TAIL ALGEBRAS OF QUANTUM EXCHANGEABLE RANDOM VARIABLES

KENNETH J. DYKEMA AND CLAUS KÖSTLER

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ABSTRACT. We show that any countably generated von Neumann algebra with specified normal faithful state can arise as the tail algebra of a quantum exchangeable sequence of noncommutative random variables. We also characterize the cases when the state corresponds to a limit of convex combinations of free product states.

1. INTRODUCTION

Exchangeability is a basic distributional symmetry in probability. It means that the distribution of a sequence of random variables is invariant under finite permutations of these random variables. The de Finetti theorem characterizes an exchangeable infinite sequence of random variables to be identically distributed and conditionally independent over its tail σ -algebra. An alternative formulation of this famous theorem is that the law of such a sequence is a mixture of infinite product measures, where the mixture is specified by a certain random probability measure (see e.g. [4]). Hewitt and Savage [3] cast this theorem in terms of symmetric measures on infinite Cartesian products of a compact Hausdorff space S , and inspired Størmer's transfer [8] of their approach to a C^* -algebraic setting: symmetric states on an infinite tensor product of a C^* -algebra with itself are identified as a mixture of infinite product states. In analogy with the classical situation, here the mixture is specified by a probability measure on the state space of the C^* -algebra.

Recently a noncommutative de Finetti theorem was discovered by Köstler and Speicher [6] in the realm of Voiculescu's free probability theory (see [9]). Replacing random variables by operators and the role of permutations by the natural coaction of Wang's quantum permutations [10], the notion of a quantum exchangeable sequence was introduced in a framework of noncommutative probability spaces. In close analogy to the classical case, a quantum exchangeable infinite sequence is characterized as being identically distributed and 'conditionally free' over its tail algebra. Here 'conditional freeness' means 'freeness with amalgamation'.

In this paper we take a closer look at tail algebras of quantum exchangeable sequences. Our main results are:

- ◊ Any countably generated von Neumann algebra may appear as the tail algebra of a quantum exchangeable sequence of self-adjoint operators (see Theorem 3.3).

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- ◇ The tail algebra of a quantum exchangeable sequence lies in the center of the von Neumann algebra generated by the sequence if and only if the corresponding state is a limit of convex combinations of free product states (see Proposition 4.7).
- ◇ There exist quantum exchangeable sequences of projections generating a finite factor and whose tail algebra is nontrivial and abelian (see Example 4.8). Thus, the corresponding state is not a limit of convex combinations of free product states.

Altogether these results show that, as to be expected, the structure of tail algebras of quantum exchangeable infinite sequences has a much higher complexity than those of tail σ -algebras of exchangeable sequences in probability theory.

2. PRELIMINARIES

A *noncommutative probability space* is a pair (A, ϕ) , where A is a unital algebra over the complex numbers and ϕ is a linear functional on A sending 1 to 1, and the elements of A are called *noncommutative random variables*. A W^* -*noncommutative probability space* is one in which A is a von Neumann algebra and ϕ is a normal state. In this paper, we only consider W^* -noncommutative probability spaces (A, ϕ) where the state ϕ is faithful. The *Noncommutative de Finetti Theorem* of Köstler and Speicher [6] states that a sequence $(x_i)_{i=1}^\infty$ of noncommutative random variables in a W^* -noncommutative probability space is quantum exchangeable if and only if the sequence is free over the tail algebra of the sequence, with respect to the ϕ -preserving conditional expectation. Quantum exchangeability of the sequence is defined in terms of a natural coaction of the quantum permutation group of Wang [10]. (See [6] for more on this.) The *tail algebra* is the von Neumann algebra $\bigcap_{n=1}^\infty W^*(\{x_i \mid i \geq n\})$, and the ϕ -preserving conditional expectation E from $W^*(\{x_i \mid i \geq 1\})$ onto the tail algebra is guaranteed to exist. This was proved in [5]; see also Proposition 4.2 of [6].

The following is the “hard part” of the noncommutative de Finetti theorem (Thm. 1.1 of [6]).

Theorem 2.1 ([6]). *Let $(x_i)_{i=1}^\infty$ be a quantum exchangeable sequence in the W^* -noncommutative probability space $(\mathcal{M}, \phi)$, where ϕ is faithful, and suppose $\mathcal{M}$ is generated by $\{x_i \mid i \in I\}$. Let $\mathcal{N}$ be the tail algebra and let $E : \mathcal{M} \rightarrow \mathcal{N}$ be the ϕ -preserving conditional expectation onto $\mathcal{N}$. Let $\mathcal{A}_i$ be the von Neumann subalgebra of $\mathcal{M}$ generated by $\mathcal{N} \cup \{x_i\}$. Then the family $(\mathcal{A}_i)_{i \in I}$ is free with amalgamation over $\mathcal{N}$, with respect to E and, therefore,*

$$(1) \quad (\mathcal{M}, E) \cong (*_{\mathcal{N}})_{i=1}^\infty (\mathcal{A}_i, E_i)$$

is isomorphic to the W^ -amalgamated free product of countably infinitely many copies of $(\mathcal{A}_1, E_1)$, where $E_i : \mathcal{A}_i \rightarrow \mathcal{N}$ is the restriction to $\mathcal{A}_i$ of E .*

The next result is the “easy part” of the noncommutative de Finetti theorem (Prop. 3.1 of [6]):

Proposition 2.2 ([6]). *Let*

$$(\mathcal{M}, E) \cong (*_{\mathcal{B}})_{i=1}^\infty (\mathcal{A}_i, E_i)$$

be an amalgamated free product of von Neumann algebras, where every E_i is faithful and normal. Let ϕ be any normal, faithful state on $\mathcal{B}$, and denote also by ϕ the

state $\phi \circ E$ on $\mathcal{M}$. If $x_i \in \mathcal{A}_i$ is such that the moments $E_i(x_i b_1 x_i b_2 \cdots b_{n-1} x_i)$ for all $n \in \mathbf{N}$ and $b_1, \dots, b_{n-1} \in \mathcal{B}$ are independent of i , then the sequence $(x_i)_{i=1}^\infty$ in the noncommutative probability space $(\mathcal{M}, \phi)$ is quantum exchangeable.

In this note, we will prove (Theorem 3.3) that every countably generated von Neumann algebra $\mathcal{N}$ with specified normal faithful state ϕ can arise as the tail algebra of a quantum exchangeable sequence in a W^* -noncommutative probability space $(\mathcal{M}, \phi_{\mathcal{M}})$ in such a way that the restriction of $\phi_{\mathcal{M}}$ to $\mathcal{N}$ is ϕ .

We also investigate the question, given a quantum exchangeable sequence $(x_i)_{i=1}^\infty$ in a W^* -noncommutative probability space $(\mathcal{M}, \phi)$, of when the distribution of $(x_i)_{i=1}^\infty$ under ϕ is a limit of convex combinations of equidistributed free product states. Naturally enough, this occurs if and only if the tail algebra of the sequence commutes with all x_i . See Section 4 for more details. This may be compared to Størmer's result [8], where in the commutative context all symmetric states are limits of convex combinations of tensor powers.

3. EXAMPLES OF TAIL ALGEBRAS

Now we investigate the content of the tail algebra in the construction of Proposition 2.2.

Proposition 3.1. *In the setting of Proposition 2.2, the tail algebra $\mathcal{N}$ of the sequence $(x_i)_{i=1}^\infty$ is a von Neumann subalgebra of $\mathcal{B}$ and is, in fact, the smallest unital von Neumann subalgebra $\mathcal{B}_0$ of $\mathcal{B}$ satisfying*

$$(2) \quad E(b_0 x^{k_1} b_1 x^{k_2} \cdots b_{n-1} x^{k_n} b_n) \in \mathcal{B}_0$$

whenever $n \in \mathbf{N}$, $k_1, \dots, k_n \in \mathbf{N}$, $b_0, b_1, \dots, b_n \in \mathcal{B}_0$ and $x = x_i$. (Note that the above expression is independent of the choice of $i \in \mathbf{N}$.)

Proof. If we consider the structure of $L^2(\mathcal{M}, \phi)$, we easily see that the tail algebra $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{B}$. Indeed, from Voiculescu's free product construction (see [1] for a more detailed discussion in the case of von Neumann algebras),

$$(3) \quad L^2(\mathcal{M}, \phi) = L^2(\mathcal{B}, \phi) \oplus \bigoplus_{\substack{n \geq 1 \\ i_1, \dots, i_n \geq 1 \\ i_j \neq i_{j+1}}} F_{i_1} \otimes_{\mathcal{B}} F_{i_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} F_{i_n} \otimes_{\pi_\phi} L^2(\mathcal{B}, \phi),$$

where F_i is the Hilbert $\mathcal{B}$ -module $L^2(\mathcal{A}_i, E_i) \ominus \mathcal{B}$ and π_ϕ is the Gelfand–Naimark–Segal representation of $\mathcal{B}$ on $L^2(\mathcal{B}, \phi)$. In particular, the image in $L^2(\mathcal{M}, \phi)$ of the von Neumann algebra generated by $\{x_i \mid i \geq N\}$ is contained in

$$L^2(\mathcal{B}, \phi) \oplus \bigoplus_{\substack{n \geq 1 \\ i_1, \dots, i_n \geq N \\ i_j \neq i_{j+1}}} F_{i_1} \otimes_{\mathcal{B}} F_{i_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} F_{i_n} \otimes_{\pi_\phi} L^2(\mathcal{B}, \phi)$$

and the intersection of these spaces is $L^2(\mathcal{B}, \phi)$. We must, therefore, have $\mathcal{N} \subseteq \mathcal{B}$.

Since $\mathcal{N} \subseteq \mathcal{B} \cap W^*(\{x_i \mid i \geq 1\})$, we have

$$\mathcal{N} \subseteq E(W^*(\{x_i \mid i \geq 1\})).$$

Thus, we have

$$(4) \quad \mathcal{N} \subseteq W^*(\{E(x_{i_1} x_{i_2} \cdots x_{i_n}) \mid n \geq 1, i_1, \dots, i_n \geq 1\}).$$

To see that we have equality in (4), take $j_1, \dots, j_N \geq 1$ and let $M_1 = \min(j_1, \dots, j_N)$ and $M_2 = \max(j_1, \dots, j_N)$. Then $x_{j_1}x_{j_2} \cdots x_{j_N} = b + y$, where $b = E(x_{j_1}x_{j_2} \cdots x_{j_N})$ and where the element $\hat{y}$ of $L^2(\mathcal{M}, \phi)$ corresponding to y belongs to

$$\bigoplus_{\substack{1 \leq n \leq N \\ M_1 \leq i_1, \dots, i_n \leq M_2 \\ i_j \neq i_{j+1}}} F_{i_1} \otimes_{\mathcal{B}} F_{i_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} F_{i_n} \otimes_{\pi_\phi} L^2(\mathcal{B}, \phi).$$

Since the sequence $(x_i)_{i \geq 1}$ is quantum exchangeable, it is also exchangeable, i.e., the joint moments of this sequence are invariant under arbitrary permutations of $\mathbf{N}$. Thus, for each $p \geq 0$ we have $E(x_{j_1+p}x_{j_2+p} \cdots x_{j_N+p}) = E(x_{j_1}x_{j_2} \cdots x_{j_N})$ and, therefore, $x_{j_1+p}x_{j_2+p} \cdots x_{j_N+p} = b + y_p$, where

$$(5) \quad \hat{y}_p \in \bigoplus_{\substack{1 \leq n \leq N \\ M_1+p \leq i_1, \dots, i_n \leq M_2+p \\ i_j \neq i_{j+1}}} F_{i_1} \otimes_{\mathcal{B}} F_{i_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} F_{i_n} \otimes_{\pi_\phi} L^2(\mathcal{B}, \phi).$$

Now since the subspaces (5) corresponding to values of p that differ by more than $M_2 - M_1$ are orthogonal to each other, we see that for each $q \in \mathbf{N}$, the ergodic averages

$$\frac{1}{K} \sum_{p=q}^{q+K-1} x_{i_1+p}x_{i_2+p} \cdots x_{i_N+p}$$

converge in $L^2(\mathcal{M}, \phi)$ to b as $K \rightarrow \infty$. So $b \in W^*(\{x_i \mid i \geq q\})$ for all $q \geq 1$. Letting $q \rightarrow \infty$, we get $b = E(x_{j_1}x_{j_2} \cdots x_{j_N}) \in \mathcal{N}$. This proves

$$(6) \quad \mathcal{N} = W^*(\{E(x_{i_1}x_{i_2} \cdots x_{i_n}) \mid n \geq 1, i_1, \dots, i_n \geq 1\}).$$

It remains to show $\mathcal{N} = \mathcal{B}_0$. Using (6), we have $E(b_0x^{k_1}b_1x^{k_2} \cdots b_{n-1}x^{k_n}b_n) \in \mathcal{N}$ whenever $n \in \mathbf{N}$, $k_1, \dots, k_n \in \mathbf{N}$, $b_0, b_1, \dots, b_n \in \mathcal{N}$ and $x = x_i$. By the characterization of $\mathcal{B}_0$, this implies $\mathcal{B}_0 \subseteq \mathcal{N}$. So it suffices to see that we have $E(x_{i_1}x_{i_2} \cdots x_{i_n}) \in \mathcal{B}_0$ for all $n \geq 1$ and $i_1, \dots, i_n \geq 1$. However, this follows by considering how mixed moments of the free variables $x_1, x_2, \dots$ can be evaluated in terms of the moments of the individual x_i . Speicher's operator-valued cumulants [7] can be used to give a careful proof of this fact. Indeed, using the expression of the cumulants in terms of moments and the Möbius function and using the defining property (2) of $\mathcal{B}_0$, we see that for each individual $x = x_i$, the operator-valued cumulant $\kappa[xb_1, xb_2, \dots, xb_{n-1}, x]$ belongs to $\mathcal{B}_0$ for every $b_1, \dots, b_{n-1} \in \mathcal{B}_0$. Now using the moment cumulant formula for $E(x_{i_1}x_{i_2} \cdots x_{i_n})$ and the vanishing of mixed cumulants, we obtain $E(x_{i_1}x_{i_2} \cdots x_{i_n}) \in \mathcal{B}_0$. $\square$

Lemma 3.2. *Let $\mathcal{N}$ be a countably generated von Neumann algebra equipped with a normal faithful state ϕ . Then there is a von Neumann algebra $\mathcal{A}$ containing $\mathcal{N}$ as a unital von Neumann subalgebra and possessing a ϕ -preserving conditional expectation $E : \mathcal{A} \rightarrow \mathcal{N}$ onto $\mathcal{N}$, and there is a self-adjoint element $a \in \mathcal{A}$ with the property that $\{E(a^k) \mid k \in \mathbf{N}\}$ generates $\mathcal{N}$ as a von Neumann algebra.*

Proof. One easily sees (using the spectral theorem) that the von Neumann algebra $\mathcal{N}$ is also generated by a finite or countable collection $(p_i)_{i \in I}$ of projections. Let $\mathcal{A} = \bigoplus_{i \in I} \mathcal{N}$, with $\mathcal{N} \subseteq \mathcal{A}$ identified with the constant sequences and with the

conditional expectation E given by some strictly positive weights $(\alpha_i)_{i \in I}$ that sum to 1:

$$E((x_i)_{i \in I}) = \sum_i \alpha_i x_i.$$

We let $a = (\beta_i p_i)_{i \in I} \in \mathcal{A}$ for some bounded family $\beta_i \in \mathbf{R}$; thus, we have

$$E(a^k) = \sum_i \alpha_i \beta_i^k p_i.$$

If I is finite, then by choosing all the β_i to be distinct, the determinant of the matrix $(\beta_i^j)_{i \in I, 0 \leq j < |I|}$, being a Vandermonde determinant, is seen to be nonzero. Thus, we recover $\{p_i \mid i \in I\}$ by taking linear combinations of $(E(a^k))_{0 \leq k < |I|}$.

Suppose I is infinite, identify it with $\mathbf{N}_0$ and let $\beta_i = 2^{-i}$. Let

$$\mathcal{D} = W^*(\{E(a^k) \mid k \in \mathbf{N}\}).$$

Then

$$\lim_{k \rightarrow \infty} E(a^k) = \lim_{k \rightarrow \infty} \left(\alpha_0 p_0 + \sum_{i=1}^{\infty} \alpha_i 2^{-ki} p_i \right) = \alpha_0 p_0,$$

where the convergence is in norm topology, and $p_0 \in \mathcal{D}$. Similarly,

$$\begin{aligned} \lim_{k \rightarrow \infty} 2^k (E(a^k) - \alpha_0 p_0) &= \alpha_1 p_1, \\ \lim_{k \rightarrow \infty} 2^{2k} (E(a^k) - \alpha_0 p_0 - 2^{-1} \alpha_1 p_1) &= \alpha_2 p_2, \end{aligned}$$

and so on. Thus, we get $p_0, p_1, p_2, \dots \in \mathcal{D}$ and $\mathcal{D}$ is all of $\mathcal{N}$. $\square$

Theorem 3.3. *Let $\mathcal{N}$ be a countably generated von Neumann algebra with a normal, faithful state ϕ . Then there is a quantum exchangeable sequence $x_1, x_2, \dots$ in some W^* -noncommutative probability space $(\mathcal{M}, \phi_{\mathcal{M}})$ whose tail algebra is a copy of $\mathcal{N}$ in $\mathcal{M}$, with the restriction of $\phi_{\mathcal{M}}$ to $\mathcal{N}$ being ϕ .*

Proof. Let $\mathcal{A} \supseteq \mathcal{N}$ and $a \in \mathcal{A}$ be as from Lemma 3.2. For each $j \in \mathbf{N}$, let $\mathcal{A}_j$ be a copy of $\mathcal{A}$ with the ϕ -preserving conditional expectation onto $\mathcal{N}$ denoted by E_j . Let

$$(\mathcal{M}, E) = (*_{\mathcal{N}})_{j=1}^{\infty} (\mathcal{A}_j, E_j)$$

be their amalgamated free product of von Neumann algebra and, of course, let $\phi_{\mathcal{M}} = \phi \circ E$. Let x_j be the copy of a in the j th copy $\mathcal{A}_j$ of $\mathcal{A}$. By Proposition 2.2, $(x_j)_{j=1}^{\infty}$ is a quantum exchangeable sequence. By Lemma 3.2, the set

$$(7) \quad \{E(x_j^k) \mid k \in \mathbf{N}\}$$

generates all of $\mathcal{N}$. By Proposition 3.1, the tail algebra of the sequence $(x_i)_{i \in I}$ is equal to $\mathcal{N}$. $\square$

4. FREE PRODUCT STATES

Given a C^* -algebra A with state ϕ and with self-adjoint elements $x_i \in A$ ($i \in I$), we wish to regard the variables x_i abstractly, independently of their realization in A . Consider the $*$ -algebra $\mathbf{C}\langle X \rangle = \mathbf{C}\langle X_i \mid i \in I \rangle$ of polynomials in noncommuting variables X_i , with the involution defined so that $X_i^* = X_i$, and consider the unital $*$ -algebra representation $\alpha : \mathbf{C}\langle X \rangle \rightarrow A$ that sends X_i to x_i .

Definition 4.1. By a *free product state* on $\mathbf{C}\langle X \rangle$ we will mean a functional ψ of $\mathbf{C}\langle X \rangle$ such that $\psi(1) = 1$ and the variables $(X_i)_{i \in I}$ are free with respect to ψ . If, in addition, for all k the moments $\psi(X_i^k)$ are independent of i , then we say ψ is an *equidistributed free product state* on $\mathbf{C}\langle X \rangle$. We will say that a linear functional ϕ of $\mathbf{C}\langle X \rangle$ is a *limit of convex combinations of uniformly bounded free product states* (respectively, of uniformly bounded equidistributed free product states) if for some choice of constants $C_i > 0$, the functional ϕ is the limit in the topology of pointwise convergence on $\mathbf{C}\langle X \rangle$ of convex combinations of free product states (respectively, of equidistributed free product states) ψ on $\mathbf{C}\langle X \rangle$ that satisfy $|\psi(X_i^k)| \leq C_i^k$ for all $i \in I$ and $k \in \mathbf{N}$.

We will need a few lemmas about freeness. Here is an easy result about freeness over a subalgebra of the center.

Lemma 4.2. *Let $D = C(\Omega)$ be a commutative, unital C^* -algebra, let A be a unital C^* -algebra with D embedded as a unital C^* -subalgebra of the center of A and suppose $E : A \rightarrow D$ is a conditional expectation. Let I be a set and suppose for every $i \in I$ there is a C^* -algebra $A_i \in I$ with $D \subseteq A_i$. For each $\omega \in \Omega$, consider the character $\text{ev}_\omega : f \mapsto f(\omega)$ of D and let $\rho_\omega = \text{ev}_\omega \circ E$. Then the family $(A_i)_{i \in I}$ is free (over D) with respect to E if and only if for every $\omega \in \Omega$, the family $(A_i)_{i \in I}$ is free (over $\mathbf{C}$) with respect to ρ_ω .*

Proof. First suppose the family is free with respect to E , fix $\omega \in \Omega$ and let us show freeness with respect to ρ_ω . Suppose $a_j \in A_{i_j} \cap \ker \rho_\omega$ for $j = 1, \dots, n$ and $i_j \neq i_{j+1}$ for all $1 \leq j < n$. We must show $\rho_\omega(a_1 a_2 \cdots a_n) = 0$. Let $\epsilon > 0$. Since $E(a_j) \in C(\Omega)$ vanishes at ω , by Tietze's extension theorem there is $f \in C(\Omega)$ such that $f(\omega) = 1$, $\|f\|_\infty = 1$ and $\|E(a_j)f\| < \epsilon$ for all j . Let $f_j = E(a_j)f$. Then freeness with respect to E implies $E((fa_1 - f_1)(fa_2 - f_2) \cdots (fa_n - f_n)) = 0$, so we have

$$\rho_\omega((fa_1 - f_1)(fa_2 - f_2) \cdots (fa_n - f_n)) = 0.$$

But if $M = 1 + \epsilon + \max_j \|a_j\|$, then

$$\begin{aligned} & |\rho_\omega((fa_1)(fa_2) \cdots (fa_n))| \\ &= |\rho_\omega((fa_1)(fa_2) \cdots (fa_n)) - \rho_\omega((fa_1 - f_1) \cdots (fa_n - f_n))| \leq nM^{n-1}\epsilon. \end{aligned}$$

However, we have $\rho_\omega(a_1 a_2 \cdots a_n) = \rho_\omega((f^n) a_1 a_2 \cdots a_n) = \rho_\omega((fa_1)(fa_2) \cdots (fa_n))$, so letting $\epsilon \rightarrow 0$ finishes the proof of freeness with respect to ρ_ω .

Now suppose the family is free with respect to ρ_ω for all ω and let us show freeness with respect to E . Suppose $a_i \in A_{i_j} \cap \ker E$ for i_j as above and let $d = E(a_1 a_2 \cdots a_n)$. For every ω and every j , we have $\rho_\omega(a_j) = 0$, so by hypothesis, $d(\omega) = \rho_\omega(a_1 a_2 \cdots a_n) = 0$. So $d = 0$. $\square$

Here is an easy result about conditional expectations and Gelfand–Naimark–Segal (GNS) constructions, whose proof we include for convenience.

Lemma 4.3. *Let A be a unital C^* -algebra and $B \subseteq A$ a unital C^* -subalgebra. Suppose $E : A \rightarrow B$ is a conditional expectation onto B . Suppose ϕ is a state on A such that $\phi \circ E = \phi$. Let $(\pi_\phi, \mathcal{H}_\phi)$ be the GNS representation of A arising from ϕ and let $a \mapsto \hat{a}$ denote the linear mapping $A \rightarrow \mathcal{H}_\phi$ arising in the GNS construction.*

- (i) Then there is a self-adjoint projection P_ϕ on $\mathcal{H}_\phi$ such that $E(x)^\wedge = P_\phi \hat{x}$ and
- $$(8) \quad \pi_\phi(E(x))P_\phi = P_\phi \pi_\phi(x)P_\phi$$

for all $x \in A$.

- (ii) If B lies in the center of A , then there is a conditional expectation $E_\phi : \pi_\phi(A) \rightarrow \pi_\phi(B)$ satisfying $E_\phi(\pi_\phi(x)) = \pi_\phi(E(x))$ for all $x \in A$.

Proof. If $b \in B$, $x \in A$ and $E(x) = 0$, then

$$\langle \hat{x}, \hat{b} \rangle_\phi := \phi(b^*x) = \phi(E(b^*x)) = \phi(b^*E(x)) = 0.$$

Thus, given $a \in A$ and writing $a = E(a) + (a - E(a))$ we get for the norm in $\mathcal{H}_\phi$

$$\|\hat{a}\|_\phi^2 = \|E(a)^\wedge\|_\phi^2 + \|(a - E(a))^\wedge\|_\phi^2,$$

and the map $\hat{a} \mapsto E(a)^\wedge$ is an idempotent, linear map that is contractive with respect to $\|\cdot\|_\phi$ and, hence, extends to a self-adjoint projection P_ϕ from $\mathcal{H}_\phi$ onto $\{\hat{b} \mid b \in B\}$. For $a, x \in A$ we have

$$\pi_\phi(E(x))P_\phi \hat{a} = (E(x)E(a))^\wedge = E(xE(a))^\wedge = P_\phi(xE(a))^\wedge = P_\phi \pi_\phi(x)P_\phi \hat{a},$$

which proves (8).

For (ii), assume B is in the center of A . Note that if $b \in B$, then for every $a \in A$ we have

$$\begin{aligned} \|\pi_\phi(b)\hat{a}\|_\phi^2 &= \phi(a^*b^*ba) = \phi(E(a^*b^*ba)) = \phi(E(a^*a)b^*b) \\ &= \phi(b^*E(a^*a)b) = \|\pi_\phi(b)(E(a^*a)^{1/2})^\wedge\|_\phi^2, \end{aligned}$$

so if $\pi_\phi(b)P_\phi = 0$, then $\pi_\phi(b) = 0$. Hence, the $*$ -homomorphism $\pi_\phi(B) \ni \pi_\phi(b) \mapsto \pi_\phi(b)P_\phi$ is isometric. Thus, we have, for $x \in A$,

$$\|\pi_\phi(E(x))\| = \|\pi_\phi(E(x))P_\phi\| = \|P_\phi \pi_\phi(x)P_\phi\| \leq \|\pi_\phi(x)\|,$$

and $\pi_\phi(x) \mapsto \pi_\phi(E(x))$ is a contractive, linear, idempotent map from $\pi_\phi(A)$ onto $\pi_\phi(B)$; this is the desired conditional expectation E_ϕ . $\square$

Remark 4.4. In the above lemma, it is not true that faithfulness of E implies faithfulness of E_ϕ . For example, let $A = M_2(C([0, 1]))$, let B be the center of A , identified with $C([0, 1])$ and let $E : A \rightarrow B$ be the conditional expectation given by, for $a = (a_{ij})_{1 \leq i, j \leq 2} \in A$ with $a_{ij} \in C([0, 1])$,

$$E(a)(t) = ta_{11}(t) + (1 - t)a_{22}(t) \quad (t \in [0, 1]).$$

Then E is faithful. Letting ϕ be the state on A determined by $\phi = \phi \circ E$ and $\phi(b) = b(0)$ for $b \in B$, we find that E_ϕ is the state $M_2(\mathbf{C}) \rightarrow \mathbf{C}$ sending $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ to 0, and so is not faithful.

Lemma 4.5. *Let A be a unital C^* -algebra and B a unital C^* -subalgebra of the center of A with a conditional expectation $E : A \rightarrow B$. Let I be a set and suppose that for every $i \in I$, A_i is a C^* -subalgebra of A that contains B and the family $(A_i)_{i \in I}$ is free (over B) with respect to E . Let ϕ be a state on A satisfying $\phi \circ E = \phi$ and let $E_\phi : \pi_\phi(A) \rightarrow \pi_\phi(B)$ be the conditional expectation from Lemma 4.3. Then the family $(\pi_\phi(A_i))_{i \in I}$ is free (over $\pi_\phi(B)$) with respect to E_ϕ .*

Proof. We have $B = C(X)$ for some compact Hausdorff space X and $\pi_\phi(B) = C(Y)$ for a closed subspace Y of X , where the $*$ -homomorphism $\pi_\phi|_B : C(X) \rightarrow C(Y)$ sends a function to its restriction to Y . For $y \in Y$, let $\text{ev}_y : \pi_\phi(B) \rightarrow \mathbf{C}$ and $\text{ev}_y^{(X)} : B \rightarrow \mathbf{C}$ be the homomorphisms of evaluation at y . By Lemma 4.2, it will suffice to show that for every $y \in Y$, the family $(\pi_\phi(A_i))_{i \in I}$ is free (over $\mathbf{C}$) with respect to $\text{ev}_y \circ E_\phi$. However, we have

$$\text{ev}_y \circ E_\phi \circ \pi_\phi = \text{ev}_y^{(X)} \circ E.$$

From Lemma 4.2, we have freeness of $(A_i)_{i \in I}$ with respect to $\text{ev}_y^{(X)} \circ E$, and from this follows the freeness of $(\pi_\phi(A_i))_{i \in I}$ with respect to $\text{ev}_y \circ E_\phi$. $\square$

The following result shows that a state that is a limit of convex combinations of uniformly bounded free product states always arises from the situation of a free product with amalgamation over a subalgebra of the center.

Proposition 4.6. *Let A be a C^* -algebra with faithful state ϕ . Suppose $x_i \in A$ ($i \in I$), for I countable, are self-adjoint elements that together generate A as a C^* -algebra and so that $\tilde{\phi} := \phi \circ \alpha$ is a limit of convex combinations of uniformly bounded free product states, where $\alpha : \mathbf{C}\langle X \rangle \rightarrow A$ is the $*$ -homomorphism given by $X_i \mapsto x_i$. Then there is a unital C^* -algebra B with a unital C^* -subalgebra $D \subseteq Z(B)$ of the center of B and with a faithful conditional expectation $E : B \rightarrow D$, and there is a unital $*$ -homomorphism $\pi : A \rightarrow B$ and a faithful tracial state $\sigma = \sigma \circ E$ on B so that $\phi = \sigma \circ \pi$ and the family $(\pi(x_i))_{i \in I}$ is free with respect to E .*

Furthermore, if $\tilde{\phi}$ is a limit of convex combinations of uniformly bounded equidistributed free product states with respect to $(x_i)_{i \in I}$, then $E : B \rightarrow D$ and $\pi : A \rightarrow B$ may be chosen so that for all n and all $d_0, \dots, d_n \in D$, the corresponding moment $E(d_0 \pi(x_i) d_1 \cdots \pi(x_i) d_n)$ of the variable $\pi(x_i)$ is independent of i .

Proof. We fix constants C_i as in Definition 4.1. If ψ is a free product state on $\mathbf{C}\langle X \rangle$, then the usual GNS construction yields a $*$ -representation $\pi_\psi : \mathbf{C}\langle X \rangle \rightarrow B(\mathcal{H}_\psi)$ with $\|\pi_\psi(X_i)\| \leq C_i$. Let $B_\psi \subseteq B(\mathcal{H}_\psi)$ be the unital C^* -algebra generated by the image of π_ψ and denote also by ψ the state on B_ψ so that $\psi \circ \pi_\psi$ is the original functional ψ on $\mathbf{C}\langle X \rangle$. Of course, the condition that ψ is a free product state on $\mathbf{C}\langle X \rangle$ implies that the variables $(\pi_\psi(X_i))_{i \in I}$ are free in B_ψ with respect to ψ . Moreover, B_ψ is isomorphic to the free product over the scalars of the abelian C^* -algebras generated by $\pi_\psi(X_i)$. Since the restrictions of ψ to these abelian C^* -algebras are faithful, and since the free product of faithful states is faithful [2] and since the free product of traces is a trace (see [9]), it follows that ψ is a faithful trace on B_ψ .

By hypothesis, we may write $\tilde{\phi}$ as the limit of a net of convex combinations of free product states and consider the set F of all the free product states appearing with nonzero coefficients in these convex combinations. Let

$$\tilde{B} = \prod_{\psi \in F} B_\psi = \{(b_\psi)_{\psi \in F} \mid b_\psi \in B_\psi, \sup_{\psi} \|b_\psi\| < \infty\}$$

be the C^* -algebra direct product. Then $\tilde{\beta} : \mathbf{C}\langle X \rangle \rightarrow \tilde{B}$ defined by $\tilde{\beta}(p) = (\pi_\psi(p))$ is a $*$ -representation. Let $\tilde{D} \subseteq \tilde{B}$ be the subalgebra consisting of all sequences of scalars, i.e. all $(\lambda_\psi 1)_{\psi \in F}$ for $\lambda_\psi \in \mathbf{C}$. Clearly, $\tilde{D} \cong \ell^\infty(F)$ is a subalgebra of the center of $\tilde{B}$, and the map $\tilde{E} : \tilde{B} \rightarrow \tilde{D}$ given by $\tilde{E}((b_\psi)_{\psi \in F}) = (\psi(b_\psi))_{\psi \in F}$ is

a conditional expectation that is faithful because each tracial state ψ is faithful on B_ψ . The variables $(\tilde{\beta}(X_i))_{i \in I}$ are free with respect to $\tilde{E}$, because if for some $n \in \mathbf{N}$ and $i(1), \dots, i(n) \in \mathbf{N}$ with $i(j) \neq i(j+1)$, $p_j(X_{i(j)})$ is a polynomial in $X_{i(j)}$ with coefficients from $\tilde{B}$ and $E(p_j(X_{i(j)})) = 0$, then $\psi(p_j(X_{i(j)})) = 0$ for every $\psi \in F$. By freeness $\psi(p_1(X_{i(1)}) \cdots p_n(X_{i(n)})) = 0$ for every $\psi \in F$, so also $\tilde{E}(p_1(X_{i(1)}) \cdots p_n(X_{i(n)})) = 0$. Note that $\tilde{E}$ has the property $\tilde{E}(xy) = \tilde{E}(yx)$ for all $x, y \in \tilde{B}$, owing to the fact that each ψ is a trace.

In the above construction, if all free product states $\psi \in F$ are equidistributed, then the moments

$$(9) \quad \tilde{E}(d_0 \tilde{\beta}(X_i) d_1 \cdots \tilde{\beta}(X_i) d_n)$$

of the variables $\tilde{\beta}(X_i)$ are independent of i .

To a convex combination $\rho = \sum t_\psi \psi$ (with finite support) of elements of F , consider the state $\hat{\rho}$ of $\tilde{D}$ given by the corresponding weighted average, $\hat{\rho} : (\lambda_\psi 1)_{\psi \in F} \mapsto \sum t_\psi \lambda_\psi$. Let $(\rho_j)_{j \in J}$ denote a net of convex combinations of free product states on $\mathbf{C}\langle X \rangle$ that converges (in the topology of pointwise convergence on $\mathbf{C}\langle X \rangle$) to $\tilde{\phi}$. Replacing this net by a subnet, if necessary, we may without loss of generality assume that the corresponding net $\hat{\rho}_j$ converges in the weak*-topology on $\tilde{D}^*$ to a state $\tilde{\sigma}$ on $\tilde{D}$. Then we have $\tilde{\phi} = \tilde{\sigma} \circ \tilde{E} \circ \tilde{\beta}$.

We are almost done, except that $\tilde{\sigma} \circ \tilde{E}$ need not be faithful. Let $\pi_{\tilde{\sigma} \circ \tilde{E}}$ denote the GNS representation associated to the tracial state $\tilde{\sigma} \circ \tilde{E}$ on $\tilde{B}$, and let $B = \pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{B})$ and $D = \pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{D})$. Then the corresponding state σ on B is a faithful trace. By Lemma 4.3, there is a conditional expectation $E : B \rightarrow D$ so that

$$E(\pi_{\tilde{\sigma} \circ \tilde{E}}(x)) = \pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{E}(x)) \quad (x \in \tilde{B}),$$

and by Lemma 4.5 and freeness of $(\tilde{\beta}(X_i))_{i \in I}$ with respect to $\tilde{E}$, the family

$$(10) \quad (\pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{\beta}(X_i)))_{i \in I}$$

is free (over D) with respect to E .

Let $\beta = \pi_{\tilde{\sigma} \circ \tilde{E}} \circ \tilde{\beta} : \mathbf{C}\langle X \rangle \rightarrow B$. Then $\tilde{\phi} = \phi \circ \alpha = \sigma \circ \beta$. We will now define a $*$ -homomorphism $\pi : A \rightarrow B$ so that all triangles in the diagram

$$\begin{array}{ccc} \mathbf{C}\langle X \rangle & \xrightarrow{\alpha} & A \\ \beta \downarrow & \nearrow \pi & \downarrow \phi \\ B & \xrightarrow{\sigma} & \mathbf{C} \end{array}$$

commute. Indeed, since ϕ and σ are faithful, we have

$$\begin{aligned} \|a\| &= \limsup_{n \rightarrow \infty} |\phi((a^* a)^n)|^{1/2n} & (a \in A), \\ \|b\| &= \limsup_{n \rightarrow \infty} |\sigma((b^* b)^n)|^{1/2n} & (b \in B). \end{aligned}$$

But this implies $\|\alpha(p)\| = \|\beta(p)\|$ for all $p \in \mathbf{C}\langle X \rangle$. So the $*$ -homomorphism defined on the image of α by $\alpha(p) \mapsto \beta(p)$ is isometric and extends to an isometric $*$ -homomorphism $\pi : A \rightarrow B$, as required. Since $\pi(x_i) = \beta(X_i)$, by freeness of the family (10) we have that $(\pi(x_i))_{i \in I}$ is free with respect to E .

In the case that all free product states are equidistributed, the observation above regarding the moments (9) of $\tilde{\beta}(X_i)$ with respect to $\tilde{E}$ implies that the moments of $\pi(x_i)$ with respect to E are independent of i . $\square$

Proposition 4.7. *Let $(x_i)_{i \in I}$ be quantum exchangeable random variables in $(\mathcal{M}, \phi)$, suppose $\mathcal{M}$ is generated by $\{x_i \mid i \in I\}$ and let $\alpha : \mathbf{C}\langle X \rangle \rightarrow \mathcal{M}$ be the $*$ -homomorphism given by $\alpha(X_i) = x_i$. Then $\phi \circ \alpha$ is a limit of convex combinations of uniformly bounded equidistributed free product states with respect to $(x_i)_{i \in I}$ if and only if the tail algebra $\mathcal{N}$ lies in the center of $\mathcal{M}$.*

Proof. Suppose the tail algebra lies in the center of $\mathcal{M}$. Then by Theorem 2.1, $\mathcal{M}$ is the free product with amalgamation over the tail algebra $\mathcal{N}$, and $\phi = \phi|_{\mathcal{N}} \circ E$, where $E : \mathcal{M} \rightarrow \mathcal{N}$ is the ϕ -preserving conditional expectation with respect to which the algebras $\mathcal{A}_i = W^*(\mathcal{N} \cup \{x_i\})$ are free over $\mathcal{N}$. Regarding $\mathcal{N}$ as a commutative C^* -algebra, by the Gelfand theorem we have $\mathcal{N} \cong C(\Omega)$. Then, by a classical result, every state on $C(\Omega)$ lies in the closed convex hull of the set of point evaluation maps $\{\text{ev}_\omega \mid \omega \in \Omega\}$. By Lemma 4.2, every functional $\text{ev}_\omega \circ E \circ \alpha$ is an equidistributed free product state on $\mathbf{C}\langle X \rangle$ and clearly the boundedness criterion is satisfied with constants $C_i = \|x_i\|$. Taking convex combinations of the ev_ω that approximate $\phi|_{\mathcal{N}}$, we easily see that $\phi \circ \alpha$ is a limit of convex combinations of equidistributed free product states.

Conversely, suppose that $\phi \circ \alpha$ is a limit of convex combinations of uniformly bounded equidistributed free product states. Consider the C^* -subalgebra $C^*(\{x_i \mid i \in I\})$ of $\mathcal{M}$ generated by the x_i . By Proposition 4.6, there is a unital C^* -algebra B and a unital subalgebra D of the center of B with a faithful conditional expectation $E : B \rightarrow D$ and a unital, injective $*$ -homomorphism

$$\pi : C^*(\{x_i \mid i \in I\}) \rightarrow B$$

so that the elements $(\pi(x_i))_{i \in I}$ are free with respect to E and so that the moments of $\pi(x_i)$ are independent of i . Furthermore, there is a faithful state σ on D so that $\sigma \circ E \circ \pi = \phi|_{\mathfrak{A}}$. Let $\mathfrak{A}$ be the C^* -subalgebra of B generated by $D \cup \{\pi(x_i) \mid i \in I\}$. Then $\mathfrak{A}$ is an amalgamated free product of C^* -algebras $C^*(D \cup \{x_i\})$, with amalgamation over D . We take the Hilbert space representation ρ of $\mathfrak{A}$ that is the GNS construction for the restriction of the state $\sigma \circ E$ to $\mathfrak{A}$. The strong-operator-topology closure of $\rho(\mathfrak{A})$ is a von Neumann algebra, $\mathcal{Q}$, that is isomorphic to a free product

$$(11) \quad (*_{\mathcal{D}})_{i \in I}(W^*(\mathcal{D} \cup \{\rho(x_i)\}, \overline{E}))$$

with amalgamation over the strong-operator-topology closure $\mathcal{D}$ of $\rho(D)$. Taking strong-operator-topology limits (using Kaplansky's density theorem), we easily see that $\mathcal{D}$ lies in the center of $\mathcal{Q}$. Since $\sigma \circ E$ and ϕ are faithful, the composition $\rho \circ \pi$, when compressed to a Hilbert subspace, equals the GNS construction of the restriction of ϕ to $\mathfrak{A}$. Therefore, this mapping of $\mathfrak{A}$ into the strong-operator-closure of $\mathfrak{A}$ is canonically isomorphic to the inclusion of $\mathfrak{A}$ in $\mathcal{M}$, and we may regard $\mathcal{M}$ as embedded in the amalgamated free product (11). By Proposition 3.1, the tail algebra of $\{x_i \mid i \in I\}$ lies in $\mathcal{D}$. Thus, the tail algebra is in the center of $\mathcal{M}$. $\square$

The following easy example shows that the tail algebra can be commutative without lying in the center of the algebra generated by the quantum exchangeable sequence.

Example 4.8. Let $\mathcal{B} = \mathbf{C} \oplus \mathbf{C}$ embed into $M_2(\mathbf{C})$ as the diagonal matrices. For each $i \in \mathbf{N}$, let $\mathcal{A}_i$ be a copy of $M_2(\mathbf{C})$ and let $E_i : \mathcal{A}_i \rightarrow \mathcal{B}$ be the conditional expectation taking a matrix to its diagonal. Let

$$(\mathcal{M}, E) = (*_{\mathcal{B}})_{i=1}^{\infty}(\mathcal{A}_i, E_i)$$

be the amalgamated free product of von Neumann algebras. We note that $\mathcal{M}$ is easily seen to be isomorphic to the free group factor $L(\mathbf{F}_{\infty})$. Let ψ be any faithful state on $\mathcal{B}$. Then $\phi = \psi \circ E$ is a normal faithful state on $\mathcal{M}$. Fix $0 < t < 1/2$ and let x_i be the copy of the projection $\begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$ in $\mathcal{A}_i$. By Proposition 2.2, the sequence $(x_i)_{i=1}^{\infty}$ is quantum exchangeable. Applying Proposition 3.1, we see that the tail algebra $\mathcal{N}$ of the sequence is $\mathcal{B} \cong \mathbf{C} \oplus \mathbf{C}$ and, incidentally, the von Neumann algebra generated by the sequence $\{x_i \mid i \in \mathbf{N}\}$ is all of $\mathcal{M}$.

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843-3368

E-mail address: kdykema@math.tamu.edu

INSTITUTE OF MATHEMATICAL AND PHYSICAL SCIENCES, ABERYSTWYTH UNIVERSITY, ABERYSTWYTH SY23 3BZ, WALES, UNITED KINGDOM

E-mail address: cck@aber.ac.uk