A NOTE ON APPROXIMATION PROPERTIES OF THE OSELEDETS SPLITTING

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ABSTRACT. We prove that the Oseledets splitting, mean angle and independence number of an ergodic hyperbolic measure of a C^{1+r} diffeomorphism can be approximated by those of atomic measures on hyperbolic periodic orbits. This removes the assumption on simple spectrum in an earlier paper by the authors and strengthens Katok's closing lemma by presenting more information about not only the state space but also its linearization.

1. INTRODUCTION AND RESULTS

In dynamical systems hyperbolicity is always accompanied by abundant periodic orbits, and thus the study on periodic approximation properties is an important topic. Among these are the approximation of Lyapunov exponents and approximation of an angle between Oseledets bundles. In [10], Wang and Sun proved that the Lyapunov exponents of hyperbolic measures can be approximated by those of nearby periodic orbits in the setting of nonuniformly hyperbolic systems, and in [1] Dai considered the approximation of Lyapunov exponents for linear cocycles. In [4], Liang, Liu and Sun studied more properties on periodic points: Oseledets splitting, mean angle, independence number and Liao style number. These approximation properties all together contribute more extensive information on the linearization of diffeomorphisms and thus strengthen Katok's closing lemma. However, the results in [4] are proved only for diffeomorphisms with simple spectrum due to the technique limitation: a given measure in a state manifold has only finitely many covering measures on the Oseledets frame bundle by Sun and Vargas [9].

In the present note, we prove the general approximation properties on Oseledets splitting without the assumption of simple spectrum. To overcome the obstruction, we need a conceptually different approach. We consider the Oseledets splitting, rather than a column of vectors, as an element, construct the Oseledets flag but not the frame bundle, and prove the isolation of the covering measure on the flag

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for a given measure on the state manifold. This uniqueness instead of the finiteness of covering measures in [4] allows the absence of the condition on simple spectrum in [4] for all the periodic approximation properties.

Let M be a compact connected d -dimensional Riemannian manifold without boundary. For $1 \leq i \leq d$, $x \in M$, let $G(i, x)$ denote the Grassmann manifold formed by all i -dimensional linear spaces in $T_x M$. Given integers $n_1, n_2, \dots, n_s$ with $1 \leq n_i \leq d$ and $\sum_{i=1}^s n_i = d$, we define a bundle

$$\mathcal{U}(n_1, \dots, n_s) = \bigcup_{x \in M} \mathcal{U}(n_1, \dots, n_s; x),$$

where the fiber over x is

$$\mathcal{U}(n_1, \dots, n_s; x) = \{(E_1, \dots, E_s) \in G(n_1, x) \times \dots \times G(n_s, x)\}.$$

Set $\mathcal{V}(n_1, \dots, n_s) = \bigcup_{x \in M} \mathcal{V}(n_1, \dots, n_s; x)$, where the fiber over x is

$$\mathcal{V}(n_1, \dots, n_s; x) = \{(E_1, \dots, E_s) \in \mathcal{U}(n_1, \dots, n_s; x) \mid E_1 \oplus \dots \oplus E_s = T_x M\}.$$

It is immediate that by definition, $\mathcal{V}(n_1, \dots, n_s)$ is open and

$$\mathcal{U}(n_1, \dots, n_s) = \overline{\mathcal{V}(n_1, \dots, n_s)}.$$

Given integers $1 \leq l_1 < l_2 < \dots < l_k < d$, we define a flag bundle $\mathcal{W}(l_1, \dots, l_k) = \bigcup_{x \in M} \mathcal{W}(l_1, \dots, l_k; x)$, where the fiber over x is

$$\begin{aligned} \mathcal{W}(l_1, \dots, l_k; x) = \{ (F_1, \dots, F_k; H_1, \dots, H_k) \mid & F_i \in G(l_i, x), H_i \in G(d - l_i, x), \\ & F_i \subset F_{i+1}, H_{i+1} \subset H_i \}. \end{aligned}$$

Clearly, by the definition, $\mathcal{W}(l_1, \dots, l_k)$ is compact.

Throughout this paper, we assume the following relations:

$$k = s - 1, \quad l_i = \sum_{j=1}^i n_j, \quad 1 \leq i \leq s - 1.$$

For the sake of the statement, we write $\mathcal{U} = \mathcal{U}(n_1, \dots, n_s)$, $\mathcal{V} = \mathcal{V}(n_1, \dots, n_s)$ and $\mathcal{W} = \mathcal{W}(l_1, \dots, l_k)$. Given $(E_1, \dots, E_s) \in \mathcal{V}$, we define

$$\chi((E_1, \dots, E_s)) = (E_1, E_1 \oplus E_2, \dots, \bigoplus_{i=1}^{s-1} E_i; \bigoplus_{i=2}^s E_i, \dots, E_{s-1} \oplus E_s, E_s) \in \mathcal{W}.$$

Then $\chi: \mathcal{V} \rightarrow \mathcal{W}$ is a homeomorphism. Denote $\mathcal{W}_1 = \chi(\mathcal{V})$ and $\chi^{-1} = \chi^{-1}|_{\mathcal{W}_1}$.

For a p -frame $\xi = (u_1, u_2, \dots, u_p)$ which spans a p -dimensional space E , we define $\text{vol}(\xi) = \text{vol}(u_1, u_2, \dots, u_p)$ as the volume of the parallelepiped generated by the vectors $u_1, u_2, \dots, u_p$. More precisely, we choose an orthonormal p -frame $\zeta = (w_1, w_2, \dots, w_p)$, $w_i \in T_x M$, $i = 1, \dots, p$, which generates the same linear subspace of $T_x M$ as ξ does and take a unique $p \times p$ matrix A with $\xi = \zeta A$. Then we define the volume of ξ by

$$\text{vol}(\xi) := |\det A|.$$

We remark that the volume $\text{vol}(\xi)$ does not depend on the choice of ζ , since the determinate of a transition matrix between two orthonormal frames is ± 1 . Hence,

$\text{vol}(\xi)$ is well defined. Given any linear surjective transformation Φ from one Euclid space E to another Euclid space F , we define

$$\det(\Phi) = \frac{\text{vol}(\Phi(\xi))}{\text{vol}(\xi)},$$

where ξ is a frame spanning E . The definition does not depend on the choice of ξ .

Let f be a C^1 diffeomorphism of M . For $1 \leq i \leq k$, we define functions ϕ_i, ψ_i ($1 \leq i \leq k$) on $\mathcal{W}(l_1, \dots, l_k)$:

$$\begin{aligned} (1) \quad \phi_i(\gamma) &= \log |\det(Df_x|_{F_i})|, \\ (2) \quad \psi_i(\gamma) &= \log |\det(Df_x^{-1}|_{H_i})| = -\log |\det((Df_{f^{-1}x})|_{(Df_{f^{-1}x})^{-1}H_i})|, \end{aligned}$$

where $\gamma = (F_1, \dots, F_k; H_1, \dots, H_k) \in \mathcal{W}(l_1, \dots, l_k; x)$. Then ϕ_i, ψ_i ($1 \leq i \leq k$) are bounded and continuous on $\mathcal{W}(l_1, \dots, l_k)$. For $1 \leq i \leq k$, we define projections

$$\pi_i(\gamma) = F_i, \quad \widehat{\pi}_i(\gamma) = H_i.$$

In what follows, we suppose f preserves an ergodic measure ω . Then by the Oseledets Theorem [5] there exist

- (a) real numbers $\lambda_1 < \dots < \lambda_s$ ($s \leq d$);
- (b) positive integers $n_1, \dots, n_s$, satisfying $n_1 + \dots + n_s = d$;
- (c) a Borel set $O(\omega)$, called an Oseledets basin of ω , satisfying $f(O(\omega)) = O(\omega)$ and $\omega(O(\omega)) = 1$;
- (d) a measurable splitting, called an Oseledets splitting,

$$T_x M = E_1(x) \oplus \dots \oplus E_s(x)$$

$$\text{with } \dim E_i(x) = n_i \text{ and } Df(E_i(x)) = E_i(fx),$$

such that

$$\lim_{n \rightarrow \pm\infty} \frac{\log \|Df^n v\|}{n} = \lambda_i,$$

for $\forall x \in O(\omega)$, $v \in E_i(x) \setminus \{0\}$, $i = 1, 2, \dots, s$.

(a) and (d) above allow us to arrange the Oseledets splitting according to the increasing order of the Lyapunov exponents. To avoid excessive terminology, we will arrange the Oseledets splitting at every point in the Oseledets basin in this way throughout this paper without explanation. We call the measure ω hyperbolic if none of its Lyapunov exponents is zero.

From now on, suppose $f : M \rightarrow M$ is a C^{1+r} diffeomorphism, i.e., the derivative Df is r -Hölder continuous. We give a quick review concerning some notions and results of Pesin theory. We point the reader to [3], [8], [7] for more details.

Definition 1.1. Given $b_1, b_2 \gg \epsilon > 0$, and for all $k \in \mathbb{Z}^+$, the Pesin block $\Lambda_k = \Lambda_k(b_1, b_2; \epsilon)$ consists of all points $x \in M$ for which there is a splitting $T_x M = E_x^s \oplus E_x^u$ with the invariance property $(Df^m)E_x^s = E_{f^m x}^s$ and $(Df^m)E_x^u = E_{f^m x}^u$, and satisfying:

- (a) $\|Df^n|_{E_{f^m x}^s}\| \leq e^{\epsilon k} e^{-(b_2 - \epsilon)n} e^{\epsilon|m|}, \forall m \in \mathbb{Z}, n \geq 1$;
- (b) $\|Df^{-n}|_{E_{f^m x}^u}\| \leq e^{\epsilon k} e^{-(b_1 - \epsilon)n} e^{\epsilon|m|}, \forall m \in \mathbb{Z}, n \geq 1$; and
- (c) $\tan \angle(E_{f^m x}^s, E_{f^m x}^u) \geq e^{-\epsilon k} e^{-\epsilon|m|}, \forall m \in \mathbb{Z}$.

Definition 1.2. $\Lambda(b_1, b_2; \epsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k(b_1, b_2; \epsilon)$ is a Pesin set.

We say an invariant measure μ is associated with the Pesin set $\Lambda = \Lambda(b_1, b_2; \epsilon)$ if $\mu(\Lambda(b_1, b_2; \epsilon)) = 1$. Take and fix a hyperbolic ergodic measure ω . Let b_1, b_2 be small such that ω has no Lyapunov exponents in $[-b_2, b_1]$; then $\omega(\Lambda(b_1, b_2; \epsilon)) = 1$.

Definition 1.3. Let E and F be two Df -invariant sub-bundles of TM . The angle between $E(x)$ and $F(x)$, $x \in M$, is defined as follows:

$$\sin \angle(E(x), F(x)) := \inf_{0 \neq u \in E(x), 0 \neq v \in F(x)} \sin \angle(u, v) = \inf_{0 \neq u \in E(x), 0 \neq v \in F(x)} \frac{\|u \wedge v\|}{\|u\| \|v\|},$$

where $\wedge$ denotes the wedge product. For almost all $x \in M$, the limit

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \angle(Df^i E(x), Df^i F(x))$$

exists. We call it the mean angle between E and F at x and denote the limit by $m\angle(E(x), F(x))$.

This definition applies to the case of zero-dimensional intersection. We need the following notion to discuss the angle between two invariant bundles with higher dimensions.

Given $v \in \mathbb{R}^d$ and a linear subspace $E \subset \mathbb{R}^d$, define

$$Proj(v, E) = \begin{cases} \text{the orthogonal projection of } v/\|v\| \text{ onto } E, & v \neq 0, \\ 0, & v = 0. \end{cases}$$

We adopt the classic metric d_G on the Grassmann bundle $\bigcup_{0 \leq i \leq d} G(i)$:

$$d_G(F, F') = \sup_{v \in \mathbb{R}^d} \|Proj(v, F) - Proj(v, F')\|,$$

where $F, F' \in \bigcup_{0 \leq i \leq d} G(i)$. When F, F' are disjoint except for the origin, d_G is equivalent to the angle $\angle$. Note that M can be embedded into some Euclid space $\mathbb{R}^q$, so d_G induces a metric on the Grassmann bundle $\bigcup_{x \in M} \bigcup_{0 \leq i \leq d} G(x, i)$ as follows:

$$d_G(F(x), F'(y)) = d(x, y) + d_G(\overline{F}(x), \overline{F'}(y)),$$

where $F(x) \in \bigcup_{0 \leq i \leq d} G(x, i)$, $F'(y) \in \bigcup_{0 \leq i \leq d} G(y, i)$ and $\overline{F}(z) \ni 0$ is the translation of $F(z)$ in $\mathbb{R}^q$.

Definition 1.4. Let E and F be two Df -invariant sub-bundles of $T_x M$. The limit

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_G(Df^i E(x), Df^i F(x))$$

exists for almost every $x \in M$. We call the limit the mean distance between E and F at x and denote it by $md_G(E(x), F(x))$.

Recall that f preserves an ergodic measure ω with its Oseledets splitting

$$(3) \quad T_x M = E_1(x) \oplus \cdots \oplus E_s(x), \quad s \leq d = \dim M, \quad x \in O(\omega).$$

By the Birkhoff Ergodic Theorem, there is an ω -full measure subset in $O(\omega)$ such that every point x in this subset satisfies

$$md_G(E_i(x), E_j(x)) = \int d_G(E_i(y), E_j(y)) d\omega(y).$$

Since different subspaces tangent at the same point x in the Oseledets splitting intersect only at the origin, $d_G(E_i(x), E_j(x))$ coincides with the angle between

$E_i(x)$ and $E_j(x)$. Thus we call $\int d_G(E_i(y), E_j(y))d\omega(y)$ **the mean angle** between $E_i := \bigcup_{x \in O(\omega)} E_i(x)$ and $E_j := \bigcup_{x \in O(\omega)} E_j(x)$ and write it as $md_G^\omega(E_i, E_j)$, $\forall 1 \leq i \neq j \leq s$.

We call the flag of sub-bundles

$$\begin{aligned} & (E_1, E_1 \oplus E_2, \dots, \bigoplus_{i=1}^{s-1} E_i; \bigoplus_{i=2}^s E_i, \dots, E_{s-1} \oplus E_s, E_s) \\ &= \bigcup_{x \in O(\omega)} \chi((E_1(x), \dots, E_s(x))) \end{aligned}$$

the Oseledets flag.

Given $\gamma = (F_1, \dots, F_t)$ with $F_j \in \bigcup_{0 \leq i \leq d} G(i, x)$ ($1 \leq j \leq t$), denote by $A(\gamma)$ the matrix

$$(\cos d_G(F_i, F_j))_{t \times t}.$$

Let $\sigma(\gamma)$ denote the set of all eigenvalues of $A(\gamma)$ and let $\tau(\gamma)$ be the smallest eigenvalue. We call $\tau(\gamma)$ the independence number of γ . Note that $A(\gamma)$ is a real positive-definite symmetric matrix; therefore, $\sigma(\gamma) \subset (0, +\infty)$.

For $x \in O(\omega)$, we define the independence number of x by the independence number of bundles at x whose elements are all on different Oseledets bundles. More precisely, take $\gamma(x) = (E_1(x), E_2(x), \dots, E_s(x))$. Then we define

$$\tau(x) := \tau(\gamma),$$

the smallest eigenvalue of $A(\gamma)$. Clearly, $\tau(x)$ is well defined for $x \in O(\omega)$. Moreover, by the Birkhoff Ergodic Theorem, the equation

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \tau(f^i x) = \int \tau(y) d\omega(y)$$

holds on an ω -full measure subset of $O(\omega)$. Therefore, we can define the independence number of ω by

$$\tilde{\tau}(\omega) := \int \tau(y) d\omega(y).$$

The following proposition is about the continuity of τ . Note that this result is about the continuity of the smallest eigenvalue of matrices, analogous to Proposition 2.6 in [4], and hence we will give the proof in the appendix.

Proposition 1.5. *Given a positive number $s \in \mathbb{Z}$ and a column of integers $n_1, n_2, \dots, n_s$ with $1 \leq n_i \leq d$ and $\sum_{i=1}^s n_i = d$, the function $\tau : \mathcal{V}(n_1, n_2, \dots, n_s) \rightarrow \mathbb{R}$ is continuous.*

Let $\gamma_1 = (E_1, E_2, \dots, E_s)$ and $\gamma_2 = (F_1, F_2, \dots, F_s)$ be two elements in the bundle $\mathcal{V}(n_1, n_2, \dots, n_s)$. We define the distance between γ_1 and γ_2 by

$$d_G(\gamma_1, \gamma_2) := \max_{1 \leq i \leq s} d_G(E_i, F_i).$$

Recall that f preserves an ergodic measure ω with its Oseledets splitting

$$T_x M = E_1(x) \oplus \dots \oplus E_s(x), \quad s \leq d = \dim M, \quad x \in O(\omega),$$

arranged according to the increasing order of Lyapunov exponents.

Suppose there is another ergodic hyperbolic measure ω' with a Df -invariant (both forward and backward invariant) splitting

$$(4) \quad T_y M = E_1(y) \oplus E_2(y) \oplus \dots \oplus E_s(y), \quad \dim(E_j) = n_j, \quad 1 \leq j \leq s, \quad y \in O(\omega').$$

In order to correspond to the Oseledets splitting of ω , throughout this note we only consider the invariant splittings which are arranged according to the increasing order of Lyapunov exponents, i.e., for any nonzero vectors $v \in E_i(y), w \in E_j(y)$, $1 \leq i < j \leq s$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|Df^n v\| < \liminf_{n \rightarrow \infty} \frac{1}{n} \log \|Df^n w\|.$$

In this setting, each $E_j(y)$ ($1 \leq j \leq s$) may not be an Oseledets subspace but is necessarily constituted by some adjacent Oseledets subspaces.

Definition 1.6. Let $\eta > 0$. The Oseledets splitting (3) of ω is η approximated by (4) of ω' , if there exists a measurable subset Γ satisfying:

- (a) $\omega(\Gamma) > 1 - \eta$;
- (b) for any $x \in \Gamma$, there exist a point $z = z(x) \in \text{supp}(\omega') \cap O(\omega')$ and $\varsigma(z) = (E_1(z), \dots, E_s(z)) \in \mathcal{V}(n_1, \dots, n_s; z)$ such that

$$d_G(\gamma, \varsigma) < \eta,$$

$$\text{where } \gamma(x) = (E_1(x), \dots, E_s(x)) \in \mathcal{V}(n_1, \dots, n_s; x).$$

Remark 1.7. Here the loss of small measure $1 - \omega(\Gamma)$ is necessary, since we cannot ensure in general that the approximation property holds on a full-measured set. Take the Lorenz attractor as an example. There are many periodic points arbitrarily close to its singularity in the measure theoretic sense. Let ω' be the atomic measure on the singularity and ω be a periodic measure η -close to ω' . Then we find that only some of the points on the periodic orbit can be approximated by the singularity, the same as their Oseledets splittings.

Remark 1.8. Note that some subspace $E_i(z)$ in $\varsigma(z)$ may be a direct sum of adjacent Oseledets subspaces at z . Hence, the Oseledets splitting of ω is η -approximated by that of ω' , but not vice versa. In other words, η is not a ‘distance’ between the Oseledets splitting of ω and that of ω' . This approximation is transitive but not symmetric. This is also the reason that we fix the dimensions of subspaces in Proposition 1.5.

Remark 1.9. We can do some modification in Definition 1.6 such that η is defined transitively and symmetric, i.e., a distance.

Given $\eta > 0$, define the η -bundle distance of ω, ω' , with respect to $(n_1, \dots, n_s)$, by

$$d_\eta(\omega, \omega') = \inf_{\omega(\Gamma) > 1 - \eta, \omega'(\Gamma') > 1 - \eta} d_H(\Gamma_\omega, \Gamma'_{\omega'}),$$

where

$$\Gamma_\omega = \{\gamma(x) = (E_1(x), \dots, E_s(x)) \in \mathcal{V}(n_1, \dots, n_s; x) \mid x \in \Gamma\},$$

$$\Gamma'_{\omega'} = \{\varsigma(x) = (E_1(x), \dots, E_s(x)) \in \mathcal{V}(n_1, \dots, n_s; x) \mid x \in \Gamma'\},$$

$$d_H(\Gamma_\omega, \Gamma'_{\omega'}) = \max\left\{\sup_{\gamma \in \Gamma_\omega} \inf_{\varsigma \in \Gamma'_{\omega'}} d_G(\gamma, \varsigma), \sup_{\varsigma \in \Gamma'_{\omega'}} \inf_{\gamma \in \Gamma_\omega} d_G(\gamma, \varsigma)\right\}.$$

The Oseledets splitting (3) of ω is η approximated by (4) of ω' if there exists η satisfying:

$$d_\eta(\omega, \omega') < \eta.$$

It may be worth pointing out that while d_η is symmetric for each η and each $(n_1, \dots, n_s)$ in Definition 1.6, we fix a $(n_1, \dots, n_s)$ that depends on ω , but not ω' , and by choosing a different $(n_1, \dots, n_s)$, the distance may increase.

Now we state our main result of this paper.

Theorem 1.10. *Let $f : M \rightarrow M$ be a C^{1+r} diffeomorphism preserving an ergodic hyperbolic measure ω with its Oseledets splitting*

$$T_\Lambda M = E_1(\Lambda) \oplus E_2(\Lambda) \oplus \cdots \oplus E_s(\Lambda), \quad \dim(E_i) = n_i, \quad 1 \leq i \leq s,$$

arranged according to the increasing order of Lyapunov exponents of ω , where $\Lambda = \bigcup_{k \geq 1} \Lambda_k$ is the Pesin set associated with ω . Given $\varepsilon > 0$, there is a hyperbolic periodic orbit $\text{orb}(z, f)$ of period $p = p(\varepsilon)$ together with a Df -invariant splitting

$$T_{f^j z} M = E_1(f^j z) \oplus E_2(f^j z) \oplus \cdots \oplus E_s(f^j z), \quad \dim(E_i) = n_i, \quad 1 \leq i \leq s, \quad 1 \leq j \leq p,$$

at $f^j z$ arranged according to the increasing order of Lyapunov exponents of the orbit $\text{orb}(z)$ such that the atomic measure ω_z supported on $\text{orb}(z, f)$ is ε -close to ω in the weak-topology and satisfies the following properties:*

(i) *The mean angle of ω and of ω_z are ε -close, that is,*

$$|md_G^\omega(E_i(\Lambda), E_j(\Lambda)) - md_G^{\omega_z}(E_i(\text{orb}(z)), E_j(\text{orb}(z)))| < \varepsilon, \quad \forall 1 \leq i \neq j \leq s.$$

(ii) *The independence numbers of ω and of ω_z are ε -close, that is,*

$$|\tilde{\tau}(\omega) - \tilde{\tau}(\omega_z)| < \varepsilon.$$

(iii) *The Oseledets splitting of ω is ε -approximated by that of ω_z .*

This result discusses the same approximation properties as stated in Theorem 1.12 of [4], but without the hypotheses of simple spectrum.

2. PROOF OF THE THEOREM

In the beginning, we will find an isolated ergodic invariant measure on $\mathcal{W}$ covering ω , which is needed in the sequel. Denote by π the projection of $\mathcal{W}$ to the state space M . For any $\gamma = (F_1, \dots, F_k; H_1, \dots, H_k) \in \mathcal{W}(x)$, define

$$Df\gamma = (Df_x F_1, \dots, Df_x F_k; Df_x H_1, \dots, Df_x H_k).$$

Given a homeomorphism $T : X \rightarrow X$ of a compact metric space X , and an ergodic measure m preserved by T , define the basin of m by

$$Q_m(X, T) = \{x \in X \mid \lim_{n \rightarrow \pm\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i x) = \int_X \phi dm, \forall \phi \in C^0(X, \mathbb{R})\}.$$

By the Birkhoff Ergodic Theorem, $m(Q_m(X, T)) = 1$.

Lemma 2.1. *There exists one and only one invariant measure $\tilde{\omega} \in \mathcal{M}_{inv}(\mathcal{W}, Df)$ such that $\pi_*(\tilde{\omega}) = \omega$ and*

$$\int_{\mathcal{W}} \phi_i d\tilde{\omega} = \sum_{j=1}^i n_j \lambda_j, \quad \int_{\mathcal{W}} \psi_i d\tilde{\omega} = - \sum_{j=i+1}^s n_j \lambda_j, \quad 1 \leq i \leq s-1,$$

where ϕ_i, ψ_i are defined in (1). In addition, $\tilde{\omega}$ is ergodic and $\tilde{\omega}(\mathcal{W}_1) = 1$.

Proof. Take and fix a point $x \in O(\omega)$. Let $\gamma_0 = (E_1(x), \dots, E_s(x)) \in \mathcal{V}$ so that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(Df^j \gamma_1) &= \sum_{t=1}^i n_t \lambda_t, \quad 1 \leq i \leq s-1, \\ \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_i(Df^j \gamma_1) &= - \sum_{t=i+1}^s n_t \lambda_t, \quad 1 \leq i \leq s-1, \end{aligned}$$

where $\gamma_1 = \chi(\gamma_0) \in \mathcal{W}$. Define a sequence of measures μ_n on $\mathcal{W}$ by

$$\int \phi d\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} \phi(Df^j \gamma_1), \quad \forall \phi \in C^0(\mathcal{W}, \mathbb{R}).$$

By taking a subsequence when necessary we can assume that $\mu_n \rightarrow \nu_0$. It is standard to verify that ν_0 is a Df -invariant measure and ν_0 covers ω , i.e., $\pi_*(\nu_0) = \omega$. Set

$$Q(\mathcal{W}, Df) := \bigcup_{\nu \in \mathcal{M}_{erg}(\mathcal{W}, Df)} Q_\nu(\mathcal{W}, Df).$$

Then $Q(\mathcal{W}, Df)$ is a Df -invariant total measure subset in $\mathcal{W}$. We have

$$\begin{aligned} &\omega(Q_\omega(M, f) \cap \pi Q(\mathcal{W}, Df)) \\ &\geq \nu_0(\pi^{-1} Q_\omega(M, f) \cap Q(\mathcal{W}, Df)) \\ &= 1. \end{aligned}$$

Then the set

$$\begin{aligned} \mathcal{A} := \{ \mu \in \mathcal{M}_{erg}(\mathcal{W}, Df) \mid \exists \gamma \in Q(\mathcal{W}, Df), \pi(\gamma) \in Q_\omega(M, f), \text{ s. t.} \\ \lim_{n \rightarrow \pm\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(Df^i \gamma) = \int_{\mathcal{W}} \phi d\mu, \quad \forall \phi \in C^0(\mathcal{W}, \mathbb{R}) \} \end{aligned}$$

is nonempty. It is clear that μ covers ω , $\pi_*(\mu) = \omega$, for all $\mu \in \mathcal{A}$; and we claim that $\mathcal{A}$ coincides with the set of all the measures in $\mathcal{M}_{erg}(\mathcal{W}, Df)$ that cover ω . In fact, if $\mu \in \mathcal{M}_{erg}(\mathcal{W}, Df)$ covers ω , $\pi_*\mu = \omega$, from the fact that $\mu(Q_\mu(\mathcal{W}, Df)) = 1$, we have

$$\begin{aligned} &\omega(Q_\omega(M, f) \cap \pi Q_\mu(\mathcal{W}, Df)) \\ &\geq \mu(\pi^{-1} Q_\omega(M, f) \cap Q_\mu(\mathcal{W}, Df)) \\ &= 1. \end{aligned}$$

Thus there is $\gamma' \in Q_\mu(\mathcal{W}, Df)$ with $\pi(\gamma') \in Q_\omega(M, f)$, which means $\mu \in \mathcal{A}$. Therefore,

$$\mathcal{A} = \{ \mu \in \mathcal{M}_{erg}(\mathcal{W}, Df) \mid \pi_*(\mu) = \omega \}.$$

Assume the ergodic decomposition of ν_0 is of the form

$$\nu_0 = \int_{\mathcal{M}_{erg}(\mathcal{W}, Df)} d\tau_{\nu_0}(m).$$

Then $\tau_{\nu_0}(\mathcal{A}) = 1$. Since $\mu_n \rightarrow \nu_0$ and ϕ_i, ψ_i ($1 \leq i \leq s-1$) are continuous,

$$\begin{aligned}\int_{\mathcal{W}} \phi_i d\nu_0 &= \lim_{n \rightarrow +\infty} \int_{\mathcal{W}} \phi_i d\mu_n = \sum_{t=1}^i n_t \lambda_t, \\ \int_{\mathcal{W}} \psi_i d\nu_0 &= \lim_{n \rightarrow +\infty} \int_{\mathcal{W}} \psi_i d\mu_n = - \sum_{t=i+1}^s n_t \lambda_t.\end{aligned}$$

Using the Ergodic Decomposition Theorem, we obtain

$$\begin{aligned}(5) \quad \int_{\mathcal{M}_{erg}(\mathcal{W}, Df)} \int_{\mathcal{W}} \phi_i dm d\tau_{\nu_0}(m) &= \int_{\mathcal{W}} \phi_i d\nu_0 = \sum_{t=1}^i n_t \lambda_t, \quad 1 \leq i \leq s-1, \\ (6) \quad \int_{\mathcal{M}_{erg}(\mathcal{W}, Df)} \int_{\mathcal{W}} \psi_i dm d\tau_{\nu_0}(m) &= \int_{\mathcal{W}} \psi_i d\nu_0 = - \sum_{t=i+1}^s n_t \lambda_t, \quad 1 \leq i \leq s-1.\end{aligned}$$

Observe that for any $m \in \mathcal{A}$, for m -almost every $\gamma(x) \in \mathcal{W}$,

$$\int_{\mathcal{W}} \phi_i dm = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(Df^j \gamma) \geq \sum_{t=1}^i n_t \lambda_t,$$

because $\pi_i(\gamma(x))$ has dimension $n_1 + \cdots + n_i$ and any vector in

$$\pi_i(\gamma(x)) \setminus \bigoplus_{t=1}^i E_t(x)$$

converges under Df^j to an Oseledets subspace with a Lyapunov exponent larger than λ_i . The equality holds if and only if $\pi_i(\gamma(x)) = \bigoplus_{t=1}^i E_t(x)$. Hence, by (5), for τ_{ν_0} -almost every $m \in \mathcal{A}$, for m -almost every $\gamma(x) \in \mathcal{W}$,

$$\int_{\mathcal{W}} \phi_i dm = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(Df^j \gamma) = \sum_{t=1}^i n_t \lambda_t, \quad 1 \leq i \leq s-1.$$

In the same manner, applying the above argument for Df^{-1} , for any $m \in \mathcal{A}$, m -almost every $\gamma(x) \in \mathcal{W}$,

$$\int_{\mathcal{W}} \psi_i dm = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_i(Df^{-j} \gamma) \geq - \sum_{t=i+1}^s n_t \lambda_t,$$

and the equality holds if and only if $\widehat{\pi}_i(\gamma(x)) = \bigoplus_{t=i+1}^s E_t(x)$. It follows that for τ_{ν_0} -almost every $m \in \mathcal{A}$, for m -almost every $\gamma(x) \in \mathcal{W}$,

$$\int_{\mathcal{W}} \psi_i dm = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_i(Df^{-j} \gamma) = - \sum_{t=i+1}^s n_t \lambda_t, \quad 1 \leq i \leq s-1.$$

Additionally, if $\nu'_0 \in \mathcal{M}_{inv}(\mathcal{W}, Df)$ satisfying the fact that $\pi_*(\nu'_0) = \omega$ and

$$\int_{\mathcal{W}} \phi_i d\nu'_0 = \sum_{j=1}^i n_j \lambda_j, \quad \int_{\mathcal{W}} \psi_i d\nu'_0 = - \sum_{j=i+1}^s n_j \lambda_j, \quad 1 \leq i \leq s-1,$$

then by the above proof we can see that for ν'_0 -almost every $\gamma(x) \in \mathcal{W}$,

$$\pi_i(\gamma(x)) = \bigoplus_{t=1}^i E_t(x) \quad \text{and} \quad \widehat{\pi}_i(\gamma(x)) = \bigoplus_{t=i+1}^s E_t(x), \quad \forall 1 \leq i \leq s-1.$$

So $\nu'_0 = \nu_0$. That is, ν_0 is unique and ergodic and $\nu_0(\mathcal{W}_1) = 1$. Let $\tilde{\omega} = \nu_0$. $\square$

Remark 2.2. Lemma 2.1 implies that there is a unique invariant measure on $\mathcal{W}_1 = \mathcal{W}_1(\ell_1, \ell_2, \dots, \ell_s)$ satisfying the requirement. From the proof, we can also tell that the equations hold only on the Oseledets flag (if ℓ_i is equal to the dimension of the direct sum of multiple Oseledets spaces with bottom i Lyapunov exponents for every $1 \leq i \leq s$). That is to say, there is some correspondence between invariant splittings and invariant measures. We have discussed how to find an invariant measure supported on a given bundle to cover the given ergodic measure on the state space M in Theorem 2.7 of [4]. Here, $\tilde{\omega}$ is the invariant measure on $\mathcal{W}_1$ corresponding to the Oseledets flag indeed.

Remark 2.3. In the proof of the above lemma, γ_0 can be a column of all the Oseledets subspaces, or a column of invariant subspaces, some of which are direct sums of adjacent Oseledets subspaces. For this kind of coarser invariant splitting, Lemma 2.1 is still true, since the hyperbolic structure remains. The main idea that any subspace that does not have one of the bottom (rep. top) i (rep. $s-i$) Lyapunov exponents will be attracted to a subspace with an upper (rep. lower) Lyapunov exponent upon forward iteration is still applicable. The only difference is that we should talk about another bundle $\mathcal{W}$ whose elements have a decreasing number of subspaces and an increasing dimension of subspaces.

Before proceeding to the proof of Theorem 1.10, we cite a crucial result on measure and Lyapunov exponents approximation in the following, which is a combination of the main result in [10] and Theorem 1.8 in [4].

Theorem 2.4. *Let $f : M \rightarrow M$ be a C^{1+r} ($r > 0$) diffeomorphism on a compact manifold preserving an ergodic hyperbolic measure ω . Then there exists an f -invariant set $\tilde{\Lambda}$ of ω -full measure such that for any number $\varepsilon > 0$ and any measure μ in $\mathcal{M}_{inv}(\tilde{\Lambda})$, the set of all f -invariant measures supported by $\tilde{\Lambda}$, there is a hyperbolic periodic orbit satisfying the following:*

- (1) *the atomic measure supported on this hyperbolic periodic orbit is ε -close to μ in the weak*-topology,*
- (2) *the Lyapunov exponents of the hyperbolic periodic orbit are ε -approximating those of μ :*

$$|\lambda_j - \tilde{\lambda}_j| < \varepsilon, \quad j = 1, \dots, d,$$

where $\lambda_1 \leq \dots \leq \lambda_d$ are the Lyapunov exponents of μ and $\tilde{\lambda}_1 \leq \dots \leq \tilde{\lambda}_d$ are the Lyapunov exponents of the hyperbolic periodic orbit.

Proof. Let us sketch the proof here. Let $\Lambda = \bigcup_{\ell=1}^{\infty} \Lambda_{\ell}$ be the Pesin set associated with ω . We denote by $\omega|_{\Lambda_{\ell}}$ the conditional measure of ω on Λ_{ℓ} . Set $\tilde{\Lambda}_{\ell} = \text{supp}(\omega|_{\Lambda_{\ell}})$ and $\tilde{\Lambda} = \bigcup_{\ell=1}^{\infty} \tilde{\Lambda}_{\ell}$. Then $\tilde{\Lambda}$ will meet the conditions of our theorem (we can intersect $\tilde{\Lambda}$ with Λ of Theorem 3.1 in [10], if necessary).

We can construct a pseudo-orbit enough to approximate μ with any pre-assigned scale ε in measure sense and find a hyperbolic periodic point shadowing the pseudo-orbit as in the proof of Theorem 1.8 in [4]. Hence condition (1) is satisfied. Note

that the obtained periodic point for the main result in [10] is chosen by Katok's closing lemma (see [2]); we can directly cite the related proof here to verify that it also meets condition (2). $\square$

Proof of Theorem 1.10. Take a decreasing sequence $\{\varepsilon_n\}_{n=1}^\infty$ which approaches zero. We further assume that

$$\varepsilon_1 < \frac{1}{2} \min \{ |\lambda_i - \lambda_j| \mid 1 \leq i \neq j \leq s, \varepsilon \}.$$

For each n , applying Theorem 2.4 to the ergodic measure ω , we can find a hyperbolic periodic point z_n with period p_n which satisfies the following properties:

- (a) The atomic measure $\omega_n = \frac{1}{p_n} \sum_{i=0}^{p_n-1} \delta_{f^i z_n}$ is ε_n -close to the measure ω in the weak*-topology.
- (b) Without loss of generality we assume that ω_n has an invariant splitting $E_1^n \oplus E_2^n \oplus \cdots \oplus E_s^n$ on $T_{\text{orb}(z_n, f)} M$ such that $\dim(E_i^n) = n_i$ ($1 \leq i \leq s$), and for any $0 \neq v \in E_i^n(z_n)$, the Lyapunov exponents

$$(7) \quad \lambda_i - \varepsilon_n < \liminf_{k \rightarrow \infty} \frac{1}{k} \log \|Df_{z_n}^k v\| \leq \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|Df_{z_n}^k v\| < \lambda_i + \varepsilon_n.$$

Remark 2.5. As we discussed in Remark 1.8, some subspace E_i^n may actually be a direct sum of multiple Oseledets subspaces for ω_n . That is, a single Oseledets subspace for ω with multiplicity at least 2 may split into multiple subspaces (with very similar Lyapunov exponents) under the approximation by ω_n , and we group these subspaces together in order to approximate the splitting.

Let $\gamma_n(x) = (E_1^n(x), E_2^n(x), \dots, E_s^n(x))$ for $x \in \text{orb}(z_n)$. Thus, there is an ergodic invariant measure $\tilde{\omega}_n \in \mathcal{M}_{\text{erg}}(\mathcal{W}_1, Df)$ such that $\pi_*(\tilde{\omega}_n) = \omega_n$ and

$$\tilde{\omega}_n(\chi(\gamma_n(f^t z_n))) = \frac{1}{p_n}, \quad 0 \leq t \leq p_n - 1.$$

At most taking a subsequence, we suppose that $\tilde{\omega}_n$ converges to $\mu \in \mathcal{M}_{\text{inv}}(\mathcal{W}, Df)$. Since $\pi_*(\tilde{\omega}_n) = \omega_n$ and $\lim_{n \rightarrow +\infty} \omega_n = \omega$, we have $\pi_*(\mu) = \omega$. By (7), it is verified that

$$\begin{aligned} \sum_{j=1}^i n_j (\lambda_j - \varepsilon_n) &< \int_{\mathcal{W}} \phi_i d\tilde{\omega}_n < \sum_{j=1}^i n_j (\lambda_j + \varepsilon_n), \quad 1 \leq i \leq s-1, \\ \sum_{j=i+1}^s n_j (-\lambda_j - \varepsilon_n) &< \int_{\mathcal{W}} \psi_i d\tilde{\omega}_n < \sum_{j=i+1}^s n_j (-\lambda_j + \varepsilon_n), \quad 1 \leq i \leq s-1. \end{aligned}$$

Letting $n \rightarrow \infty$, we deduce that

$$\begin{aligned} \int_{\mathcal{W}} \phi_i d\mu &= \sum_{j=1}^i n_j \lambda_j, \quad 1 \leq i \leq s-1, \\ \int_{\mathcal{W}} \psi_i d\mu &= - \sum_{j=i+1}^s n_j \lambda_j, \quad 1 \leq i \leq s-1. \end{aligned}$$

By Lemma 2.1, it follows that $\mu = \tilde{\omega}$.

Denote $\rho_0 = \chi_*^{-1}(\tilde{\omega})$, $\rho_n = \chi_*^{-1}(\tilde{\omega}_n)$. Then $\rho_0, \rho_n \in \mathcal{M}_{\text{erg}}(\mathcal{V}, Df)$ and $\rho_n \rightarrow \rho_0$ as $n \rightarrow +\infty$.

Proof of (i). We need show that for any $1 \leq i \neq j \leq s$,

$$(8) \quad \lim_{n \rightarrow \infty} \frac{1}{p_n} \sum_{k=0}^{p_n-1} d_G(E_i^n(f^k(z_n)), E_j^n(f^k(z_n))) = md_G^\omega(E_i, E_j),$$

where $E_1^n(\text{orb}(z)) \oplus \cdots \oplus E_s^n(\text{orb}(z))$ is the invariant splitting given as in (b) above. We define $\phi_{ij} : \mathcal{U} \rightarrow \mathbb{R}$, $(E_1, \cdots, E_s) \rightarrow d_G(E_i, E_j)$. Then ϕ_{ij} is bounded and continuous. Noting that $\rho_n \rightarrow \rho_0$, we get (8).

Proof of (ii). We recall the following relations:

$$\omega_n \rightarrow \omega, \quad \rho_n \rightarrow \rho_0, \quad (\pi \circ \chi)_*(\rho_n) = \omega_n, \quad (\pi \circ \chi)_*(\rho_0) = \omega.$$

Since the function $\tau : \mathcal{V} \rightarrow \mathbb{R}$ is continuous by Proposition 1.5, we have

$$\int \tau d\rho_n \rightarrow \int \tau d\rho_0.$$

Moreover, note that the ergodic measure $\tilde{\omega} \in \mathcal{M}_{erg}(\mathcal{W}, Df)$ is unique in the sense of Lemma 2.1. By the discussion in Remark 2.2 and the definition of ρ_0 , it holds that ρ_0 is supported on the bundle consisting of all Oseledets subspaces in order. Hence, we have that

$$\int \tau d\rho_0 = \int \tau d\omega.$$

Also note that ω_n is an ergodic measure whose spectra are increasingly arranged as $E_1^n \oplus \cdots \oplus E_s^n$. Though the splitting of ω_n may be coarser than the Oseledets splitting as discussed in Remark 2.5, it holds that

$$\int \tau d\rho_n = \int \tau d\omega_n,$$

as talked about in Remark 2.3. Hence we have that $\int \tau d\omega_n \rightarrow \int \tau d\omega$ or $\tilde{\tau}(\omega_n) \rightarrow \tilde{\tau}(\omega)$. We thus obtain (ii).

Proof of (iii). Given $\varepsilon > 0$, take l large so that

$$(4.12) \quad \omega(\Lambda_l(\omega)) > 1 - \varepsilon,$$

where $\Lambda_l(\omega)$ denotes the l -th Pesin block associated with ω . Since the splitting

$$x \rightarrow E_1(x) \oplus \cdots \oplus E_s(x)$$

depends continuously on $x \in \Lambda_l(\omega)$, we can choose a uniform constant $\eta > 0$ satisfying

$$(4.13) \quad \eta < \min_{\substack{i \neq j \\ x \in \Lambda_l(\omega)}} \left\{ \frac{1}{10} d_G(E_i(x), E_j(x)), \varepsilon \right\}.$$

For each $x \in \Lambda_l(\omega) \cap \text{supp}(\omega)$, we take and fix $\gamma_0(x) = (E_1(x), \cdots, E_s(x))$. Denote by $B(\gamma_0(x), \eta)$ the η -neighborhood of $\gamma_0(x)$ under the metric on the Grassman bundle. Then $\rho_0(B(\gamma_0(x), \eta)) > 0$. Recalling that $\rho_n \rightarrow \rho_0$, we have

$$\liminf_{n \rightarrow +\infty} \rho_n(B(\gamma_0(x), \eta)) \geq \rho_0(B(\gamma_0(x), \eta)) > 0.$$

Therefore, we can take an integer $N(x) = N(\gamma_0(x)) > 0$ such that

$$\rho_n(B(\gamma_0(x), \eta)) > 0, \quad \forall n \geq N(x).$$

This implies the existence of an element $\beta_n(x)$ in $Q_{\rho_n}(\mathcal{U}, Df) \cap \text{supp}(\rho_n) \cap B(\gamma_0(x), \eta)$ for each $n \geq N(x)$. Observing that ρ_n covers the atomic measure ω_n , we can deduce that $\beta_n(x)$ must be an element based on a periodic point $z(x, n) \in \text{orb}(z_n)$, where z_n is the periodic point chosen in (a) and (b). By the uniqueness of ρ_n , we know that $\beta_n(x) = (E_1^n(z(x, n)), \dots, E_s^n(z(x, n)))$. Thus the Oseledets splitting of ω at x is η approximated by the invariant splitting $E_1^n(z(x, n)) \oplus \dots \oplus E_s^n(z(x, n))$ of ω_n at $z(x, n)$, $n \geq N(x)$. Note that the number $N(x)$ may vary with x ; we need to find a number N_1 , independent of the choice of $x \in \Lambda_l(\omega)$, such that ρ_{N_1} meets (iii). This can be done by the compactness of $\Lambda_l(\omega)$ and continuity of the Oseledets splitting on it.

Hence we have completed the proof of Theorem 1.10. $\square$

3. APPENDIX

Before proving Proposition 1.5, we need some lemmas. Denote by $\mathcal{L}(\mathbb{R}^s)$ the space of all linear maps on $\mathbb{R}^s$ and by $\sigma(L)$ the set of all eigenvalues of a linear map $L \in \mathcal{L}(\mathbb{R}^s)$. For any $\epsilon > 0$, set

$$\sigma_\epsilon(L) := \{\tau \in \mathbb{C} \mid \exists \tau' \in \sigma(L), \text{ s.t. } |\tau - \tau'| < \epsilon\}.$$

Lemma 3.1 and Lemma 3.2 are cited from [6].

Lemma 3.1. *For a given $L \in \mathcal{L}(\mathbb{R}^s)$ and $\epsilon > 0$ there is $\delta > 0$ such that $\sigma(T) \subset \sigma_\epsilon(L)$ whenever $\|T - L\| < \delta$.*

Lemma 3.2. *Let $L \in \mathcal{L}(\mathbb{R}^s)$ and $\tau \in \sigma(L)$. There exist $\epsilon > 0$ and $\delta > 0$ with the property that for any $T \in \mathcal{L}(\mathbb{R}^s)$ with $\|T - L\| < \delta$ the cardinality of $\sigma(T) \cap B_\epsilon(\tau)$ is not more than the multiplicity of τ , where $B_\epsilon(\tau) = \{\tau' \in \mathbb{C} \mid |\tau - \tau'| < \epsilon\}$.*

Proof of Proposition 1.5. We assume by contrary that τ is not continuous at a frame $\alpha_0 \in \mathcal{V}(n_1, n_2, \dots, n_s)$. Then there must exist a constant $\epsilon_0 > 0$ such that for any $\delta > 0$ we can find another frame $\beta = \beta(\delta) \in \mathcal{V}(n_1, n_2, \dots, n_s)$ satisfying $d_G(\alpha_0, \beta) < \delta$ but $|\tau(\alpha_0) - \tau(\beta)| > 2\epsilon_0$. Arrange the eigenvalues of $A(\alpha_0)$ in decreasing order as $\{\tau_1 > \tau_2 > \dots > \tau_t > \tau(\alpha_0)\}$, $t \leq s - 1$. By Lemma 3.2, there are real numbers $\epsilon_i > 0$, $\delta_1 > 0$ and $\delta' > 0$ such that for any $\gamma \in \mathcal{V}(n_1, n_2, \dots, n_s)$ with $d(\gamma, \alpha_0) < \delta'$, the following hold:

- (a) the cardinality $\#[\sigma(\gamma) \cap B_{\epsilon_i}(\tau_i)]$ is not more than the multiplicity of τ_i , and
- (b) the cardinality $\#[\sigma(\gamma) \cap B_{\delta_1}(\tau(\alpha_0))]$ is not more than the multiplicity of $\tau(\alpha_0)$, and
- (c) $B_{\epsilon_i}(\tau_i) \cap B_{\delta_1}(\tau(\alpha_0)) = \emptyset$, $i = 1, \dots, t$.

Take

$$0 < \epsilon \leq \min\{\epsilon_i, \delta_1; i = 0, 1, \dots, t\}.$$

Choose $\delta < \min\{\epsilon, \delta'\}$ by Lemma 3.1 such that each $\gamma \in \mathcal{V}(n_1, n_2, \dots, n_s)$ with $d(\gamma, \alpha_0) < \delta$ satisfies $\tau(\gamma) \in \bigcup_{i=1}^t B_\epsilon(\tau_i) \cup B_\epsilon(\tau(\alpha_0))$. From the choice of ϵ_0 and $\beta = \beta(\delta)$ we have $\tau(\beta) \in \bigcup_{i=1}^t B_{\epsilon_i}(\tau_i)$. Recall that $\tau(\beta)$ is the smallest eigenvalue of $A(\beta)$, and hence there is no eigenvalue of $A(\beta)$ inside $B_\epsilon(\tau(\alpha_0))$. So we deduce

by Lemma 3.1 and Lemma 3.2 that

$$\begin{aligned} s &= \#\sigma(\beta) = \#[\sigma(\beta) \cap B_\epsilon(\tau(\alpha_0))] + \#[\sigma(\beta) \cap \bigcup_{i=1}^t B_\epsilon(\tau_i)] \\ &= \#[\sigma(\beta) \cap \bigcup_{i=1}^t B_\epsilon(\tau_i)] \\ &< \#\sigma(\alpha_0) = s, \end{aligned}$$

a contradiction. This implies that the function τ is continuous. $\square$

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