 and Lemma 2.5 that

$$((\Sigma_{n,i}, T_{n,i}))_{i=1}^l \quad \text{and} \quad ((b_{n,i}, c_{n,i}))_{i=1}^l$$

are η -chains of length l in S_n and U_n respectively. An application of Corollary 3.8 concludes the proof of Theorem 3.1.

4. RANKED REGULAR RINGS

If $n \in \mathbb{N}$, denote by $\mathbb{M}_n$ the algebra of $n \times n$ matrices over $\mathbb{C}$ and by rk the normalized rank on $\mathbb{M}_n$. Thus, if $A \in \mathbb{M}_n$, $\text{rk}(A)$ is the rank of A divided by n . In [2], Elek considered metric ultraproducts of the matrix algebras over $\mathbb{C}$ with respect to the *rank metric*

$$d(a, b) = \text{rk}(a - b).$$

If $\sigma \in S_n$, denote, as before, by A_σ the permutation matrix associated to σ , regarded as an element of $\mathbb{M}_n$. It is easily seen that

$$(4.1) \quad \text{rk}(I_n - A_\sigma) = 1 - \frac{l}{n},$$

where l is the number of possibly trivial cycles of σ .

It can be deduced from Lemma 2.4 that, for any increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers, the sequence $(\mathbb{M}_{k_n})_{n \in \mathbb{N}}$ of matrix algebras endowed with the rank metric has the order property witnessed by the formula $\varphi(x_1, x_2, y_1, y_2)$ defined by

$$\min \{3d(x_1 y_2, y_2 x_1), 1\}.$$

The proof of this fact is left to the reader, being very similar to the proof of Lemma 2.5, using (4.1) instead of (1.1). The following proposition, that answers a question of Elek, can now be obtained by direct application of Corollary 3.8.

Proposition 4.1. *If CH fails, then for every increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers there are 2^c many metric ultraproducts of the sequence $(\mathbb{M}_{k_n})_{n \in \mathbb{N}}$ with respect to the rank metric whose multiplicative semigroups are pairwise non-isomorphic.*

Proposition 4.1 can in fact be generalized to direct sequences of finite sums of matrix algebras obtained from a Bratteli diagram and a harmonic function as in [2], Section 3. Moreover, the same holds without change and with the same proof if $\mathbb{C}$ is replaced by any other field or by any von Neumann regular ring R endowed with a rank function N . The easy details are left to the interested reader. An exhaustive treatment of von Neumann regular rings and ranked von Neumann regular rings can be found in [10].

5. THE Σ_2 -THEORIES OF UNIVERSAL SOFIC AND HYPERLINEAR GROUPS

If CH holds, then every universal sofic group and every universal hyperlinear group is saturated when regarded as a metric structure; hence two such groups are isometrically isomorphic if and only if their metric theories coincide. (Again assuming CH, it is currently not known whether there exist algebraically non-isomorphic universal sofic groups or algebraically non-isomorphic universal hyperlinear groups; in [12], Thomas asked whether all the universal sofic groups were elementarily equivalent when regarded as first order structures.) In this section, we will prove the partial results that all universal sofic groups have the same metric Σ_2 -theories and that all universal hyperlinear groups have the same metric Σ_2 -theories. This is equivalent to the statement that for any Σ_2 formula φ in the language of metric groups, the sequences of real numbers given by evaluation of φ in the symmetric groups and, respectively, in the unitary groups converge. Since a formula φ is Π_2 iff $1 - \varphi$ is Σ_2 , this implies that universal sofic, and respectively hyperlinear, groups have the same Π_2 theories as well.

The proof will make use of a general lemma, roughly asserting that if $(M_n)_{n \in \mathbb{N}}$ is a sequence of structures such that, given $m \in \mathbb{N}$, the elements of M_n , for n large enough, can be arbitrarily well approximated by elements in the range of some approximate embedding of M_m into M_n , then for every Σ_2 sentence φ the sequence $(\varphi^{M_n})_{n \in \mathbb{N}}$ converges. In order to precisely state and prove the lemma, we need to introduce the following terminology. A function $\iota : M \rightarrow N$ between structures in a metric language is said to *preserve all the function and relation symbols up to* $\delta \geq 0$ if, for every n -ary function symbol f and $a_1, \dots, a_n \in M$,

$$d^N(f^N(\iota(a_1), \dots, \iota(a_n)), \iota(f^M(a_1, \dots, a_n))) \leq \delta$$

and for every n -ary relation symbol R and $a_1, \dots, a_n \in M$,

$$|R^N(\iota(a_1), \dots, \iota(a_n)) - R^M(a_1, \dots, a_n)| \leq \delta.$$

If $\varphi(x_1, \dots, x_n)$ is a formula, the function ι is said to *preserve φ up to δ* if

$$|\varphi^N(\iota(a_1), \dots, \iota(a_n)) - \varphi^M(a_1, \dots, a_n)| \leq \delta.$$

Theorem 3.5 of [1] asserts that the interpretation of a formula φ in any structure is uniformly continuous in every variable, with uniform continuity modulus independent from the other variables and from the structure. It follows that for every $\epsilon > 0$ there exists $\delta > 0$ such that every embedding ι of a structure M into a structure N preserving all the function and relation symbols up to δ also preserves φ up to ϵ .

Observe that, if $\psi(y, x)$ is a quantifier-free formula, then

$$\inf_x \sup_y \psi(y, x)$$

is a Σ_2 sentence. Conversely, by [1], Theorem 6.3, Proposition 6.6 and Proposition 6.9, the set of such sentences is dense in the set of all Σ_2 sentences. Thus, there is no loss of generality in considering only this type of Σ_2 sentence.

Lemma 5.1. *Assume that $(M_n)_{n \in \mathbb{N}}$ is a sequence of structures in a metric language $\mathcal{L}$. Suppose that, $\forall \delta > 0$, $\exists m_0 \in \mathbb{N}$ such that, $\forall m \geq m_0$, $\exists k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$, $\forall a \in M_k$ there exists an embedding ι_m^k (possibly depending on a) of M_m into M_k satisfying the following properties:*

- ι_m^k preserves all the relation and function symbols up to δ ,
- there is $\tilde{a} \in M_m$ such that $d(\iota_m^k(\tilde{a}), a) < \delta$.

Then if ψ is a quantifier-free $\mathcal{L}$ -formula and φ is the $\mathcal{L}$ -formula

$$\inf_x \sup_{y_1, \dots, y_n} \psi(y_1, \dots, y_n, x),$$

then the sequence $(\varphi^{M_m})_{m \in \mathbb{N}}$ converges.

Proof. Fix $\varepsilon > 0$ and define $\delta > 0$ such that any embedding preserving all the function and relation symbols up to δ preserves ψ up to ε , and moreover $\delta < \omega(\varepsilon)$, where ω is a uniform continuity modulus for ψ in the last variable. Consider $m_0, m \geq m_0, k_0$ and $k \geq k_0$ as in the statement. If $a \in M_k$, then there exists an embedding i_m^k of M_m into M_k that preserves all the functions and relation symbols up to δ and such that $d(\iota_m^k(\tilde{a}), a) < \delta$ for some $\tilde{a} \in M_m$. Hence,

$$\begin{aligned} \sup_{y_1, \dots, y_n \in M_k} \psi(y_1, \dots, y_n, a) &\geq \sup_{y_1, \dots, y_n \in M_k} \psi(y_1, \dots, y_n, \iota_m^k(\tilde{a})) - \varepsilon \\ &\geq \sup_{y_1, \dots, y_n \in M_m} \psi(\iota_m^k(y_1), \dots, \iota_m^k(y_n), \iota_m^k(\tilde{a})) - \varepsilon \\ &\geq \sup_{y_1, \dots, y_n \in M_m} \psi(y_1, \dots, y_n, \tilde{a}) - 2\varepsilon \\ &\geq \inf_{x \in M_m} \sup_{y_1, \dots, y_n \in M_m} \psi(y_1, \dots, y_n, x) - 2\varepsilon \\ &= \psi^{M_m} - 2\varepsilon. \end{aligned}$$

Since this is true for every $a \in M_k$,

$$\varphi^{M_k} = \inf_{x \in M_k} \sup_{y_1, \dots, y_n \in M_k} \psi(y_1, \dots, y_n, x) \geq \varphi^{M_m} - 2\varepsilon.$$

Since this is true for every $k \geq k_0$,

$$\liminf_{k \rightarrow +\infty} \psi^{M_k} \geq \psi^{M_m} - 2\varepsilon,$$

and since this is true for every $m \geq m_0$,

$$\liminf_{k \rightarrow +\infty} \psi^{M_k} \geq \limsup_{m \rightarrow +\infty} \psi^{M_m} - 2\varepsilon.$$

Finally, letting ε go to 0, one gets

$$\liminf_{k \rightarrow +\infty} \psi^{M_k} \geq \limsup_{m \rightarrow +\infty} \psi^{M_m}.$$

This concludes the proof. $\square$

We will now prove that the sequence $(S_n)_{n \in \mathbb{N}}$ of symmetric groups satisfies the hypothesis of Lemma 5.1. The key observation is that a permutation $\sigma \in S_{km}$ belongs to the image of some isometric embedding of S_m into S_{km} if, for every $l \geq 1$, the number l -cycles of σ is a multiple of k (here and in the following, we consider 0 to be multiple of any natural number). Thus, it is enough to prove that one can “chop up” any permutation $\sigma \in S_{km}$, obtaining another permutation $\tilde{\sigma}$ close to σ with the number of its l -cycles a multiple of k for every $l \in \mathbb{N}$. Lemma 5.2 and Lemma 5.3 essentially show that any permutation is close to a permutation with only “small” cycles, while Lemma 5.4 shows that a permutation with only small cycles is close to one with the number of its l -cycles a multiple of k .

In order to simplify the discussion, we will introduce the following notation: If $\sigma \in S_n$ and $l \geq 1$, define $C_l(\sigma)$ to be the set of cycles of σ of length l and $w(\sigma)$ the greatest l such that $C_l(\sigma)$ is non-empty. In particular, $C_1(\sigma)$ is the set of fixed points of σ .

Lemma 5.2. *For every $m, n \in \mathbb{N}$ such that $m|n$, if $\sigma \in S_n$, then there exists $\tau \in S_n$ such that $w(\tau) \leq m$, $C_1(\tau) \supset C_1(\sigma)$ and $d(\sigma, \tau) \leq \frac{2}{m}$.*

Proof. Suppose $k \in \mathbb{N}$ is such that $n = km$. If σ is a cycle of length n , there is a product of k cycles of length m at distance $\frac{1}{m}$ from σ . Suppose σ has no cycle of length n . If $m < l < n$, then $l = \lambda m + \rho$ for some $0 \leq \rho < m$ and $1 \leq \lambda < k$. Pick $c \in C_l(\sigma)$ and consider the permutation σ' obtained by σ breaking up c into λ cycles of length m and, if $\rho > 0$, one cycle of length ρ . Thus,

$$km \cdot d(\sigma, \sigma') \leq \lambda + 1.$$

Consider the permutation τ obtained by σ repeating this process for any element of $\bigcup_{k > m} C_k(\sigma)$. Then, $w(\tau) \leq m$. Define, for $\lambda \in \{1, 2, \dots, k-1\}$ and $\rho \in \{0, 1, \dots, m-1\}$,

$$n_{\lambda, \rho} = |C_{\lambda m + \rho}(\sigma)|.$$

Observe that

$$\sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} \lambda m n_{\lambda, \rho} \leq \sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} (\lambda m + \rho) n_{\lambda, \rho} \leq n = km,$$

and hence

$$\sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} \lambda n_{\lambda, \rho} \leq k.$$

Now, we have

$$km \cdot d(\sigma, \tau) \leq \sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} (\lambda + 1) n_{\lambda, \rho} \leq 2 \sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} \lambda n_{\lambda, \rho} \leq 2k,$$

and hence $d(\sigma, \tau) \leq \frac{2}{m}$. □

Lemma 5.3. *For every $m, n \in \mathbb{N}$ such that $m|n$ and $\beta \in (0, 1)$, if $\sigma \in S_n$, then there exists $\tau \in S_n$ such that $C_1(\tau) \supset C_1(\sigma)$, $w(\tau) \leq \lceil m^\beta \rceil$ and $d(\sigma, \tau) \leq \frac{8}{m^\beta}$.*

Proof. Define $k = \frac{n}{m}$ and $N = \lceil \frac{km}{m^\beta} \rceil \lceil m^\beta \rceil$. Observe that

$$N \leq \left(\frac{km}{m^\beta} + 1 \right) (m^\beta + 1) = km + \frac{km}{m^\beta} + m^\beta + 1 \leq 4km.$$

If $\sigma \in S_{km}$, consider the element $\tilde{\sigma}$ of S_N acting as σ on $\{1, 2, \dots, km\}$ and fixing $\{km+1, \dots, N\}$ pointwise. By Lemma 5.2, there is $\tilde{\tau} \in S_N$ such that $C_1(\tilde{\tau}) \supset C_1(\tilde{\sigma})$, $w(\tilde{\tau}) \leq \lceil m^\beta \rceil$ and $d(\tilde{\sigma}, \tilde{\tau}) \leq \frac{2}{\lceil m^\beta \rceil}$. Now consider the element τ of S_{km} obtained restricting $\tilde{\tau}$ to $\{1, 2, \dots, km\}$. Now, $w(\tau) \leq \lceil m^\beta \rceil$, $C_1(\tau) \supset C_1(\sigma)$ and

$$d(\sigma, \tau) = \frac{N}{km} d(\tilde{\sigma}, \tilde{\tau}) \leq \frac{8}{m^\beta}. \quad \square$$

Lemma 5.4. *For every $\beta > 0$ there exists $m_0 \in \mathbb{N}$ such that, if $m \geq m_0$, $k \in \mathbb{N}$, $n = km$ and $\tau \in S_n$ is such that $w(\tau) \leq \lceil m^\beta \rceil$, then there exists $\rho \in S_n$ such that $w(\rho) \leq \lceil m^\beta \rceil$, $C_1(\rho) \supset C_1(\sigma)$, $d(\rho, \tau) < m^{2\beta-1}$ and $C_i(\rho)$ a multiple of k for every $i \in \mathbb{N}$.*

Proof. Pick $m_0 \in \mathbb{N}$ such that, for $m \geq m_0$,

$$\sum_{i=1}^{\lceil m^\beta \rceil} i \leq \frac{m^{2\beta} + 3m^\beta + 2}{2} \leq m^{2\beta}.$$

Suppose $m \geq m_0$. Define, for $i \in \{2, \dots, \lceil m^\beta \rceil\}$,

$$|C_i(\tau)| = n_i = t_i k + r_i,$$

where $0 \leq r_i < k$ and $t_i \geq 0$. Observe that

$$\sum_{i=2}^{\lceil m^\beta \rceil} i r_i \leq k \sum_{i=2}^{\lceil m^\beta \rceil} i \leq k m^{2\beta}.$$

Consider the permutation ρ obtained by dropping r_i i -cycles from σ for every $i \in \{2, 3, \dots, \lceil m^\beta \rceil\}$. Observe that $|C_i(\rho)| = k t_i$ for $i = 2, 3, \dots, \lceil m^\beta \rceil$. Moreover,

$$|C_1(\rho)| = km - \sum_{i=2}^{\lceil m^\beta \rceil} C_i(\rho) = \left(m - \sum_{i=2}^{\lceil m^\beta \rceil} t_i \right) k$$

and $|C_i(\rho)| = 0$ for $i > \lceil m^\beta \rceil$. Finally,

$$km \cdot d(\tau, \rho) \leq \sum_{i=2}^{\lceil m^\beta \rceil} i r_i \leq k m^{2\beta},$$

and hence $d(\tau, \rho) \leq m^{2\beta-1}$. $\square$

Proposition 5.5. *There exists $m_0 \in \mathbb{N}$ such that, for every $m \geq m_0$ and $k \in \mathbb{N}$, if $\sigma \in S_{km}$, then there exists $\rho \in S_{km}$ such that $C_1(\rho) \supset C_1(\sigma)$, $w(\rho) \leq \lceil \sqrt[3]{m} \rceil$, $d(\rho, \sigma) \leq \frac{9}{\sqrt[3]{m}}$ and $\rho = \Phi(\tilde{\rho})$ for some $\tilde{\rho} \in S_m$ and isometric embedding $\Phi : S_m \rightarrow S_{km}$.*

Proof. By Lemma 5.3, there is $\tau \in S_{km}$ such that $d(\tau, \sigma) < \frac{8}{\sqrt[3]{m}}$, $w(\tau) \leq \lceil \sqrt[3]{m} \rceil$ and $C_1(\tau) \supset C_1(\sigma)$. By Lemma 5.4, there is $\rho \in S_{km}$ such that $C_1(\rho) \supset C_1(\tau)$, $d(\rho, \sigma) \leq \frac{1}{\sqrt[3]{m}}$, $w(\rho) \leq \lceil \sqrt[3]{m} \rceil$ and $|C_i(\sigma)|$ is a multiple of k for every $i \in \mathbb{N}$. Thus, $\rho = \Phi(\tilde{\rho})$ for some $\tilde{\rho} \in S_m$ and isometric embedding $\Phi : S_m \rightarrow S_{km}$. Finally,

$$d(\sigma, \rho) \leq d(\tau, \rho) + d(\tau, \sigma) \leq \frac{1}{\sqrt[3]{m}} + \frac{8}{\sqrt[3]{m}} \leq \frac{9}{\sqrt[3]{m}}. \quad \square$$

If $k, m \in \mathbb{N}$ and $0 \leq r < m$, define ι to be the injective group homomorphism of S_{km} into S_{km+r} obtained by sending a permutation σ to the permutation that acts as σ on $\{1, 2, \dots, km\}$ and fixes pointwise $\{km+1, \dots, km+r\}$. Since ι preserves distances up to $1 - \frac{1}{m}$, and any element of S_{km+r} is at distance at most $1 - \frac{1}{m}$ from some element in the range of ι , it follows from Proposition 5.5 that the sequence $(S_n)_{n \in \mathbb{N}}$ satisfies the hypothesis of Lemma 5.1. This concludes the proof of:

Theorem 5.6. *If φ is a formula of the form*

$$\inf_x \sup_{y_1, \dots, y_n} \varphi(y_1, \dots, y_n, x)$$

in the language of bi-invariant metric groups, then the sequence $(\varphi^{S_n})_{n \in \mathbb{N}}$ converges.

The analogue of Proposition 5.5 in the case of unitary groups has been proved by von Neumann in [13] using the spectral theorem for normal matrices and an averaging argument on the eigenvalues. The precise statement is reported here for convenience of the reader. Observe that if $W \in U_{km}$, then the function from U_m to U_{km} sending B to

$$W(I_k \otimes B)W^*$$

is an isometric embedding. Here, $\otimes$ denotes the usual tensor product of matrices.

Proposition 5.7. *If $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and $m \geq n_0$, if $A \in \mathbb{M}_{km}$ is a normal matrix with operator norm at most 1, there exists $B \in M_m$ of operator norm at most 1 and $W \in U_{km}$ such that $\|A - W(I_k \otimes B)W^*\|_2 < \varepsilon$, where $\|\cdot\|_2$ is the normalized Hilbert-Schmidt norm. Moreover, if A is Hermitian (resp. unitary), B can be chosen Hermitian (resp. unitary).*

In order to show that the sequence of unitary groups satisfies the hypothesis of Lemma 5.1 it remains only to show that if $k, m \in \mathbb{N}$ and $0 \leq r < k$, then there is an injective group homomorphism ι of U_{km} into U_{km+r} that almost preserves the metric and such that any element of U_{km+r} is close to some element in the range of ι . This is done in the following lemma.

Lemma 5.8. *If $k, m \in \mathbb{N}$ and $0 \leq r < m$, then the function ι from U_{km} to U_{km+r} sending A to*

$$\begin{pmatrix} A & 0 \\ 0 & I_r \end{pmatrix}$$

(where I_r is the $r \times r$ identity matrix) is an injective group homomorphism that preserves the metric up to $\frac{1}{k}$. Moreover, any element of U_{km+r} is at distance at most $\frac{4}{\sqrt[4]{k}}$ from some element in the range of ι .

Proof. Denote $km + r$ by n . Direct calculation shows that if $A, B \in U_{km}$,

$$0 \leq d(A, B) - d(\iota(A), \iota(B)) = 1 - \sqrt{\frac{km}{n}} \leq \frac{1}{k}.$$

Suppose now that $C \in U_n$ and define A to be the element of $\mathbb{M}_{km}$ such that $A_{i,j} = C_{i,j}$ for $1 \leq i, j \leq km$. It is easy to see that, since U is unitary,

$$\|A^*A - I_{km}\|_2^2 \leq \frac{1}{k}.$$

By [9], Corollary 1, there exists $B \in U_{km}$ such that

$$\|A - B\|_2^2 \leq 36 \|A^*A - I_{km}\|_2 \leq \frac{36}{\sqrt{k}}.$$

Thus,

$$d(\iota(B), C) \leq \frac{1}{2} \sqrt{\frac{km \|A - B\|_2^2 + 6r}{n}} \leq \frac{4}{\sqrt[4]{k}}. \quad \square$$

This concludes the proof of:

Theorem 5.9. *If φ is a formula of the form*

$$\inf_x \sup_{y_1, \dots, y_n} \varphi(y_1, \dots, y_n, x)$$

in the language of bi-invariant metric groups, then the sequence $(\varphi^{U_n})_{n \in \mathbb{N}}$ converges.

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