ON SUPPORT POINTS OF THE CLASS $S^0(B^n)$

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Abstract. We consider support points of the class $S^0(B^n)$ introduced by G. Kohr and prove that, given a normalized Loewner chain $f(z,t)$ such that $f(\cdot,0)$ is a support point of $S^0(B^n)$, all elements of the chain are support points of $S^0(B^n)$. Also, we prove a similar result for Loewner chains that come from the Roper–Suffridge extension operator.

1. Introduction and results

Let $D \subset \mathbb{C}^n$ be a domain. We denote by $\mathcal{H}(D,\mathbb{C}^n)$ the set of all holomorphic functions which map from $D$ to $\mathbb{C}^n$. Furthermore we denote by $\|z\|$ the Euclidean norm of $z \in \mathbb{C}^n$. Let $B^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ be the unit ball and let $S(B^n)$ be the set of all normalized biholomorphic mappings on $B^n$, i.e.,

$$S(B^n) := \{f \in \mathcal{H}(B^n,\mathbb{C}^n) \mid f \text{ biholomorphic with } f(0) = 0, Df(0) = I_n\}.$$ 

Compared to the case $n \geq 2$, the class $S := S(B^1)$ is well understood, and a powerful tool to describe this class is the so-called Loewner theory: every element $f$ of $S$ can be embedded in a Loewner chain, which can be described by a differential equation. This equation, in turn, can be used to study properties of $f$. In higher dimensions, the situation is more complicated and G. Kohr introduced the subclass $S^0(B^n) \subseteq S(B^n)$ in [Koh01] by the method of Loewner chains in order to get a class having similar properties as $S$.

We recall the definition and some properties of $S^0(B^n)$ (see [GK03] or [GHKK12]). For $f, g \in \mathcal{H}(B^n,\mathbb{C}^n)$, we write $f \prec g$ if there exists a Schwarz mapping $v \in \mathcal{H}(B^n,\mathbb{C}^n)$ (i.e., $\|v(z)\| \leq \|z\|$) for all $z \in B^n$ such that $f = g \circ v$.

A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a normalized Loewner chain if $e^{-t}f(\cdot,t) \in S(B^n)$ for all $t \geq 0$ and $f(\cdot, s) \prec f(\cdot, t)$ for all $s, t$ with $0 \leq s \leq t$.

A mapping $f \in S(B^n)$ is said to have parametric representation if there exists a normalized Loewner chain $f(z,t)$ with $f = f(\cdot,0)$ such that $(e^{-t}f(\cdot,t))_{t \geq 0}$ is a normal family on $B^n$. $S^0(B^n)$ is the set of all biholomorphic mappings on $B^n$ which have parametric representation.

When $n = 1$, the normalization in those Loewner chains is not really restrictive and we have $S^0(B^1) = S$. For $n \geq 2$, however, we always have $S^0(B^n) \subseteq S(B^n)$. On the other hand, $S^0(B^n)$ has some nice properties similar to $S$; e.g., it is compact in contrast to $S(B^n)$ and the elements satisfy a growth condition similar to the growth theorem for $S$ (see Corollary 8.3.9 in [GK03]). Besides, the class $S^0(B^n) \subseteq S(B^n)$ of normalized starlike mappings is contained in $S^0(B^n)$ because $f \in S^0(B^n)$ if and

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only if $e^t f(z)$ is a normalized Loewner chain. We denote by $\text{supp } S^0(B^n)$ the set of all support points of $S^0(B^n)$, i.e., those functions that maximize over $S^0(B^n)$ the real part of a continuous linear functional $L : \mathcal{H}(B^n, \mathbb{C}^n) \to \mathbb{C}$, which is nonconstant on $S^0(B^n)$. We will prove the following theorem, which is already known to be true for the case $n = 1$, i.e., for the class $S$.

**Theorem 1.1.** Let $f \in \text{supp } S^0(B^n)$ and let $f(z, t)$ be a normalized Loewner chain with $f(\cdot, 0) = f$ such that $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is a normal family on $B^n$. Then $e^{-t} f(\cdot, t) \in \text{supp } S^0(B^n)$ for all $t \geq 0$.

In order to get (nontrivial) examples for biholomorphic mappings for $n \geq 2$, one can use extension operators, which are mappings of the form $\Theta : S(B^k) \to S(B^{k+m})$, $k, m \in \mathbb{N}$. Such an operator is said to preserve normalized Loewner chains if for any normalized Loewner chain $f(z, t)$ on $B^k$ the mapping $e^t \Theta(e^{-t} f(z, t))$ is a normalized Loewner chain on $B^{k+m}$. One example of such an operator is the classical Roper–Suffridge extension operator $\Psi : S \to S(B^n)$ with $n \geq 2$ and

\[
\Psi_n(f)(z_1, \ldots, z_n) = (f(z_1), z_2 \sqrt{f'(z_1)}, \ldots, z_n \sqrt{f'(z_1)})
\]

(see Section 11.3 in [GK03]). Here, the branch of the square root is chosen such that $\sqrt{1} = 1$. We have $\Psi_n(S) \subseteq S^0(B^n)$. $\Psi_n$ has other nice properties; e.g., it maps starlike/convex/Bloch mappings again onto starlike/convex/Bloch mappings (see Section 11.1 in [GK03]).

We will prove Conjecture 3.1 in [GKP07], which says that all support points of $\Psi_n(S)$ are unbounded mappings. This implies the following result.

**Theorem 1.2.** Let $f \in S$ and $F = \Psi_n(f)$. Also let $f(z_1, t)$ be a normalized Loewner chain with $f(\cdot, 0) = f$ and let $F(z, t) = e^t \Psi_n(e^{-t} f(z, t))$. If $F \in \text{supp } \Psi_n(S)$, then $e^{-t} F(\cdot, t) \in \text{supp } \Psi_n(S)$ for all $t \geq 0$.

2. Proof of Theorem 1.1

We will prove Theorem 1.1 by generalizing ideas from a proof for the case $n = 1$, which is described in [HM84]. First, we need to take a look at Runge domains. Let $D \subseteq \mathbb{C}^n$ be a domain of holomorphicity. We let $\mathcal{P}(D, \mathbb{C}^n) \subset \mathcal{H}(D, \mathbb{C}^n)$ be the set of all polynomials with the topology induced by locally uniform convergence in $D$.

For $n = 1$ (a version of) the polynomial Runge theorem says that $\mathcal{P}(D, \mathbb{C})$ is dense in $\mathcal{H}(D, \mathbb{C})$ whenever $D$ is simply connected. In higher dimensions this is no longer true and one calls $D$ a Runge domain if $\mathcal{P}(D, \mathbb{C}^n)$ is dense in $\mathcal{H}(D, \mathbb{C}^n)$. Furthermore, if $E \subseteq \mathbb{C}^n$ is an domain with $D \subseteq E$, then the pair $(D, E)$ is called a Runge pair if $\mathcal{H}(E, \mathbb{C}^n)$ is dense in $\mathcal{H}(D, \mathbb{C}^n)$. So $D$ is a Runge domain if and only if $(D, \mathbb{C}^n)$ is a Runge pair. The unit ball $B^n$ is a simple example of a Runge domain. However, the Runge property is not invariant with respect to biholomorphic mappings, as an example of J. Wermer shows; see [Wer59].

First, we rephrase the property that $B^n$ is mapped biholomorphically onto a Runge domain by $f$ in terms of $f$.

**Lemma 2.1.** Let $f \in S(B^n)$. The following statements are equivalent:

a) $f(B^n)$ is a Runge domain.

b) $f^{-1} \in \mathcal{P}(f(B^n), \mathbb{C}^n)$.

c) For every $g \in \mathcal{H}(B^n, \mathbb{C}^n)$ there exists a sequence $p_k$ of polynomials with $g(z) = \lim_{k \to \infty} p_k(f(z))$ locally uniformly in $B^n$. 


Proof. “a) ⇒ b)”: If \( f(B^n) \) is Runge, then \( f^{-1} ∈ \mathcal{H}(f(B^n), \mathbb{C}^n) = \mathcal{P}(f(B^n), \mathbb{C}^n) \).

“b) ⇒ c)”: Let \( h_k \) be a sequence of polynomials which approximate \( f^{-1} \) locally uniformly on \( f(B^n) \) and let \( g ∈ \mathcal{H}(B^n, \mathbb{C}^n) \). As \( B^n \) is Runge, there exists a sequence \( q_k \) of polynomials which approximate \( g \) locally uniformly on \( B^n \). Then \( q_k ∘ h_k ∘ f \) approximates locally uniformly the map \( g ∘ f^{-1} ∘ f = g \).

“c) ⇒ a)” Note that the linear map \( F : \mathcal{H}(f(B^n), \mathbb{C}^n) → \mathcal{H}(B^n, \mathbb{C}^n), F(g) = g ∘ f, \) is an isomorphism, and hence locally uniform convergence of a sequence \( h_k \) in \( f(B^n) \) is equivalent to locally uniform convergence of the sequence \( h_k ∘ f \) in \( B^n \).

Let \( f ∈ S^0(B^n) \) and \( f(z, t) \) be a normalized Loewner chain with \( f = f(., 0) \) such that \( \{e^{-t}f(., t)\}_{t≥0} \) is a normal family on \( B^n \). For every \( s, t \) with \( 0 ≤ s ≤ t \) we have \( f(., s) ∼ f(., t) \) and we can write \( f(., s) = f(v(., s, t), t) \) with a unique Schwarz mapping \( v(., s, t) \), which is also called transition mapping. For \( s = 0 \) we will just write \( v_t := v(., 0, t) \). For a fixed \( s, \) the family \( \{v(., s, t)\}_{t≥s} \) satisfies the so-called Loewner ordinary differential equation (see [GIKK12]).

In [ABFW], a remarkable fact concerning solutions to this equation is shown.

**Theorem 2.2** (Proposition 5.1 in [ABFW]). \( v(B^n, s, t) \) is a Runge domain for all \( s, t \) with \( 0 ≤ s ≤ t \).

We note a simple consequence of the proof of Theorem 2.2.

**Theorem 2.3.** Every \( f ∈ S^0(B^n) \) maps \( B^n \) onto a Runge domain.

**Proof.** Let \( f ∈ S^0(B^n) \) and \( f(z, t) \) be a normalized Loewner chain with \( f = f(., 0) \) such that \( \{e^{-t}f(., t)\}_{t≥0} \) is a normal family on \( B^n \). For every \( s, t \) with \( 0 ≤ s ≤ t \) we have \( f(., s) ∼ f(., t) \), and Theorem 4.2 in [ABFW] immediately implies that \( (f(B^n, s), f(B^n, t)) \) is a Runge pair. Let \( R \) be the Loewner range of the Loewner chain \( f(z, t) \), i.e., \( R = \bigcup_{t≥0} f(B^n, t) \). One can easily show that \( (f(B^n, t), R) \) is also a Runge pair for every \( t ≥ 0 \). Now, the Koebe distortion theorem for the class \( S^0(B^n) \), which is Corollary 8.3.9 in [GK03], implies

\[
R = \bigcup_{t≥0} f(B^n, t) = \bigcup_{t≥0} \{z ∈ \mathbb{C}^n | ∥z∥ < \frac{e^t}{4}\} = \mathbb{C}^n,
\]

and consequently \( R = \mathbb{C}^n \). Hence \( (f(B^n, t), \mathbb{C}^n) \) is a Runge pair for every \( t ≥ 0 \). In particular, \( f(B^n) = f(B^n, 0) \) is a Runge domain. \( \square \)

**Corollary 2.4.** For \( n ≥ 2 \), every \( f ∈ S^0(B^n) \) can be approximated locally uniformly by automorphisms of \( \mathbb{C}^n \).

**Proof.** We just have to apply Theorem 2.1 in [AL92], which states that every biholomorphic map from a starlike domain to a Runge domain can be approximated by automorphisms of \( \mathbb{C}^n \) when \( n ≥ 2 \).

It would be nice to find out more about those \( f ∈ S^0(B^n) \) that extend to automorphisms of \( \mathbb{C}^n \). When \( n = 1 \), the only normalized univalent function that is also univalent in \( \mathbb{C} \) is the identity mapping. Corollary 2.4 suggests that the case \( n ≥ 2 \) should be more interesting.

**Remark 2.5.** Theorem 2.3 can also be shown without using the Loewner range, namely by proving Corollary 2.4 first: Every \( f ∈ S^0(B^n) \) can be calculated by \( v_t \) via

\[
f(z) = \lim_{t→∞} e^t v_t(z)
\]
locally uniformly on $B^n$; see Theorem 8.1.9 in [GK03]. By combining Theorem 2.22 with Theorem 2.1 in [AL92], we see that $f$ can be approximated locally uniformly by automorphisms of $\mathbb{C}^n$. This fact already implies that $f(B^n)$ is a Runge domain; see the remark at the end of page 372 in [AL92].

Next, we note the important fact that, given a transition mapping $v_t$ and a $G \in S^n(B^n)$, then $e^t G(v_t)$ is also in $S^n(B^n)$, which is mentioned in the proof of Theorem 2.1 in [GHKK12].

**Lemma 2.6.** Let $G \in S^n(B^n)$ and $t \geq 0$. Furthermore, let $f \in S^n(B^n)$ and $f(z,s)$ be a normalized Loewner chain with $f = f(\cdot,0)$ such that $\{e^{-s} f(\cdot,s)\}$ is a normal family. Write $f = f(v_t,t)$. Then $e^t G(v_t) \in S^n(B^n)$.

**Proof.** Let $G(z,s)$ be a normalized Loewner chain with $G(\cdot,0) = G$ such that $\{e^{-s} G(\cdot,s)\}_{s \geq 0}$ is a normal family and let $F(z,s) : B^n \times [0,\infty) \to \mathbb{C}^n$ be the mapping

$$F(z,s) = \begin{cases} e^t G(v(z,s,t)), & 0 \leq s \leq t, \\ e^t G(z,s-t), & s > t. \end{cases}$$

Then $F(z,s)$ is a normalized Loewner chain, $F(\cdot,0) = e^t G(v_t)$ and $\{e^{-s} F(\cdot,s)\}_{s \geq 0}$ is a normal family. Thus $e^t G(v_t) \in S^n(B^n)$. \hfill $\Box$

Choosing $G(z) = z$ in Lemma 2.6 shows that $e^t v_t \in S^n(B^n)$. Support points of the class $S$ map $B^1$ onto $\mathbb{C}$ minus a slit. In particular, they are unbounded mappings. It is not known whether support points of $S^n(B^n)$ or $S(B^n)$ can be bounded when $n \geq 2$. Now we will prove Conjecture 2.6 in [GHKK12], which says that $e^t v_t$ is not a support point of $S^n(B^n)$ for any $t \geq 0$.

**Lemma 2.7.** Let $v_t$ be defined as in Lemma 2.6 and let $h = e^t v_t \in S^n(B^n)$. Furthermore, let $P \in \mathcal{P}(B^n,\mathbb{C}^n)$ with $P(0) = 0$, $DP = 0$. Then there exists $\delta > 0$ such that $h + \varepsilon e^t P(e^{-t} h) \in S^n(B^n)$ for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$.

**Proof.** Let $g_\varepsilon(z) = z + \varepsilon P(z)$. Obviously we have $g_\varepsilon(0) = 0$, $Dg_\varepsilon(0) = I_n$. Now $\det(Dg_\varepsilon(z)) \to 1$ for $\varepsilon \to 0$ uniformly on $\overline{B^n}$, so $g_\varepsilon$ is locally biholomorphic for $\varepsilon$ small enough. In this case, for every $z \in \overline{B^n}$, we have:

$$[Dg_\varepsilon(z)]^{-1} = [I_n + \varepsilon DP(z)]^{-1} = I_n - \varepsilon DP(z) + \varepsilon^2 DP(z)^2 + \ldots = I_n - \varepsilon \underbrace{(DP(z) + \ldots)}_{=:U(z)\in \mathbb{C}^n\times n}.$$

Write $[Dg_\varepsilon(z)]^{-1} g_\varepsilon(z) = z + \varepsilon P(z) - \varepsilon U(z) z - \varepsilon^2 U(z) P(z) = (I_n + \varepsilon M(z)) z$, with a matrix-valued function $M(z)$. Consequently

$$\langle [Dg_\varepsilon(z)]^{-1} g_\varepsilon(z), z \rangle = \langle (I_n + \varepsilon M(z)) z, z \rangle,$$

and there is an $\delta > 0$ such that $I_n + \varepsilon M(z)$ has only eigenvalues with positive real part for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$. In this case we have

$$\text{Re} \langle [Dg_\varepsilon(z)]^{-1} g_\varepsilon(z), z \rangle > 0 \quad \forall z \in B^n \setminus \{0\},$$

and thus $g_\varepsilon \in S^n(B^n) \subset S^n(B^n)$ by Theorem 6.2.2 in [GK03]. From Lemma 2.6 it follows that $e^t g_\varepsilon(v_t) = e^t g_\varepsilon(e^{-t} h) = h + \varepsilon e^t P(e^{-t} h) \in S^n(B^n)$. \hfill $\Box$

**Proposition 2.8.** Let $v_t$ be defined as in Lemma 2.6 and let $h = e^t v_t \in S^n(B^n)$. Then $h$ is not a support point of $S^n(B^n)$.
Proof. Let \( h = e^t v_t \) be a support point of \( S^0(B^n) \); i.e., there is a continuous linear functional \( L : \mathcal{H}(B^n, \mathbb{C}^n) \to \mathbb{C} \) such that \( \text{Re} L \) is nonconstant on \( S^0(B^n) \) and

\[
\text{Re} L(h) = \max_{g \in S^0(B^n)} \text{Re} L(g).
\]

Let \( P \) be a polynomial with \( P(0) = 0 \) and \( DP(0) = 0 \). Then \( h + \varepsilon e^t P(e^{-t} h) \in S^0(B^n) \) for all \( \varepsilon \in \mathbb{C} \) small enough by Lemma 2.7. We conclude \( \text{Re} L(P(e^{-t} h)) = \text{Re} L(P(v_t)) = 0 \); otherwise we could choose \( \varepsilon \) such that \( \text{Re} L(h + \varepsilon e^t P(e^{-t} h)) > \text{Re} L(h) \). Now \( v_t(B^n) \) is a Runge domain by Theorem 2.2. Hence we can write any analytic function \( g \) in the unit ball \( B^n \) with \( g(0) = 0 \) and \( Dg(0) = 0 \) as \( g = \lim_{k \to \infty} P_k(v_t) \), where every \( P_k \) is a polynomial with \( P_k(0) = 0 \) and \( DP_k(0) = 0 \), according to Lemma 2.1(c). The continuity of \( L \) implies \( \text{Re} L(g) = 0 \). Hence \( \text{Re} L \) is constant on \( S(B^n) \), a contradiction.

Proof of Theorem 1.1 Let \( L \) be a continuous linear functional on \( \mathcal{H}(B^n, \mathbb{C}^n) \) such that \( \text{Re} L \) is nonconstant on \( S^0(B^n) \) with

\[
\text{Re} L(f) = \max_{g \in S^0(B^n)} \text{Re} L(g).
\]

Fix \( t \geq 0 \); then \( f(z) = f(v_t(z), t) \) for all \( z \in B^n \). Define the continuous linear functional

\[
J(g) := L(e^t \cdot g \circ v_t) \quad \text{for} \quad g \in \mathcal{H}(B^n, \mathbb{C}^n).
\]

Now we have

\[
J(e^{-t} f(\cdot, t)) = L(f) \quad \text{and} \quad \text{Re} J(g) \leq \text{Re} J(e^{-t} f(\cdot, t)) \quad \text{for all} \quad g \in \mathcal{H}(B^n, \mathbb{C}^n).
\]

Furthermore, \( \text{Re} J \) is not constant on \( S^0(B^n) \); as \( e^t v_t \) is not a support point of \( S^0(B^n) \), we have \( \text{Re} J(id) = \text{Re} L(e^t v_t) < \text{Re} L(f) = \text{Re} J(e^{-t} f(\cdot, t)) \). \( \Box \)

3. Proof of Theorem 1.2

In order to simplify notation, we look at only the Roper-Suffridge-Operator

\[
\Psi := \Psi_2 :
\]

\[
\Psi(f)(z_1, z_2) = \left( f(z_1), z_2 \sqrt{f'(z_1)} \right).
\]

\( \Psi \) maps \( S \) into \( S^0(B^2) \). Of course, \( \Psi(\text{supp} S) \subseteq \text{supp} \Psi(S) \), because the first coordinate of \( \Psi(f)(z_1, z_2) \) is just \( f(z_1) \).

Now we prove Conjecture 3.1 in [GKP07].

Proposition 3.1. Let \( f \in S \) be bounded. Then \( \Psi(f) \) is not a support point of \( \Psi(S) \).

Proof. Let \( f \in S \) be bounded and let \( L : \mathcal{H}(B^n, \mathbb{C}^n) \to \mathbb{C} \) be a continuous linear functional such that \( \text{Re} L \) is nonconstant on \( \Psi(S) \) and

\[
\text{Re} L(\Psi(f)) = \max_{G \in \Psi(S)} \text{Re} L(G).
\]
For every $n \geq 1$ there is an $\varepsilon_0 > 0$ such that $f + \frac{\varepsilon}{n+1} f^{n+1} \in S$, for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \varepsilon_0$, as $f$ is bounded. It follows that

$$\Psi(f + \frac{\varepsilon}{n+1} f^{n+1})(z_1, z_2) = \left( f(z_1) + \frac{\varepsilon}{n+1} f(z_1)^{n+1}, z_2 \sqrt{f'(z_1)} \sqrt{1 + \varepsilon f(z_1)^n} \right)$$

$$= \left( f(z_1), z_2 \sqrt{f'(z_1)} \right) + \varepsilon \left( \frac{1}{n+1} f(z_1)^{n+1}, z_2/2 \sqrt{f'(z_1)} f(z_1)^n \right) + (0, O(|\varepsilon|^2)) \in \Psi(S).$$

If $\text{Re}L(P_n) \neq 0$, then we can choose $\varepsilon$ such that $\text{Re}L(\Psi(f + \frac{\varepsilon}{n+1} f^{n+1})) > \text{Re}L(\Psi(f))$, a contradiction. Hence,

$$(3.1) \quad \text{Re}L \left( \frac{1}{n+1} f(z_1)^{n+1}, z_2/2 \cdot \sqrt{f'(z_1)} f(z_1)^n \right) = 0 \quad \forall n \geq 1.$$

Now we can repeat this argument for the $\varepsilon$-terms of higher order, because all coefficients of the expansion $\sqrt{1+x} = 1 + \frac{1}{2} x + ...$ are $0$, and we get

$$(3.2) \quad \text{Re}L \left( 0, z_2 \sqrt{f'(z_1)} f(z_1)^n \right) = 0 \quad \forall j \geq 2, n \geq 1.$$

Now consider an arbitrary function of the form $(0, z_2 g(z_1))$ with $g(0) = g'(0) = 0$. Write $z_2 g(z_1) = z_2 \sqrt{f'(z_1)} \frac{g(z_1)}{\sqrt{f'(z_1)}}$ and approximate the second factor by a sequence of polynomials in $f$ (see Lemma 2.1 c):

$$z_2 g(z_1) = z_2 \sqrt{f'(z_1)} \sum_{k \geq 2} a_k f(z_1)^k = \sum_{k \geq 2} a_k z_2 \sqrt{f'(z_1)} f(z_1)^k.$$

By using (3.2), it follows that

$$(3.3) \quad \text{Re}L(0, z_2 g(z_1)) = 0 \quad \text{for all} \ g \in H(B^1, \mathbb{C}) \ \text{with} \ g(0) = g'(0) = 0.$$

We can apply this to (3.1) for $n \geq 2$ to get $\text{Re}L( f(z_1)^n, 0 ) = 0$ for all $n \geq 3$, and by Runge approximation it follows that

$$(3.4) \quad \text{Re}L( g(z_1), 0 ) = 0 \quad \text{for all} \ g \in H(B^1, \mathbb{C}) \ \text{with} \ g(0) = g'(0) = g''(0) = 0.$$

Now let $H(z_1, z_2) = (h(z_1), z_2 \sqrt{h'(z_1)}) \in \Psi(S)$. If $h(z) = z + a_2 z^2 + ...$, then $\sqrt{h'(z)} = \sqrt{1 + 2a_2 z^2 + ...} = 1 + a_2 z + ...$ Thus $h(z_1), z_2 \sqrt{h'(z_1)} = (z_1 + a_2 z_1^2 + ... z_2 + a_2 z_2 z_1 + ...) \text{ and with (3.3) and (3.4):}$

$$\text{Re}L(H) = d_1 + d_2 + a_2 + d_3 + d_4 \cdot a_2$$

with $d_1 = \text{Re}L(z_1, 0)$, $d_2 = \text{Re}L(z_1^2, 0)$, $d_3 = \text{Re}L(0, z_2)$, $d_4 = \text{Re}L(0, z_2 z_1)$. Finally, use (3.1) with $n = 1$ to get

$$\text{Re}L(z_1^2, z_2 z_1) = 0 \iff d_2 + d_4 = 0,$$

which implies $\text{Re}L(H) = d_1 + d_3$. Hence, $\text{Re}L$ is constant on $\Psi(S)$ and $\Psi(f)$ cannot be a support point of $\Psi(S)$. \hfill \square

**Proof of Theorem 1.2** The proof is now quite the same as the proof of Theorem 1.1. We just have to replace $S^0(B^n)$ by $\Psi(S)$; see remark 3.1 in [GKP07]. \hfill \square
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