DIMENSION FREE BOUNDEDNESS OF RIESZ TRANSFORMS FOR THE GRUSHIN OPERATOR

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Abstract. Let \( G = -\Delta_\xi - |\xi|^2 \frac{\partial^2}{\partial^2 \eta} \) be the Grushin operator on \( \mathbb{R}^n \times \mathbb{R} \). We prove that the Riesz transforms associated to this operator are bounded on \( L^p(\mathbb{R}^{n+1}) \), \( 1 < p < \infty \), and their norms are independent of dimension \( n \).

1. Introduction

We consider the Grushin operator \( G = -\Delta_\xi - |\xi|^2 \frac{\partial^2}{\partial^2 \eta} \) on \( \mathbb{R}^n \times \mathbb{R} \) which can be written formally as

\[
G f(\xi, \eta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda \eta} H(\lambda) f^\lambda(\xi) d\lambda,
\]

where

\[
H(\lambda) = -\Delta + \lambda^2 |\xi|^2
\]

is the scaled Hermite operator on \( \mathbb{R}^n \) and

\[
f^\lambda(\xi) = \int_{\mathbb{R}} f(\xi, \eta) e^{i\lambda \eta} d\eta
\]

is the inverse Fourier transform of \( f \) in the \( \eta \) variable. In the light of this decomposition, we can define the Riesz transforms associated to the operator \( G \) as

\[
R_j f(\xi, \eta) = \int_{-\infty}^{\infty} e^{-i\lambda \eta} R_j(\lambda) f^\lambda(\xi) d\lambda
\]

and

\[
R^*_j f(\xi, \eta) = \int_{-\infty}^{\infty} e^{-i\lambda \eta} R^*_j(\lambda) f^\lambda(\xi) d\lambda
\]

for \( j = 1, 2, 3, \ldots, n \). Here,

\[
R_j(\lambda) = A_j(\lambda) H(\lambda)^{-\frac{1}{2}}, \quad R^*_j(\lambda) = A_j(\lambda)^* H(\lambda)^{-\frac{1}{2}}
\]

are the Riesz transforms associated to the Hermite operator which has the decomposition

\[
H(\lambda) = \frac{1}{2} \sum_{j=1}^{n} (A_j(\lambda) A_j(\lambda)^* + A_j(\lambda)^* A_j(\lambda)),
\]

where

\[
A_j(\lambda) = -\frac{\partial}{\partial \xi_j} + \lambda \xi_j, \quad A_j(\lambda)^* = \frac{\partial}{\partial \xi_j} + \lambda \xi_j
\]

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are the creation and annihilation operators on $\mathbb{R}^n$. The boundedness of these Riesz transforms on $L^p(\mathbb{R}^{n+1})$ is known \cite{JST,BG}. In \cite{BG}, the authors have proved the boundedness of Riesz transforms associated to a much larger class of smooth locally subelliptic diffusion operators on a smooth connected non-compact manifold. In \cite{JST} these Riesz transforms were treated as operator valued Fourier multipliers for $L^p(\mathbb{R}^n)$ valued functions on $\mathbb{R}$ and the boundedness was proved with the aid of a result of L. Weis \cite{Wei01} on the operator valued Fourier multipliers. The aim of this paper is to prove the following theorem about the dimension free boundedness of the vector of Riesz transforms $Rf$. That is, we consider the operator $R = (R_1, R_2, \cdots, R_n, R_1^*, R_2^*, \cdots, R_n^*)$ with

$$|Rf(\xi, \eta)| = \left(\sum_{j=1}^{n} |R_j f(\xi, \eta)|^2 + \sum_{j=1}^{n} |R_j^* f(\xi, \eta)|^2\right)^{1/2}$$

and prove:

**Theorem 1.1.** For each $1 < p < \infty$, there exists a constant $C_p$ independent of dimension $n$ such that for all $f \in L^p(\mathbb{R}^n \times \mathbb{R})$,

$$\|Rf\|_p \leq C_p \|f\|_p.$$
of \cite{ST12} for details. Then the proof of the Hermite Riesz transforms follows by looking at functions of the form $F(\xi, \eta) = f(\xi)e^{i\langle k, n \rangle}$ on $\mathbb{R}^n \times [0, 2\pi)$.

2. Riesz transforms for the Grushin operator

We first consider the individual Riesz transforms $R_j$ and $R_j^\epsilon$ for $j = 1, 2, \ldots, n$ and show that they satisfy dimension free bounds on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$. In order to do this we introduce the operators $R_j^\epsilon$ and $R_j^{\epsilon*}$, which we will call the truncated Riesz transforms. We only give details of $R_j^\epsilon$, as the other one is similar. Note that the Riesz transform $R_j(\lambda)$ associated to the Hermite operator $H(\lambda)$ can be written as

$$R_j(\lambda) = A_j(\lambda)H(\lambda)^{-1/2} = \frac{A_j(\lambda)}{\sqrt{\pi}} \int_0^\infty e^{-rH(\lambda)r^{-1/2}} dr.$$ 

Here $e^{-rH(\lambda)}$ is the Hermite semigroup. For $\epsilon > 0$, we define the truncated Riesz transforms $R_j^\epsilon(\lambda)$ by

$$R_j^\epsilon(\lambda) = \frac{A_j(\lambda)}{\sqrt{\pi}} \int_{\epsilon^2}^{1/\epsilon^2} e^{-rH(\lambda)r^{-1/2}} dr.$$ 

Then the truncated Riesz transforms $R_j^\epsilon$ for the Grushin operator are defined as

$$R_j^\epsilon f(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda\eta} R_j^\epsilon(\lambda) f^{\lambda}(\xi) d\lambda.$$ 

We first prove

**Proposition 2.1.** For every $f \in L^2(\mathbb{R}^{n+1})$, $R_j^\epsilon f \to R_j f$ in $L^2(\mathbb{R}^{n+1})$ as $\epsilon \to 0$.

**Proof.** It follows from the definition that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |R_j^\epsilon f(\xi, \eta) - R_j f(\xi, \eta)|^2 d\xi d\eta = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |R_j^\epsilon(\lambda) f^{\lambda}(\xi) - R_j(\lambda) f^{\lambda}(\xi)|^2 d\xi \right) d\lambda.$$ 

The proposition follows once we show that for every $\lambda \in \mathbb{R}^*$

$$\int_{\mathbb{R}^n} |R_j^\epsilon(\lambda) f^{\lambda}(\xi) - R_j(\lambda) f^{\lambda}(\xi)|^2 d\xi \to 0 \text{ as } \epsilon \to 0$$

and

$$\int_{\mathbb{R}^n} |R_j^\epsilon(\lambda) f^{\lambda}(\xi) - R_j(\lambda) f^{\lambda}(\xi)|^2 d\xi \leq 4 \int_{\mathbb{R}^n} |f^{\lambda}(\xi)|^2 d\xi.$$ 

To see these, we expand $f^{\lambda}$ in terms of scaled Hermite functions $\Phi^{\lambda}_{\alpha}$ (see \cite{Tha04} for a definition) and use the fact that

$$A_j(\lambda)\Phi^{\lambda}_{\alpha} = (2\alpha_j + 2)^{1/2} |\lambda|^{1/2} \Phi^{\lambda}_{\alpha + \epsilon_j},$$

where $e_j$ is the canonical unit vector of $\mathbb{R}^n$ with 1 in the $j^{th}$ entry and zero elsewhere. Then

$$R_j^\epsilon(\lambda) f^{\lambda}(\xi) - R_j(\lambda) f^{\lambda}(\xi)$$

$$= \sum_{\alpha \in \mathbb{N}^n} \frac{(2\alpha_j + 2)^{1/2} |\lambda|^{1/2}}{\sqrt{\pi}} \left( \int_{A_\epsilon} e^{-(2|\alpha|+n)|\lambda|r^{-1/2}} dr \right) (f^{\lambda}, \Phi^{\lambda}_{\alpha}) \Phi^{\lambda}_{\alpha + e_j}(\xi).$$
where $A_ε = (0, ε^2) ∪ (1/ε^2, ∞)$. From this, it follows that

$$
\|R_j^ε(λ) f^λ - R_j(λ) f^λ\|_2^2 \leq \frac{1}{π} \sum_{α∈N^n} \frac{2α_j + 2|λ|}{2|α| + n} |(f^λ, Φ^λ_α)|^2 \left( \int_{A_{ε,α}} e^{-r_1 r^{-1/2} dr} \right)^2,
$$

where $A_{ε,α} = (0, (2|α| + n)|λ|ε^2) ∪ ((2|α| + n)|λ|e^{-2}, ∞)$. From the above equation it is clear that

$$
\|R_j^ε(λ) f^λ - R_j(λ) f^λ\|_2 \leq 2\|f^λ\|_2.
$$

We also note that when

$$
f^λ(ξ) = \sum_{(2|α|+n)|λ|≤N} (f^λ, Φ^λ_α)Φ^λ_α(ξ)
$$
is a finite linear combination of $Φ^λ_α$,

$$
\|R_j^ε(λ) f^λ - R_j(λ) f^λ\|_2 \leq \frac{1}{π} \sum_{(2|α|+n)|λ|≤N} |(f^λ, Φ^λ_α)|^2 \left( \int_{(0,Nε^2)∪(Nε^2,∞)} e^{-r_1 r^{-1/2} dr} \right)^2,
$$

which goes to 0 as $ε → 0$. As such functions are dense in $L^2(ℝ^n)$, we get

$$
\|R_j^ε(λ) f^λ - R_j(λ) f^λ\|_2 → 0, \text{ as } ε → 0,
$$
for any $f ∈ L^2(ℝ^{n+1})$.

For the individual Riesz transforms, we have the following result:

**Theorem 2.2.** For $j = 1, 2, 3, \cdots, n$ we have

$$
\|R_j f\|_p + \|R_j^2 f\|_p ≤ C_p \|f\|_p, \quad 1 < p < ∞,
$$

for all $f ∈ L^p(ℝ^{n+1})$, where $C_p$ is independent of the dimension.

In order to prove this result, we claim that it is enough to prove

$$
\|R_j^ε f\|_p + \|R_j^ε^2 f\|_p ≤ C_p \|f\|_p
$$

for the truncated Riesz transforms. A proof of this will be given in section 5, where we will show that $(R_j^ε f)$ is Cauchy in $L^p(ℝ^{n+1})$ and hence there exists an operator $S_j$, bounded on $L^p(ℝ^{n+1})$ such that $R_j^ε f → S_j f$ as $ε → 0$. In view of Proposition 2.1, $R_j^ε f → R_j f$ for $f ∈ L^2(ℝ^{n+1})$, and hence $S_j f = R_j f$. This will prove the stated boundedness of $R_j$ (and $R_j^ε$). Now to prove the boundedness of $R_j^ε$ and $R_j^ε^2$ we express these operators as a superposition of certain truncated Hilbert transforms.

3. A REPRESENTATION FOR THE TRUNCATED RIEZ TRANSFORMS

The representation we obtain is very similar to the representation obtained in [CNZ96] for the Riesz transforms on the Heisenberg group $H^n$. Before stating this result, we recall some definitions and notation. For further details and proofs we refer to [Tha04].

Recall that as a manifold $H^n = ℂ^n × ℝ$, and hence we write $(z, t), z = x + iy ∈ ℂ^n, t ∈ ℝ,$ to denote the elements of $H^n$. The sub-Laplacian $L$ on the Heisenberg group $H^n$ can be written in terms of the differential operators $X_j = \left( \frac{∂}{∂x_j} + \frac{1}{2} y_j \frac{∂}{∂t} \right)$
and $Y_j = \left(\frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial r}\right)$ as $\mathcal{L} = \sum_{j=1}^{n}(X_j^2 + Y_j^2)$. The sub-Laplacian is homogeneous of degree 2 with respect to the non-isotropic dilations $\delta_r(z, t) = (rz, r^2t)$ of the Heisenberg group. Hence $p_s(z, t)$, the heat kernel associated to $\mathcal{L}$, satisfies

$$p_{r^2s}(z, t) = r^{-(2n+2)}p_s(z/r, t/r^2).$$

This also follows from the following explicit expression for $q_s(z, \lambda)$, the inverse Fourier transform of $p_s(z, t)$ in the $t$ variable:

$$q_s(z, \lambda) = \int_{\mathbb{R}} p_s(z, t) e^{i\lambda t} dt = (4\pi)^{-n} \left(\frac{\lambda}{\sinh \lambda s}\right)^n e^{-\frac{1}{4} \lambda \coth(s\lambda)|z|^2}.$$

By $\pi_\lambda$, we denote the Schrödinger representation of the Heisenberg group $\mathbb{H}^n$ acting on $L^2(\mathbb{R}^n)$ in the following manner:

$$\pi_\lambda(x + iy, t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x, \xi + \frac{1}{2}x \cdot y)} \phi(\xi + y).$$

When $t = 0$, we will denote $\pi_\lambda(z, t)$ by $\pi_\lambda(z)$. $\pi_\lambda$ also defines a representation of the group algebra $L^1(\mathbb{H}^n)$ as

$$\pi_\lambda(f) = \int_{\mathbb{H}^n} f(z, t) \pi_\lambda(z, t) dz dt = W_\lambda(f^\lambda),$$

where $W_\lambda(g) = \int_{\mathbb{C}^n} g(z) \pi_\lambda(z) dz$ is the Weyl transform of $g$. The representation given in [CMZ96] for the Riesz transforms associated to the sub-Laplacian $\mathcal{L}$ is

$$(X_i\mathcal{L}^{-\frac{1}{2}} f)(z, t) = -\frac{1}{4(2\pi)^{n+1}} \int_{\mathbb{H}^n} X_i p_1(w, s) H_{(w, s)} f(z, t) dw ds,$$

where $H_{(w, s)}$ is the Hilbert transform along a curve in the Heisenberg group. For the Grushin operator, we obtain a similar representation involving certain operators $T_\epsilon^{(z, t)}$ and certain differential operators $\tilde{Z}_j = i\tilde{X}_j + \tilde{Y}_j$; $\tilde{Z}_j^* = i\tilde{X}_j - \tilde{Y}_j$

with

$$\tilde{X}_j = \left(\frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial t}\right); \tilde{Y}_j = \left(\frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial t}\right).$$

**Proposition 3.1.**

$$R_j f(\xi, \eta) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{H}^n} T_\epsilon^{(z, t)} f(\xi, \eta) \tilde{Z}_j p_1(z, t) dz dt$$

and

$$R_j^{*\epsilon} f(\xi, \eta) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{H}^n} T_\epsilon^{(z, t)} f(\xi, \eta) \tilde{Z}_j^* p_1(z, t) dz dt,$$

where

$$T_\epsilon^{(z, t)} f(\xi, \eta) = \int_{|r| < \epsilon} f(\xi + ry, \eta + rx \cdot \xi + r^2 \left(t + \frac{x \cdot y}{2}\right)) \frac{dr}{r}.$$

We follow the method in [LP06] for proving this proposition. First we prove the following lemma, which will be used in the proof of Proposition 3.1.

**Lemma 3.2.** For any $\phi \in S(\mathbb{R}^n)$,

(i) $r \frac{\partial}{\partial x_j} (\pi_\lambda(rz) \phi(\xi)) = \left(\frac{\partial}{\partial y_j} + \frac{i\lambda r^2 x_j}{2}\right) (\pi_\lambda(rz) \phi(\xi)),$

(ii) $ir \lambda \xi_j \pi_\lambda(rz) \phi(\xi) = \left(\frac{\partial}{\partial x_j} - \frac{i\lambda r^2 y_j}{2}\right) \pi_\lambda(rz) \phi(\xi).$
Proof. We first look at the derivative of $\pi_\lambda(rz)$ with respect to the variable $y_j$ and see that
\[
\frac{\partial}{\partial y_j} \left( \pi_\lambda(rz) \phi(\xi) \right) = \frac{\partial}{\partial y_j} \left( e^{i\lambda(rx+\frac{1}{2}rz^2y)} \phi(\xi + ry) \right) = \frac{i\lambda r^2 x_j}{2} \pi_\lambda(rz) \phi(\xi) + e^{i\lambda(rx+\frac{1}{2}rz^2y)} \frac{\partial}{\partial y_j} \phi(\xi + ry).
\]
Writing
\[
\pi_\lambda(rz) \frac{\partial}{\partial \xi_j} (\phi(\xi)) = \left( \pi_\lambda(rz) \frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial \xi_j} \pi_\lambda(rz) \right) \phi(\xi) + \frac{\partial}{\partial \xi_j} \pi_\lambda(rz) \phi(\xi)
\]
and noting that
\[
\frac{\partial}{\partial \xi_j} \left( \pi_\lambda(rz) \phi(\xi) \right) = i\lambda r x_j \pi_\lambda(rz) \phi(\xi) + \pi_\lambda(rz) \frac{\partial}{\partial \xi_j} \phi(\xi),
\]
we see that
\[
\frac{\partial}{\partial y_j} \left( \pi_\lambda(rz) \phi(\xi) \right) = \left( r \frac{\partial}{\partial \xi_j} - \frac{i\lambda r^2 x_j}{2} \right) \left( \pi_\lambda(rz) \phi(\xi) \right),
\]
which proves part (i). Part (ii) of the lemma follows from
\[
\frac{\partial}{\partial x_j} \left( \pi_\lambda(rz) \phi(\xi) \right) = i\lambda r \xi_j \pi_\lambda(rz) \phi(\xi) + \frac{i\lambda r^2 y_j}{2} \pi_\lambda(rz) \phi(\xi).
\]
\[\square\]

Proof of Proposition 3.1. The heat kernels for the sub-Laplacian and the Hermite operators are related via the group Fourier transform on the Heisenberg group as follows:
\[
p_\lambda(z) = \int_{\mathbb{H}_n} p_\lambda(z, t) \pi_\lambda(z, t) \, dz \, dt = e^{-sH(\lambda)}.
\]
We refer to [Tha04] for a proof of this. Using the homogeneity of the heat kernel we get
\[
e^{-r^2H(\lambda)} = r^{-(2n+2)} \int_{\mathbb{H}_n} \pi_\lambda(z, t) p_1 \left( \frac{z}{r}, \frac{t}{r^2} \right) \, dz \, dt = r^{-2n} \int_{\mathbb{C}_n} \pi_\lambda(z) q_1 \left( \frac{z}{r}, -\lambda r^2 \right) \, dz = \int_{\mathbb{C}_n} \pi_\lambda(rz) q_1 \left( z, -\lambda r^2 \right) \, dz.
\]
Since $R_\lambda^f(\lambda) = \frac{1}{\sqrt{\pi}} A_j(\lambda) \int_{|z| < 1/\epsilon} e^{-r^2H(\lambda)} \, dr$, we see that
\[
R_\lambda^f(\lambda) = \frac{1}{\sqrt{\pi}} \int_{|z| < 1/\epsilon} \int_{\mathbb{C}_n} \left( -\frac{\partial}{\partial \xi_j} + \lambda \xi_j \right) \pi_\lambda(rz) q_1 \left( z, -\lambda r^2 \right) \, dz \, dr,
\]
\[
R_\lambda^{*f}(\lambda) = \frac{1}{\sqrt{\pi}} \int_{|z| < 1/\epsilon} \int_{\mathbb{C}_n} \left( \frac{\partial}{\partial \xi_j} + \lambda \xi_j \right) \pi_\lambda(rz) q_1 \left( z, -\lambda r^2 \right) \, dz \, dr.
\]
Now using the previous lemma,
\[
ir \int_{\mathbb{C}^n} \lambda \xi_j \pi_\lambda(rz) q_1 \left( \frac{z}{r}, -\lambda r^2 \right) dz
\]
\[
= \int_{\mathbb{C}^n} \left( \frac{\partial}{\partial x_j} - \frac{i \lambda r^2 y_j}{2} \right) \pi_\lambda(rz) q_1(z, -\lambda r^2) dz
\]
\[
= - \int_{\mathbb{C}^n} \pi_\lambda(rz) \left( \frac{\partial}{\partial x_j} + \frac{i \lambda r^2 y_j}{2} \right) \left( \int_{\mathbb{R}} p_1(z, t) e^{i \lambda r^2 t} dt \right) dz
\]
\[
= - \int_{\mathbb{C}^n} \pi_\lambda(rz) \left( \int_{\mathbb{R}} \left( \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial t} \right) p_1(z, t) e^{i \lambda r^2 t} dt \right) dz
\]
\[
= - \int_{\mathbb{H}^n} \pi_\lambda(rz) \hat{X}_j p_1(z, t) e^{i \lambda r^2 t} dz dt.
\]

Similarly,
\[
r \int_{\mathbb{C}^n} \left( \frac{\partial}{\partial \xi_j} \pi_\lambda(rz) \right) q_1 \left( \frac{z}{r}, -\lambda r^2 \right) dz
\]
\[
= \int_{\mathbb{C}^n} \left( \frac{\partial}{\partial y_j} + \frac{i \lambda r^2 x_j}{2} \right) \pi_\lambda(rz) q_1(z, -\lambda r^2) dz
\]
\[
= - \int_{\mathbb{H}^n} \pi_\lambda(rz) \hat{Y}_j p_1(z, t) e^{i \lambda r^2 t} dz dt.
\]

Hence
\[
R_j^e(\lambda) = \frac{1}{\sqrt{\pi}} \int_{|r|<1/\varepsilon} \int_{\mathbb{H}^n} \pi_\lambda(rz) \hat{Z}_j p_1(z, t) e^{i \lambda r^2 t} dz dt \frac{dr}{r}
\]
and
\[
R_j^{e*}(\lambda) = \frac{1}{\sqrt{\pi}} \int_{|r|<1/\varepsilon} \int_{\mathbb{H}^n} \pi_\lambda(rz) \hat{Z}_j^* p_1(z, t) e^{i \lambda r^2 t} dz dt \frac{dr}{r}.
\]

Thus
\[
R_j^e f(\xi, \eta) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \left( \int_{|r|<1/\varepsilon} \int_{\mathbb{H}^n} \pi_\lambda(rz) f^\lambda(\xi) \hat{Z}_j p_1(z, t) e^{i \lambda r^2 t} dz dt \frac{dr}{r} \right) e^{i \lambda \eta} d\lambda.
\]

Now the proposition follows from the fact that
\[
\int_{\mathbb{R}} \pi_\lambda(rz) f^\lambda(\xi) e^{i \lambda r^2} e^{i \lambda \eta} d\lambda = f \left( \xi + ry, \eta + rx \cdot \xi + r^2 \left( t + \frac{x \cdot y}{2} \right) \right).
\]

\[ \square \]

4. A TRANSFERENCE RESULT

In the previous section we obtained a representation for the Riesz transforms as the superposition of certain operators $T_{(z,t)}^\varepsilon$. To prove the boundedness of the vector of Riesz transforms, using the method of rotations, we need to prove that these operators are bounded uniformly in $(z, t)$. 

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Proposition 4.1. For $1 < p < \infty$, there exists a constant $C_p$ independent of $(z,t) \in \mathbb{H}^n$, $\epsilon > 0$ and the dimension $n$ such that

$$\left\| T_\epsilon^{(z,t)} f \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\| f \right\|_{L^p(\mathbb{R}^{n+1})}.$$  

Proof. This will be proved using Calderón’s method of transferring an operator on $L^p(X)$, for a measure space $X$, to $L^p(G)$ when the group $G$ acts on $X$ by measure preserving transformations. The measure space $\mathbb{R}^{n+1}$, the group $\mathbb{H}^n$ and the group action

$$U(x + iy, t)(\xi, \eta) = (\xi - y, \eta - t + \frac{x \cdot y}{2} - x, \xi)$$

are all the same as those used by Ratnakumar and Thangavelu in [RT98], and we follow their notation. Accordingly, for $f \in L^p(\mathbb{R}^{n+1})$ and $(\xi, \eta) \in \mathbb{R}^{n+1}$, we define the transferred function $F_{(\xi, \eta)}$ on $\mathbb{H}^n$ as

$$F_{(\xi, \eta)}(z, t) = f(U(z, t)(\xi, \eta)).$$

For $T \in \mathcal{B}(L^p(\mathbb{H}^n))$, the transferred operator $T_0$ on $\mathcal{B}(L^p(\mathbb{R}^{n+1}))$ is defined as

$$T_0 f(\xi, \eta) = (TF_{(\xi, \eta)})(0).$$

For a curve $\gamma = \{ \gamma(t) \in \mathbb{H}^n, t \in \mathbb{R} \}$ and a function $f$ on $\mathbb{H}^n$, the Hilbert transform of $f$ along $\gamma$ is defined as

$$H_{\gamma} f(w, s) = \int_{\mathbb{R}} f((w, s) \gamma(r)^{-1}) \frac{dr}{r}.$$

When $\gamma$ is the curve $(rz, r^2 t)$, we will denote $H_{\gamma}$ by $H_{(z, t)}$. By $H_{(z, t)}^\epsilon$, we denote the truncated Hilbert transform where the integration is only over $\{ r \in \mathbb{R}, \epsilon < |r| < 1/\epsilon \}$. When we transfer this operator $H_{(z, t)}^\epsilon$ to an operator on $L^p(\mathbb{R}^{n+1})$ under the $U$ action defined above, we get

$$H_{(z, t)}^\epsilon \gamma f(\xi, \eta) = \int_{\epsilon < |r| < 1/\epsilon} F_{(\xi, \eta)}(-rz, -r^2 t) \frac{dr}{r}$$

$$= \int_{\epsilon < |r| < 1/\epsilon} f \left( \xi + ry, \eta + r(x \cdot \xi) + r^2 \left( t + \frac{x \cdot y}{2} \right) \right) \frac{dr}{r}.$$

Hence we see that $(H_{(z, t)}^\epsilon \gamma) f = T_\epsilon^{(z, t)}$. Now our aim is to prove

$$\left\| T_\epsilon^{(z,t)} f \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\| f \right\|_{L^p(\mathbb{R}^{n+1})}.$$  

This can be obtained using the transference method from the uniform boundedness of the operator $H_{(z, t)}^\epsilon$. To be precise, we use the fact that there exist a finite constant $C_p$, independent of $(z,t) \in \mathbb{H}^n$, $n$ and $\epsilon > 0$ such that

$$\left\| H_{(z, t)}^\epsilon f \right\|_p \leq C_p \left\| f \right\|_p.$$  

A proof of the above fact is indicated in [CMZ96]. We will present it at the end of this section after proving the boundedness of $T_\epsilon^{(z,t)}$. Then

$$\int_{\mathbb{R}^{n+1}} |T_\epsilon^{(z,t)} f(\xi, \eta)|^p d\xi d\eta = \int_{\mathbb{R}^{n+1}} |H_{(z, t)}^\epsilon f(\xi, \eta)(0)|^p d\xi d\eta$$

$$= \int_{\mathbb{R}^{n+1}} |H_{(z, t)}^\epsilon F_{U(w, s)(\xi, \eta)(0)}|^p d\xi d\eta.$$
because of the invariance of the measure $d\xi \, d\eta$ under the $U$ action. Hence
\[
\int_{\mathbb{R}^{n+1}} |T_{\epsilon}^\gamma (z,t) f(\xi,\eta)|^p \, d\xi \, d\eta = \frac{1}{|B_R(0)|} \int_{B_R(0)} \int_{\mathbb{R}^{n+1}} |H_{\epsilon}^\gamma (z,t) F_U (w,s)(\xi,\eta)(0)|^p \, d\eta \, dw \, ds
\]
\[
= \frac{1}{|B_R(0)|} \int_{B_R(0)} \int_{\mathbb{R}^{n+1}} |H_{\epsilon}^\gamma (z,t) F_{\xi,\eta} (w,s)|^p \, d\eta \, dw \, ds,
\]
where $B_R(0)$ is the ball centred at the origin in the Heisenberg group and radius $R$ under the Koranyi norm $\|(w,s)\| = (|w| + |s|^2)^{1/4}$. When $(w,s) \in B_R(0)$ and $\epsilon < |r| < 1/\epsilon, (w,s) (\delta_r (z,t))^{-1} \in B_{R+\epsilon^2} (0)$, where $h$ is the Koranyi norm of $(z,t)$.

Hence, in the above equality $F_{\xi,\eta}$ can be replaced by $\tilde{F}_{\xi,\eta} = F_{\xi,\eta} \cdot \chi_{B_{R+h/\epsilon}} (0)$. Now by an application of Fubini, we get
\[
\|T_{\epsilon}^\gamma (z,t) f\|_p \leq \frac{1}{|B_R(0)|} \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^n} |H_{\epsilon}^\gamma (z,t) \tilde{F}_{\xi,\eta} (w,s)|^p \, dw \, ds \, d\xi \, d\eta.
\]

Now, from the uniform boundedness of the truncated Hilbert transforms, we get
\[
\|T_{\epsilon}^\gamma (z,t) f\|_p \leq C_p \frac{1}{|B_R(0)|} \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^n} |\tilde{F}_{\xi,\eta} (w,s)|^p \, dw \, ds \, d\xi \, d\eta
\]
\[
= C_p \frac{1}{|B_R(0)|} \int_{B_{R+h/\epsilon}} \int_{\mathbb{R}^{n+1}} |f (U (w,s)(\xi,\eta))|^p \, d\eta \, dw \, ds
\]
\[
= C_p \left( \frac{R + h/\epsilon}{R} \right)^{2n+2} \|f\|_p^p,
\]
again by the invariance of $d\xi \, d\eta$ under the $U$ action. Letting $R \to \infty$, we see that
\[
\|T_{\epsilon}^\gamma (z,t) f\|_{L^p (\mathbb{R}^n)} \leq C_p \|f\|_{L^p (\mathbb{R}^n)}.
\]

Coming back to the boundedness of $H_{\epsilon}^\gamma (z,t)$, we can use a technique in Lemma 3.1 of [Str91] to reduce this to the boundedness of $H_{\gamma}^\epsilon$ on $L^p (\mathbb{R}^2)$ for the curve $\gamma = \{(t,t^2), t \in \mathbb{R}\}$.

Recall that
\[
H_{z,t} f (w,s) = \int_{\mathbb{R}} f (w - rz, s - r^2 t - r \text{Im} (w \bar{z})) \frac{dr}{r}.
\]

For $\sigma \in U(n)$, define $\rho(\sigma) f (w,s) = f (\sigma w, s)$. Then
\[
H_{z,t} (\rho(\sigma) f) (w,s) = \int_{\mathbb{R}} f (\sigma w - rz, s - r^2 t - r \text{Im} (w \bar{z})) \frac{dr}{r},
\]
\[
\rho(\sigma^{-1}) H_{z,t} (\rho(\sigma) f) (w,s) = H_{z,t} (\rho(\sigma) f) (\sigma^{-1} w, s)
\]
\[
= \int_{\mathbb{R}} f (w - rz, s - r^2 t - r \text{Im} (\sigma^{-1} w \bar{z})) \frac{dr}{r}
\]
\[
= \int_{\mathbb{R}} f (w - rz, s - t - r \text{Im} (w \sigma \bar{z})) \frac{dr}{r}.
\]

Since $\|\rho(\sigma) f\|_p = \|f\|_p$ and since there exists $\sigma \in U(n)$ such that $\rho(\sigma) (x + iy) = x_1 e_1$, it is enough to consider the operator
\[
f \to \int_{\mathbb{R}} f (u_1 - r x_1, w, s - r^2 t - rv_1 x_1) \frac{dr}{r},
\]
which is equivalent to the operator
\[
T_{(x,t,v)}f(u,s) = \int_{\mathbb{R}} f(u - rx, s - r^2 t - rvx) \frac{dr}{r}
\]
acting on functions defined on \(\mathbb{R}^2\). For \(\lambda_1, \lambda_2 > 0\), let
\[
\delta_{\lambda_1, \lambda_2}f(u, s) = f(\lambda_1 u, \lambda_2 s).
\]
Then \(\|\delta_{\lambda_1, \lambda_2}f\|_p = (\lambda_1 \lambda_2)^{-1/p} \|f\|_p\) and
\[
\delta_{1/\lambda_1, 1/\lambda_2} T_{(x,t,v)} \delta_{\lambda_1, \lambda_2} f(u, s) = \int_{\mathbb{R}} f(u - r \lambda_1 x, s - r^2 \lambda_2 t - r \lambda_2 vx) \frac{dr}{r}.
\]
Since we can choose \(\lambda_1, \lambda_2\) such that \(\lambda_1 x = \lambda_2 t = 1\), it is enough to get uniform estimates for operators of the form \(T_{(1,1,a)}\). That would imply that
\[
\|T_{(x,t,v)}f\|_p = \|\delta_{\lambda_1, \lambda_2} T_{(1,1,vx/t)} \delta_{1/\lambda_1, 1/\lambda_2} f\|_p
= (\lambda_1 \lambda_2)^{-1/p} \|T_{(1,1,vx/t)} \delta_{1/\lambda_1, 1/\lambda_2} f\|_p
\leq (\lambda_1 \lambda_2)^{-1/p} C_p \|\delta_{1/\lambda_1, 1/\lambda_2} f\|_p
= (\lambda_1 \lambda_2)^{-1/p} C_p (\lambda_1 \lambda_2)^{1/p} \|f\|_p.
\]
Now consider \(\tau_a f(u, s) = f(u, s + au)\) so that \(\|\tau_a f\|_p = \|f\|_p\). Since
\[
T_{(1,1,a)}(\tau_a^{-1} f)(u, s) = \int_{\mathbb{R}} (\tau_a^{-1} f)(u - r, s - r^2 - ar) \frac{dr}{r}
= \int_{\mathbb{R}} f(u - r, s - au - r^2) \frac{dr}{r},
\]
we have
\[
\tau_a T_{(1,1,a)}(\tau_a^{-1} f)(u, s) = T_{(1,1,a)}(\tau_a^{-1} f)(u, s + au)
= \int_{\mathbb{R}} f(u - r, s - r^2) \frac{dr}{r}.
\]
This is the Hilbert transform along the parabola \(\gamma(r) = (r, r^2)\) in \(\mathbb{R}^2\). Since
\[
\gamma(r) = \begin{cases} 
\delta_+(1,1) & \text{if } r > 0, \\
0 & \text{if } r = 0, \\
\delta_-(1,1) & \text{if } r < 0
\end{cases}
\]
and the linear space spanned by \(\{\gamma(r)\}_{r > 0}\) is the same as the linear space spanned by \(\{\gamma(r)\}_{r < 0}\), namely \(\mathbb{R}^2\), \(\gamma(r)\) is a two-sided homogeneous curve (see Definition 3.1 in [SW78], p. 1261). Now we can appeal to Theorem 11 of [SW78], p. 1271] to get the uniform boundedness of the truncated Hilbert transforms \(H^\epsilon_{(z,t)}\) on \(L^p(\mathbb{H}^n)\). That is, there exist a finite constant \(C_p\), independent of \((z,t) \in \mathbb{H}^n\), \(n\) and \(\epsilon > 0\), such that
\[
\|H^\epsilon_{(z,t)} f\|_p \leq C_p \|f\|_p.
\]
\[\square\]
5. Proof of the main theorem

We first prove Theorem 2.2 regarding the boundedness of the individual Riesz transforms $R_j$ and $R^*_j$.

Proof of Theorem 2.2. We first consider the operators $R_j^s$ and $R^*_j$. In light of Propositions 3.1 and 4.1, we just need to prove that there exist a finite $C$, independent of $n$, such that

$$\|\tilde{Z}_j p_1(z,t)\|_{L^1(\mathbb{H}^n)}, \|\tilde{Z}_j^* p_1(z,t)\|_{L^1(\mathbb{H}^n)} \leq C.$$  

As mentioned in Lemma 3 of [CMZ96], this follows from the fact that

$$p^n(z_1, z_2, \cdots, z_n, t) = p^1(z_1, \cdot) * p^1(z_2, \cdot) * \cdots * p^1(z_n, \cdot)(t),$$

where $p^n$ is the heat kernel on $\mathbb{H}^n$. Now for the operators $R_j$ and $R^*_j$, we need only to show that $(R^s_j f)$, $(R^*_j f)$ are Cauchy in $L^p(\mathbb{R}^{n+1})$. We have already seen that $\|R^s_j f\|_p \leq C\|T^{(z,t)}_\epsilon f\|_p$ and so it is enough to prove that $T^{(z,t)}_\epsilon f$ is Cauchy in $L^p(\mathbb{R}^{n+1})$. During the course of the proof of Proposition 4.1, we had observed that the operator $T^{(z,t)}_\epsilon$ is obtained by applying a transference on $H^\epsilon_{(z,t)}$ and that

$$\|T^{(z,t)}_\epsilon f\|_{L^p(\mathbb{R}^{n+1})} \leq \|H^\epsilon_{(z,t)} f\|_{L^p(\mathbb{H}^n)}.$$  

Being truncated Hilbert transforms, $H^\epsilon_{(z,t)} f$ is Cauchy in $L^p(\mathbb{H}^n)$ [SW78 Theorem 11], and consequently, so is $T^{(z,t)}_\epsilon f$. \hfill \Box

Now we complete the proof of our main theorem. As mentioned during the discussion of Theorem 2.2, we only need to prove the boundedness of the operator $R^\epsilon$ obtained by replacing $R_j$ and $R^*_j$ by their truncated versions. We will be closely following [LP04] in proving this and so will skip some details. Using the property

$$\frac{1}{r} \frac{\partial p_1}{\partial r} = \frac{1}{x_j} \frac{\partial p_1}{\partial x_j} = \frac{1}{y_j} \frac{\partial p_1}{\partial y_j},$$

where $r = |z|$, we can rewrite

$$R^\epsilon_j f(\xi, \eta) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{H}^n} (ix_j + y_j) T^{(z,t)}_\epsilon f(\xi, \eta) \frac{1}{r} \frac{\partial p_1}{\partial r}(z,t) dz dt$$

$$+ \frac{1}{2\sqrt{\pi}} \int_{\mathbb{H}^n} (x_j - y_j) T^{(z,t)}_\epsilon f(\xi, \eta) \frac{\partial p_1}{\partial t}(z,t) dz dt$$

and

$$R^*_\epsilon j f(\xi, \eta) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{H}^n} (ix_j - y_j) T^{(z,t)}_\epsilon f(\xi, \eta) \frac{1}{r} \frac{\partial p_1}{\partial r}(z,t) dz dt$$

$$- \frac{1}{2\sqrt{\pi}} \int_{\mathbb{H}^n} (x_j + y_j) T^{(z,t)}_\epsilon f(\xi, \eta) \frac{\partial p_1}{\partial t}(z,t) dz dt.$$  

For a fixed $(\xi, \eta)$ we can choose $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ such that $\sum_{j=1}^{2n} |\lambda_j|^2 = 1$ and

$$|R^\epsilon f(\xi, \eta)| = \sum_{j=1}^{2n} \left( \lambda_j R_{j}^\epsilon f(\xi, \eta) + \lambda_{j+n} R_{j}^{*\epsilon} f(\xi, \eta) \right).$$
Now using the triangle inequality, Hölder’s inequality and also Lemma 2(a) of [LP04], we get

\[ |R^\varepsilon f(\xi, \eta)| \leq C \|T^{(z,t)}_\varepsilon f(\xi, \eta)\|_{L^p(\frac{1}{p} \partial p \, dz \, dt)} \|x_1\|_{L^p(\frac{1}{p} \partial p \, dz \, dt)} + C \|T^{(z,t)}_\varepsilon f(\xi, \eta)\|_{L^p(\frac{1}{p} \partial p \, dz \, dt)} \|x_1\|_{L^p(\frac{1}{p} \partial p \, dz \, dt)}, \]

where \( C \) is a universal constant. Now the theorem follows from Proposition 4.1 and an application of Minkowski inequality along with the fact (proved in [LP04]) that

\[ \int_{\mathbb{H}^n} |x_1|^p \frac{1}{r} \frac{\partial p_1}{\partial r} \, dz \, dt, \quad \int_{\mathbb{H}^n} |x_1|^p \frac{\partial p_1}{\partial t} \, dz \, dt \leq A_p, \ p \geq 0, \]

where \( A_p \) is independent of the dimension.

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