ASYMPTOTIC BEHAVIOR OF THE NONLOCAL DIFFUSION EQUATION WITH LOCALIZED SOURCE

JINGE YANG, SHUANGSHUANG ZHOU, AND SINING ZHENG

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ABSTRACT. In this paper we study the nonlocal diffusion equation with localized source: $u_t = J * u - u + a(x)u^p$ in $\mathbb{R}^N \times (0, T)$, with $a(x)$ nonnegative, continuous, and compactly supported. It is found that the localized source $a(x)$ drastically changes the asymptotic behavior of the nonlocal diffusion equation that the Fujita phenomenon happens only if $N = 1$. That is to say, the solutions must be global provided the initial data are small if $N > 1$. Furthermore, we determine the second critical exponent $b_c = \frac{1}{p-1}$ with $N = 1$, and $b_c = 0$ with $N > 1$, rather than $b_c = \frac{2}{p-1}$ for the case of homogeneous source with all $N \geq 1$. This implies that the scope of initial data for global solutions determined by the second critical exponent b_c is enlarged due to the localized factor $a(x)$. Finally, the time-decay profile of the global solutions is also studied for slow-decay initial data. It is mentioned that we need some new techniques to deal with the nonlocal diffusion in the model. For example, different from local diffusion equations, because of a lack of regularity mechanism from the nonlocal diffusion, we employ the moving plane method in integral form to deal with the mild solutions instead of the maximum principle. In addition, due to the localization of the source, we have to use precise weighted L^1 estimates for the critical situation to replace the general test function method.

1. INTRODUCTION

Consider the Cauchy problem for nonlocal diffusion equation

$$(1.1) \quad \begin{cases} u_t = J * u - u + a(x)u^p, & x \in \mathbb{R}^N, \ t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $J \in C_c(\mathbb{R}^N)$ is nonnegative, radially symmetric and decreasing, with unit integral, $*$ stands for usual convolution in $\mathbb{R}^N$, and $a(x)$ is a nonnegative continuous function with compact support containing the origin. Instead of the usual and classical local diffusion represented via the second order differential operators, the diffusion in (1.1) is nonlocal and described by the convolution $J * u - u$. Such a nonlocal diffusion equation (1.1) can be used to describe a variety of nonlocal diffusion processes for generation populations [2], deblurring-denoising of images

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The third author is the corresponding author.

[3], etc. In addition, the nonlinear source in (1.1) is heterogeneous and localized by the weight function $a(x)$.

For the case of homogeneous source with $a(x) \equiv 1$, Melián and Quirós [1] proved that (1.1) shares the critical exponent $p_c = 1 + \frac{2}{N}$ with the classical nonlinear heat equation

$$(1.2) \quad u_t = \Delta u + u^p \quad \text{in } \mathbb{R}^N \times (0, T).$$

That is, (i) If $1 < p \leq p_c$, the solutions of (1.1) with $a(x) \equiv 1$ blow up in finite time for any nonnegative and nontrivial initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$; (ii) If $p > p_c$, there exist global solutions for small initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

The interesting critical exponent p_c was first introduced by Fujita [4]. Since then the related studies have been thoroughly extended to a variety of PDEs; see the surveys [5, 6]. The localized source cases with Laplacian or nonlinear diffusion are also involved [8–11].

In the present paper we investigate the Fujita type theorems for the nonlocal diffusion equation (1.1) with localized source. It is observed that the Fujita phenomenon occurs only if $N = 1$, just like for the situation of classical local diffusion [8]. In addition, we consider the second critical exponent, introduced by Lee and Ni [12] for the heat equation (1.2), to characterize the critical size of initial data for global solutions and nonglobal solutions in the coexistence of global and nonglobal solutions. Furthermore, we also study the asymptotic profile of the global solutions under slow decay initial data similar to that for heat equations [13].

We deal with mild solutions of (1.1) and call nonnegative $u \in C^1((0, T), L^\infty(\mathbb{R}^N)) \cap C([0, T], L^\infty(\mathbb{R}^N))$ a mild solution to (1.1) if

$$(1.3) \quad u = e^{-t}u_0 + W(x, t) * u_0 + \int_0^t e^{-(t-s)} a u^p ds + \int_0^t W(x, t-s) * a u^p ds,$$

with

$$(1.4) \quad W(x, t) = e^{-t} \sum_{n=1}^{\infty} \frac{t^n}{n!} J^{*n}(x),$$

where J^{*n} represents the $(n-1)$ times convolution of J with itself. The local existence of the mild solutions can be obtained via the standard theory.

By the known results for nonnegative solutions to nonlocal diffusion equations (see Proposition 3.1 in [14]), one can easily get the comparison principle for (1.1):

Lemma 1.1 (Comparison Principle). *Assume*

$$u, v \in C^1((0, T), L^\infty(\mathbb{R}^N)) \cap C([0, T], L^\infty(\mathbb{R}^N))$$

satisfying

$$u_t - J * u + u - a(x)u^p \leq v_t - J * v + v - a(x)v^p$$

in $\mathbb{R}^N \times (0, T)$, with $u(x, 0) \leq v(x, 0)$ in $\mathbb{R}^N$. Then $u \leq v$ in $\mathbb{R}^N \times (0, T)$.

Now we state the main results of this paper. The first one concerns the critical exponent.

Theorem 1. (i) *If $N = 1$, then the critical exponent of (1.1) is $p_c = 2$; namely, there is no nontrivial global solution if $1 < p \leq p_c$, whereas there exist both global and non-global solutions if $p > p_c$, depending on the size of the initial data.*

(ii) *If $N > 1$, the critical exponent $p_c = p_0 = 1$.*

To deal with the second critical exponent, we need the following notation:

$$\mathbb{I}_b = \left\{ \phi(x) \in C_b(\mathbb{R}^N) \mid \phi(x) \geq 0, \liminf_{|x| \rightarrow \infty} |x|^b \phi(x) > 0 \right\},$$

$$\mathbb{I}^b = \left\{ \phi(x) \in C_b(\mathbb{R}^N) \mid \phi(x) \geq 0, \limsup_{|x| \rightarrow \infty} |x|^b \phi(x) < \infty \right\},$$

where $C_b(\mathbb{R}^N)$ denotes bounded continuous functions in $\mathbb{R}^N$.

Theorem 2. *Let u be a solution to (1.1), $N = 1$, $p > p_c = 2$, $u_0(x) = \lambda\varphi(x)$ with $\lambda > 0$, and $b_c = \frac{1}{p-1}$.*

- (i) *If $b \in (0, b_c)$ with $\varphi(x) \in \mathbb{I}_b$, then u blows up in finite time.*
- (ii) *If $b \in (b_c, 1)$ with $\varphi(x) \in \mathbb{I}^b$, there exists $\lambda_0 > 0$ such that the solution u is global for all $\lambda \in (0, \lambda_0)$ and satisfies with $K > 0$ that*

$$\lim_{t \rightarrow \infty} t^{\frac{b}{2}} \left\| u - \int_{\mathbb{R}} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{4t}} |y|^{-b} dy \right\|_{L^\infty(|x| \leq K\sqrt{t})} = 0.$$

Theorem 3. *Let u be a solution to (1.1), $N > 1$, $u_0(x) = \lambda\varphi(x)$ with $\lambda > 0$ and $\varphi(x) \in \mathbb{I}^b$ with $b > 0$. If λ is small enough, then u is global.*

Remark 1. Theorem 1 implies that the localized source makes a tremendous contribution to the critical exponent. More precisely, the existence of the weight function $a(x)$ cancels the Fujita phenomenon in (1.1) unless $N = 1$. This is similar to the situation with local diffusion [8]. However, there are substantial difficulties to be overcome for treating nonlocal diffusion problems, such as the complicated representation of the fundamental solution involved and the absence of a regularizing effect and scaling invariance.

Remark 2. It is well known that the classical model (1.2) possesses the second critical exponent $b_c = \frac{2}{p-1}$ [12]. The same should be true for the situation with nonlocal diffusion and $a(x) \equiv 1$. Quite differently, we prove (1.1) with the localized source that $b_c = \frac{1}{p-1}$ (if $N = 1$) by Theorem 2 or $b_c = 0$ (if $N \geq 2$) from Theorem 3. In other words, the scope of initial data for global solutions determined by the second critical exponent b_c is enlarged due to the localized factor $a(x)$. It is mentioned that the same second critical exponent can be obtained via a similar method for the local diffusion case with $a(x)$ compactly supported.

2. CRITICAL FUJITA EXPONENT

In this section, we will prove that if $1 < p \leq p_c$ with $N = 1$, then (1.1) does not admit global solutions. The involved techniques include the Kaplan method [1, 8], the moving plane method [16], together with precise weighted L^1 estimates [7]. Different from local diffusion equations, the general maximum principle fails to be true for u_r because of lack of regularity. Instead of the maximum principle, we employ the moving plane method in integral form [16] to deal with the mild solutions. We begin with a basic lemma associated with the localized source.

Lemma 2.1. *If both $u_0(x) \in L^r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $a(x) \in C_c(\mathbb{R})$ are even and nonincreasing in $(0, \infty)$, with $r \geq 1$, then the solution u of (1.1) is even and nonincreasing in $(0, \infty)$ with respect to x for $t > 0$.*

Proof. Define $u_\lambda = u(2\lambda - x, t)$ with $\lambda > 0$, $w_\lambda = u_\lambda - u$. Denote $\Sigma_\lambda = \{x|x \geq \lambda\}$, $\Sigma_\lambda^- = \{x|x \in \Sigma_\lambda, w_\lambda < 0\}$ and $\lambda_0 = \inf\{\lambda|w_\mu \geq 0, \text{ for any } \mu \geq \lambda\}$. We will prove $\lambda_0 = 0$. First, we claim $\lambda_0 < \infty$. By Lemma 2.13 of [17] and the property of J , the W in (1.3) is radially symmetric and nonincreasing, and so is $W * u_0$. We introduce the transformation

$$\begin{aligned} u_\lambda &= \int_0^t e^{-(t-s)} a u^p(x_\lambda) ds + \int_{\Sigma_\lambda} W(x_\lambda - y, t-s) a u^p(y) dy \\ &\quad + \int_{\Sigma_\lambda} W(x - y, t-s) a u^p(y_\lambda) dy. \end{aligned}$$

Noticing that $a(x)$ is compactly supported, we have for $x \geq \lambda$ with λ large enough that

$$\begin{aligned} w_\lambda &\geq \int_0^t \int_{\Sigma_\lambda} (W(x_\lambda - y_\lambda, t-s) - W(x - y_\lambda, t-s)) (a u^p(y_\lambda) - a u^p(y)) dy \\ &= \int_0^t \int_{\Sigma_\lambda^-} (W(x_\lambda - y_\lambda, t-s) - W(x - y_\lambda, t-s)) a u^p(y_\lambda) dy \geq 0. \end{aligned}$$

This leads to $\lambda_0 < \infty$.

Suppose for contradiction that $\lambda_0 > 0$. Since $a(y_{\lambda_0-\varepsilon}) \geq a(y)$ with any $\varepsilon < \lambda_0$ and $y \in \Sigma_\lambda$, we know for $x \in \Sigma_\lambda^-$ that

$$\begin{aligned} 0 &\geq u_{\lambda_0-\varepsilon} - u \\ &\geq \int_0^t \int_{\Sigma_{\lambda_0-\varepsilon}} (W(x_{\lambda_0-\varepsilon} - y_{\lambda_0-\varepsilon}, t-s) - W(x - y_{\lambda_0-\varepsilon}, t-s)) \\ &\quad \times a(y) (u^p(y_{\lambda_0-\varepsilon}) - u^p(y)) dy \\ &\geq \int_0^t \int_{\Sigma_{\lambda_0-\varepsilon}^-} W(x_{\lambda_0-\varepsilon} - y_{\lambda_0-\varepsilon}, t-s) a(y) (u^p(y_{\lambda_0-\varepsilon}) - u^p(y)) dy. \end{aligned}$$

Thus, by Young's inequality on convolution, Hölder's inequality and the fact that $\|W\|_q \leq C t^{-\frac{N}{2}(1-\frac{1}{q})}$ [20], we have

$$\begin{aligned} \|u_{\lambda_0-\varepsilon} - u\|_{L^\sigma(\Sigma_{\lambda_0-\varepsilon}^-)} &\leq C(t) \|u^{p-1}(u - u_{\lambda_0-\varepsilon})\|_{L^s(\Sigma_{\lambda_0-\varepsilon}^-)} \\ &\leq C(t) \|u\|_{L^{\frac{s\sigma(p-1)}{\sigma-s}}(\Sigma_{\lambda_0-\varepsilon}^-)} \|u_{\lambda_0-\varepsilon} - u\|_{L^\sigma(\Sigma_{\lambda_0-\varepsilon}^-)}, \end{aligned}$$

with $r \leq s < \sigma$ and $\frac{s\sigma(p-1)}{\sigma-s} \geq r$. If ε is small enough, it follows from a standard procedure in [16] that

$$\|u\|_{L^{\frac{s\sigma(p-1)}{\sigma-s}}(\Sigma_{\lambda_0-\varepsilon}^-)} \leq \frac{1}{2C(t)}.$$

Consequently, $\|u_{\lambda_0-\varepsilon} - u\|_{L^\sigma(\Sigma_{\lambda_0-\varepsilon}^-)} \leq \frac{1}{2} \|u_{\lambda_0-\varepsilon} - u\|_{L^\sigma(\Sigma_{\lambda_0-\varepsilon}^-)}$, and hence $|\Sigma_{\lambda_0-\varepsilon}^-| = 0$, or equivalently, $w_{\lambda_0-\varepsilon} \geq 0$, a contradiction to the definition of λ_0 .

The conclusion $\lambda_0 = 0$ implies $u(x, t) - u(-x, t) \geq 0$ for $x \geq 0$. A duality argument shows $u(x, t) - u(-x, t) \geq 0$ for $x \leq 0$. This proves that u is radially symmetric. The above arguments also show that u is, moreover, radially non-increasing. $\square$

Proof of Theorem 1. Without loss of generality, we assume that both $a(x)$ and u_0 achieve their maximum at $x = 0$. By comparison, it suffices to consider the case for both $a(x)$ and u_0 radially decreasing.

Let ϕ_R be the principal eigenfunction to the problem

$$\begin{cases} J * \phi_R - \phi_R = -\lambda_1(B_R)\phi_R, & x \in B_R = (-R, R), \\ \phi_R = 0, & x \in \mathbb{R} \setminus B_R, \end{cases}$$

normalized with $\|\phi_R\|_{L^1(\mathbb{R})} = 1$ and $\phi_R > 0$ in B_R . Denote $\Phi(t) = \int_{-R}^R u \phi_R dx$. By Fubini's theorem,

$$\begin{aligned} \Phi'(t) &\geq \int_{-R}^R \int_{-R}^R J(x-y)(\phi_R(y) - \phi_R(x)) dy u(x) dx + \int_{-R}^R a u^p \phi_R dx \\ &= -\lambda_1(B_R)\Phi(t) + \int_{-R}^R a u^p \phi_R dx. \end{aligned}$$

Noticing that the solution u is radially decreasing with respect to x by Lemma 2.1, we have with $\delta \in (0, 1)$ that

$$\begin{aligned} \int_{-R}^R a u^p \phi_R dx &\geq \frac{1}{2} \int_{-\delta}^{\delta} a u^p \phi_R dx + \frac{a(\delta)}{2} \int_{-\delta}^{\delta} u^p \phi_R dx \\ &\geq \frac{a(\delta)}{2} u^p(\delta, t) \int_{-\delta}^{\delta} \phi_R dx \left(\int_{\delta}^R + \int_{-R}^{-\delta} \right) \phi_R dx + \frac{a(\delta)}{2} \int_{-\delta}^{\delta} u^p \phi_R dx \\ &\geq \frac{a(\delta)}{2} \int_{-\delta}^{\delta} \phi_R dx \left(\int_{\delta}^R + \int_{-R}^{-\delta} \right) u^p \phi_R dx + \frac{a(\delta)}{2} \int_{-\delta}^{\delta} \phi_R dx \int_{-\delta}^{\delta} u^p \phi_R dx \\ (2.1) \quad &\geq \frac{a(\delta)}{2} \int_{-\delta}^{\delta} \phi_R dx \int_{-R}^R u^p \phi_R dx, \end{aligned}$$

and hence

$$\Phi'(t) \geq -\lambda_1(B_R)\Phi(t) + \frac{a(\delta)}{2} \int_{-\delta}^{\delta} \phi_R dx \Phi^p(t).$$

Consequently, Φ blows up in finite time provided

$$\left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} \phi_R dx \right)^{\frac{1}{p-1}} \Phi(0) \geq (\lambda_1(B_R))^{\frac{1}{p-1}},$$

i.e.,

$$(2.2) \quad \left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R \phi_R dx \right)^{\frac{1}{p-1}} \int_{B_R} R u_0 \phi_R dx \geq (\lambda_1(B_R))^{\frac{1}{p-1}} R^{\frac{p}{p-1}}.$$

Since ϕ_R is radially decreasing [1], we have with $R_0 \in (0, R)$ that

$$\begin{aligned} (2.3) \quad &\left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R \phi_R dx \right)^{\frac{1}{p-1}} \int_{B_R} R u_0 \phi_R dx \\ &\geq \left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R \phi_R \left(\frac{Rx}{R_0} \right) dx \right)^{\frac{1}{p-1}} \int_{B_R} R u_0 \phi_R \left(\frac{Rx}{R_0} \right) dx. \end{aligned}$$

Let Ψ be the principal eigenfunction to the problem

$$\begin{cases} -\Delta\Psi = \mu_1\Psi, & x \in B_1 = (-1, 1), \\ \Psi = 0, & x \in B_1, \end{cases}$$

with $\|\Psi\|_{B_1} = 1$. By Theorem 1.5 of [18] and Proposition 3.2 in [19],

$$(2.4) \quad R\phi_R\left(\frac{Rx}{R_0}\right) \rightarrow \Psi\left(\frac{x}{R_0}\right) \quad \text{in } L^1(B_1) \text{ as } R \rightarrow \infty.$$

By Fatou's lemma with (2.3) and (2.4), letting first $R \rightarrow \infty$ and then $R_0 \rightarrow \infty$, we get

$$(2.5) \quad \liminf_{R \rightarrow \infty} \left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R\phi_R dx \right)^{\frac{1}{p-1}} \int_{B_R} Ru_0 \phi_R dx \geq \left(\frac{\delta a(\delta)}{2} \right)^{\frac{1}{p-1}} \Psi^{\frac{p}{p-1}}(0) \int_{B_{R_1}} u_0 dx,$$

for any $R_1 > 0$.

On the other hand, we know by Theorem 1.5 of [18] that

$$(2.6) \quad \lim_{R \rightarrow \infty} R^2 \lambda_1(B_R) = \mu_1,$$

and hence

$$\lim_{R \rightarrow \infty} (\lambda_1(B_R)) R^{\frac{p}{p-1}} = \lim_{R \rightarrow \infty} (R^2 \lambda_1(B_R))^{\frac{1}{p-1}} R^{\frac{p-2}{p-1}} = 0,$$

provided $p < p_c = 2$. This with (2.5) proves (2.2) with R large enough.

Next treat the critical case $p = p_c = 2$ to prove that any nontrivial solutions of (1.1) must be nonglobal. Suppose for contradiction that there is a nonnegative nontrivial global solution u to (1.1), and hence

$$\left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R\phi_R dx \right)^{\frac{1}{p-1}} \int_{B_R} Ru(t)\phi_R dx \leq (\lambda_1(B_R))^{\frac{1}{p-1}} R^{\frac{p}{p-1}}$$

for any $t > 0$ by (2.2). Since ϕ_R is radially decreasing [1], we have with $R_0 \in (0, R)$,

$$\begin{aligned} & \left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R\phi_R dx \right)^{\frac{1}{p-1}} \int_{B_R} Ru(t)\phi_R dx \\ & \geq \left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R\phi_R\left(\frac{Rx}{R_0}\right) dx \right)^{\frac{1}{p-1}} \int_{B_R} Ru(t)\phi_R\left(\frac{Rx}{R_0}\right) dx. \end{aligned}$$

Letting first $R \rightarrow \infty$ and then $R_0 \rightarrow \infty$, by Fatou's lemma and (2.4), we get

$$\liminf_{R \rightarrow \infty} \left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R\phi_R dx \right)^{\frac{1}{p-1}} \int_{B_R} Ru(t)\phi_R dx \geq \left(\frac{\delta a(\delta)}{2} \right)^{\frac{1}{p-1}} \Psi^{\frac{p}{p-1}}(0) \int_{\mathbb{R}} u(t) dx.$$

Noticing

$$\lim_{R \rightarrow \infty} (\lambda_1(B_R)) R^{\frac{p}{p-1}} = \lim_{R \rightarrow \infty} (R^2 \lambda_1(B_R))^{\frac{1}{p-1}} R^{\frac{p-2}{p-1}} = C > 0,$$

we conclude that

$$(2.7) \quad \sup_{t > 0} \int_{\mathbb{R}} u(\cdot, t) dx \leq C.$$

Denote

$$M = \sup_{R, t > 0} \Psi(t) = \sup_{t > 0} \int_{\mathbb{R}} u(x, t) dx > 0.$$

For any $\varepsilon \in (0, M)$, there exist $R_0, t_0 > 0$ such that

$$(2.8) \quad \Psi_{\frac{R_0}{2}}(t_0) \geq M - \varepsilon.$$

Integrate (1.1) in $\mathbb{R} \times (t_0, t)$ to get

$$(2.9) \quad \int_{t_0}^t \int_{\mathbb{R}} au^p dx = \int_{\mathbb{R}} u(x, t) dx - \int_{\mathbb{R}} u(x, t_0) dx \leq \varepsilon.$$

Choose $\xi \in C_c^\infty(\mathbb{R})$ radially decreasing, $0 \leq \xi \leq 1$, with

$$\xi \equiv 1 \text{ for } x \in [-\frac{1}{2}, \frac{1}{2}], \quad \xi \equiv 0 \text{ for } x \notin (-1, 1).$$

Denote $\Theta_R(t) = \int_{-\infty}^{\infty} u \xi_R dx$ with $\xi_R = \xi(\frac{x}{R})$. Take $a > 0$ such that $(-a, a) \supset \text{supp } J$. By Fubini's theorem, we have

$$(2.10) \quad \begin{aligned} \Theta'_R(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(x-y)(\xi_R(y) - \xi_R(x)) dy u(x) dx + \int_{-\infty}^{\infty} au^p \xi_R dx \\ &= \int_{-\infty}^{\infty} \int_{-a}^a J(z)(\xi_R(x+z) - \xi_R(x)) dz u(x) dx + \int_{-\infty}^{\infty} au^p \xi_R dx. \end{aligned}$$

Thanks to Taylor's formula,

$$\xi_R(x+z) - \xi_R(x) = \sum_{i=1}^N \frac{\partial \xi_R}{\partial z_i}(x) z_i + \sum_{i,j=1}^N \frac{\partial^2 \xi_R}{\partial z_i \partial z_j}(x) z_i z_j + O(\frac{1}{R^3}).$$

Since J is radially symmetric, we have

$$\int_{-a}^a J(z) z_i dz = 0, \quad \int_{-a}^a J(z) z_i z_j dz = \frac{1}{N} \int_{-a}^a J(z) z^2 dz \sigma_{ij},$$

with

$$\sigma_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Obviously, $|\Delta \xi_R| \leq \frac{C}{R^2}$, and consequently,

$$(2.11) \quad \Theta'_R(t) \geq -\frac{C}{R^2} \int_{(-R, R) \setminus (-\frac{R}{2}, \frac{R}{2})} u dx + \int_{-R}^R au^p \xi_R dx.$$

Similarly to obtaining (2.1), we get

$$(2.12) \quad \int_{-R}^R au^p \xi_R dx \geq \frac{a(\delta)\delta}{R^2} \Theta_R^p(t).$$

Therefore,

$$(2.13) \quad \Theta'_R(t) \geq -\frac{C}{R^2} \int_{(-R, R) \setminus (-\frac{R}{2}, \frac{R}{2})} u dx + \frac{a(\delta)\delta}{R^2} \Theta_R^p(t).$$

Let $\eta_R = 1 - \xi_R$. Similarly to (2.11), we have

$$\begin{aligned} \left(\int_{\mathbb{R}} u \eta_R dx \right)' &\leq \frac{C}{R^2} \int_{(-R, R) \setminus (-\frac{R}{2}, \frac{R}{2})} u dx + \int_{\mathbb{R}} a u^p \eta_R dx \\ (2.14) \qquad \qquad \qquad &\leq \frac{C}{R^2} \Theta_{2R}(t) + \int_{\mathbb{R}} a u^p dx. \end{aligned}$$

By Cauchy's inequality with (2.12),

$$\begin{aligned} \left(\int_{\mathbb{R}} u \eta_R dx \right)' &\leq \frac{C}{R^2} R \left(\int_{-R}^R a u^p \xi_{2R} dx \right)^{\frac{1}{2}} + \int_{\mathbb{R}} a u^p dx \\ (2.15) \qquad \qquad \qquad &\leq C_0 R^{-2} + \frac{3}{2} \int_{\mathbb{R}} a u^p dx, \end{aligned}$$

with C_0 independent of R . Integrating (2.15) over (t_0, t) , we have from (2.8) and (2.9) that

$$\begin{aligned} \int_{\mathbb{R}} u(x, t) \eta_{\frac{R}{2}}(x) dx &\leq \int_{\mathbb{R}^N} u(x, t_0) \eta_{\frac{R}{2}}(x) dx + C \left(\frac{R}{2} \right)^{-2} (t - t_0) + \frac{3}{2} \int_{t_0}^t \int_{\mathbb{R}} a u^p dx \\ &\leq \frac{5}{2} \varepsilon + C_0 \left(\frac{R}{2} \right)^{-2} (t - t_0), \end{aligned}$$

and consequently,

$$\begin{aligned} \Theta'_{R_0}(t) &\geq -\frac{C}{R_0^2} \int_{\mathbb{R}} u \eta_{\frac{R_0}{2}} dx + \frac{a(\delta)\delta}{R_0^2} \Theta_{R_0}^p(t) \\ &\geq \frac{C}{R_0^2} (\Theta_{R_0}^p(t) - \frac{5}{2} C_1 \varepsilon - C_0 \left(\frac{R_0}{2} \right)^{-2} (t - t_0)) \end{aligned}$$

by (2.13), if $t \geq t_0$. Fix ε and M_0 such that $\frac{5}{2} C_1 \varepsilon + M_0 \leq \frac{1}{2} (M - \varepsilon)^p$. Then

$$\Theta'_{R_0}(t) \geq \frac{C}{2R_0^2} \Theta_{R_0}^p(t),$$

where $t_0 \leq t \leq t_1$ with $t_1 = t_0 + \frac{M_0}{C_0} \left(\frac{R_0}{2} \right)^2$, and hence

$$\begin{aligned} \Theta_{R_0}(t_1) &\geq \Theta_{R_0}(t_0) + \frac{C}{2R_0^2} (M - \varepsilon)^p \frac{M_0}{C_0} \left(\frac{R_0}{2} \right)^2 \\ &= \Theta_{R_0}(t_0) + \frac{C(M - \varepsilon)^p}{2^3 C_0} \\ &\geq M - \varepsilon + \frac{C(M - \varepsilon)^p}{2^3 C_0}. \end{aligned}$$

Since $\Theta_{R_0}(t_1) = \Theta_{\frac{2R_0}{2}}(t_1) > M - \varepsilon$, we get by the same argument that

$$\Theta_{2R_0}(t_2) \geq \Theta_{2R_0}(t_1) + \frac{C(M - \varepsilon)^p}{2^3 C_0} \geq M - \varepsilon + 2 \frac{C(M - \varepsilon)^p}{2^3 C_0},$$

and deductively,

$$\Theta_{2^{i-1}R_0}(t_i) \geq M - \varepsilon + i \frac{C(M - \varepsilon)^p}{2^3 C_0}, \quad i \in \mathbb{N}^+,$$

with $t_i = t_{i-1} + \frac{Ma}{C_0}(2^{i-2})^2$. Let $i \rightarrow \infty$ to conclude that

$$\sup_{t>0} \int_{\mathbb{R}} u(x, t) dx = +\infty,$$

a contradiction.

The coexistence of global and nonglobal solutions under $p > p_c$ with $N = 1$, contained in item (i), will be quantitatively treated in the next section on the second critical exponent, while item (ii) with $N > 1$ will be treated in the proof of Theorem 3. $\square$

3. SECOND CRITICAL EXPONENT AND ASYMPTOTIC PROFILE

In this section we mainly prove that the second critical exponent is $b_c = \frac{1}{p-1}$ for the case of $N = 1$.

Proof of Theorem 2. (i) Assume $0 < b < b_c$. We at first show that the solutions blow up in finite time under large initial data. By (2.2) and Proposition 2.1 in [20], it suffices to prove

$$\left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R\phi_R dx \right)^{\frac{1}{p-1}} \int_{B_R} R\phi_R R^{b-1} u_0 dx \geq (\lambda_1(B_R))^{\frac{1}{p-1}} R^{\frac{p}{p-1}+b-1}.$$

Indeed, similar to (2.5), we have

$$\liminf_{R \rightarrow \infty} \left(\frac{a(\delta)}{2} \int_{-\delta}^{\delta} R\phi_R dx \right)^{\frac{1}{p-1}} \int_{B_R} R\phi_R R^{b-1} u_0 dx \geq C.$$

On the other hand,

$$\lim_{R \rightarrow \infty} (\lambda_1(B_R))^{\frac{1}{p-1}} R^{\frac{p}{p-1}+b-1} = \lim_{R \rightarrow \infty} (R^2 \lambda_1(B_R))^{\frac{1}{p-1}} R^{b-\frac{1}{p-1}} = 0$$

by (2.6).

(ii) Next, assume $b_c < b < 1$. We should prove global mild solutions under small initial data.

Define the Banach space $X = \{u \mid \sup_{t>0} (1+t)^{\frac{b}{2}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \varepsilon\}$ with $\varepsilon > 0$, and the operator

$$(3.1) \quad \mathcal{M}(u) := e^{-t} u_0 + W(x, t) * u_0 + \int_0^t e^{-(t-s)} a u^p ds + \int_0^t W(x, t-s) * a u^p ds, \quad u \in X,$$

with W defined in (1.4). Clearly, we have for $u \in X$ that

$$(3.2) \quad \begin{aligned} \|\mathcal{M}(u)\|_\infty &\leq e^{-t} \|u_0\|_\infty + \left\| \int_{\mathbb{R}} W(x-y, t) u_0 dy \right\|_\infty \\ &\quad + \int_0^t e^{-(t-s)} \|a u^p(s)\|_\infty ds + \int_0^t \left\| \int_{\mathbb{R}} W(x-y, t-s) a u^p dy \right\|_\infty ds. \end{aligned}$$

By Proposition 1 in [21],

$$(3.3) \quad \left\| \int_{\mathbb{R}} W(x-y, t) u_0 dy \right\|_\infty \leq C(1+t)^{-\frac{b}{2}} (\|u_0\|_\infty + \|u_0\|_{b,\infty}),$$

with $\|u_0\|_{b,\infty} = \limsup_{|x| \rightarrow \infty} |x|^b u_0(x)$. Consequently,

$$(3.4) \quad \int_0^t e^{-(t-s)} \|au(s)\|_\infty^p ds \leq C e^{-\frac{t}{2}} \|u\|_X^p + C(1+t)^{-\frac{bp}{2}} \int_{\frac{t}{2}}^t e^{-(t-s)} ds \|u\|_X^p.$$

Noticing $\|W\|_1 \leq 1$ and $\|W\|_\infty \leq C t^{-\frac{1}{2}}$ [20], we have by Young's inequality with convolution that

$$(3.5) \quad \begin{aligned} \int_0^t \left\| \int_{\mathbb{R}} W(x-y, t-s) au^p dy \right\|_\infty ds &\leq \int_0^t \|u^p\|_\infty \left\| \int_{\mathbb{R}} W(x-y, t-s) a(y) dy \right\|_\infty ds \\ &\leq \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) (1+s)^{-\frac{bp}{2}} (1+t-s)^{-\frac{1}{2}} ds \|u\|_X^p \\ &\leq C((1+t)^{-\frac{1}{2}} + (1+t)^{-\frac{bp+1}{2}}) \|u\|_X^p. \end{aligned}$$

It follows from (3.2)–(3.5) with $b > b_c = \frac{1}{p-1}$ that

$$\|u\|_\infty \leq C(1+t)^{-\frac{b}{2}} (\|u_0\|_\infty + \|u_0\|_{b,\infty} + \|u\|_X^p).$$

Choose λ and ε small enough such that $\mathcal{M}$ maps X into itself. Furthermore, it can be shown that $\mathcal{M}$ is a strict contraction with ε even small. This ensures a unique mild solution to (1.1).

It follows from (3.4) and (3.5) that

$$\limsup_{t \rightarrow \infty} t^{\frac{b}{2}} \left(\int_0^t e^{-(t-s)} \|au^p(s)\|_\infty ds + \int_0^t \left\| \int_{\mathbb{R}} W(x-y, t-s) au^p dy \right\|_\infty ds \right) = 0.$$

Noticing

$$\limsup_{t \rightarrow \infty} t^{\frac{b}{2}} \|e^{-t} u_0 + W(x, t) * u_0 - \int_{\mathbb{R}} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{4t}} |y|^{-b} dy\|_{L^\infty(|x| \leq K\sqrt{t})} = 0$$

by Theorem 2.1 in [20], we conclude with $K > 0$ that

$$\lim_{t \rightarrow \infty} t^{\frac{b}{2}} \|u - \int_{\mathbb{R}} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{4t}} |y|^{-b} dy\|_{L^\infty(|x| \leq K\sqrt{t})} = 0. \quad \square$$

Finally, consider the case of $N > 1$ to prove Theorem 3.

Proof of Theorem 3. Assume $0 < b < N$ with $N > 1$. Similarly to the case of $N = 1$, define the Banach space $Y = \{u \mid \sup_{t>0} (1+t)^{\frac{b}{2}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon\}$ with $\varepsilon > 0$, and the operator (3.1) for $u \in Y$. We have by Young's inequality with convolution and $\|W\|_\infty \leq C t^{-\frac{N}{2}}$ that

$$\begin{aligned} \int_0^t \left\| \int_{\mathbb{R}^N} W(x-y, t-s) au^p dy \right\|_\infty ds &\leq \int_0^t \|u^p\|_\infty \left\| \int_{\mathbb{R}^N} W(x-y, t-s) a(y) dy \right\|_\infty ds \\ &\leq \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) (1+s)^{-\frac{bp}{2}} (1+t-s)^{-\frac{N}{2}} ds \|u\|_Y^p \\ &\leq C((1+t)^{-\frac{N}{2}} + (1+t)^{-\frac{bp}{2}} \log(2+t)) \|u\|_Y^p, \end{aligned}$$

where the exponent $-\frac{bp}{2} < -\frac{b}{2}$ is different from $-\frac{bp}{2} + \frac{1}{2} < -\frac{b}{2}$ for (3.4) in the case $N = 1$. So, the influence of $b > 0$ here for $N > 1$ is equivalent to that of $b > \frac{1}{p-1}$ for $N = 1$. Together with (3.2)–(3.4) related to Banach space Y , we can obtain that $\mathcal{M}$ is contracting in Y with $\lambda, \varepsilon > 0$ small enough.

The conclusion for $b \geq N$ can be obtained by comparison. $\square$

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SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN 116024, PEOPLE'S REPUBLIC OF CHINA

Current address: Department of Science, Nanchang Institute of Technology, Nanchang 330099, People's Republic of China – and – School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, People's Republic of China

SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN 116024, PEOPLE'S REPUBLIC OF CHINA – AND – SCHOOL OF MATHEMATICS AND COMPUTATIONAL SCIENCE, HUNAN CITY UNIVERSITY, YIYANG 413000, PEOPLE'S REPUBLIC OF CHINA

SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN 116024, PEOPLE'S REPUBLIC OF CHINA

E-mail address: `snzheng@dlut.edu.cn`