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A FOURIER RESTRICTION THEOREM BASED ON CONVOLUTION POWERS

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Abstract. We prove a Fourier restriction estimate under the assumption that certain convolution power of the measure admits an r-integrable density.

Introduction

Let $\mathcal{F}$ be the Fourier transform defined on the Schwartz space by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\langle \xi, x\rangle} f(x)dx$$

where $\langle \xi, x\rangle$ is the Euclidean inner product. We are interested in Borel measures μ defined on $\mathbb{R}^d$ for which $\mathcal{F}$ maps $L^p(\mathbb{R}^d)$ boundedly to $L^2(\mu)$, i.e.

$$\|\hat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \forall f \in \mathcal{S}(\mathbb{R}^d). \tag{1}$$

Here "$\lesssim$" means the left-hand side is bounded by the right-hand side multiplied by a positive constant that is independent of f.

If μ is a singular measure, then such a result can be interpreted as a restriction property of the Fourier transform. Such restriction estimates for singular measures were first obtained by Stein in the 1960's. If μ is the surface measure on the sphere, the Stein-Tomas theorem [12], [13] states that (1) holds for $1 \leq p \leq \frac{2(d+1)}{d+3}$. Mockenhaupt [10] and Mitsis [9] have shown that Tomas's argument in [12] can be used to obtain an L^2-Fourier restriction theorem for a general class of finite Borel measures satisfying

$$|\hat{\mu}(\xi)|^2 \lesssim |\xi|^{-\beta}, \forall \xi \in \mathbb{R}^d, \tag{2}$$

$$\mu(B(x,r)) \lesssim r^{\alpha}, \forall x \in \mathbb{R}^d, r > 0, \tag{3}$$

where $0 < \alpha, \beta < d$; they showed that (1) holds for $1 \leq p < p_0 = \frac{4(d-\alpha)+2\beta}{4(d-\alpha)+\beta}$. Bak and Seeger [2] proved the same result for the endpoint p_0 and further strengthened it by replacing the L^{p_0}-norm with the $L^{p_0,2}$-Lorentz norm.

It is well known that if μ is the surface measure on a compact C^∞ manifold, then the sharpness can be tested by some version of Knapp's homogeneity argument. See e.g. the work by Iosevich and Lu [6], who proved that if μ is the surface measure on a compact hypersurface and if $\mathcal{F} : L^{p_0} \to L^2(\mu)$, $p_0 = \frac{2(d+1)}{d+3}$, then the Fourier decay assumption (2) is satisfied with $\alpha = d - 1$. For general measures satisfying

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(2) and (3), there is no Knapp argument available to prove the sharpness of p_0. Here we show that indeed for certain measures the restriction estimate (1) holds in a range of p beyond the range given above. This will follow from a restriction estimate based on an assumption on the n-fold convolution $\mu^{*n} = \mu * \cdots * \mu$.

Theorem 1. *Let μ be a Borel probability measure on $\mathbb{R}^d$, let $1 \leq r \leq \infty$ and assume that $\mu^{*n} \in L^r(\mathbb{R}^d)$. Let $1 \leq p \leq \frac{2n}{2n-1}$ if $r \geq 2$, and $1 \leq p \leq \frac{nr'}{nr'-1}$ if $1 \leq r \leq 2$, and let $1 \leq q \leq \frac{p'}{nr'}$. Then*

$$\|\hat{f}\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \forall f \in \mathcal{S}(\mathbb{R}^d). \tag{4}$$

Applying Theorem 1 with $n = 2$, $r = \infty$, one obtains the following.

Corollary 1. *Let μ be a Borel probability measure on $\mathbb{R}^1$ such that $\mu * \mu \in L^\infty(\mathbb{R}^1)$. Then* (1) *holds for* $1 \leq p \leq 4/3$.

Remarks. (i) It is not easy to construct measures supported on lower dimensional sets for which Corollary 1 applies. Remarkably, Körner showed by a combination of Baire category and probabilistic argument that there exist "many" Borel probability measures μ supported on compact sets of Hausdorff dimension 1/2 so that $\mu * \mu \in C_c(\mathbb{R}^1)$.

(ii) In Corollary 1, since $\mu * \mu$ satisfies (3) with $\alpha = 1$, μ satisfies (3) with $\alpha = 1/2$ (see Proposition 1). Suppose μ is supported on a compact set of Hausdorff dimension γ. It follows that $\gamma \geq 1/2$ (cf. [14], Proposition 8.2). Furthermore, if $\gamma < 1$, then β and α in (2) and (3) cannot exceed γ (cf. [14], Corollary 8.7).

(iii) Under the above situation, since $\alpha, \beta \leq \gamma$, the range of p in (1) obtained from [10], [9], [2] is no larger than $1 \leq p \leq \frac{6-4\epsilon}{5-6\epsilon}$ where $\epsilon = \gamma - 1/2$, while Corollary 1 gives the range $1 \leq p \leq 4/3$, which is an improvement if $\gamma < 2/3$. However, we do not know of any example of such a measure μ with β (and α) close to γ.

(iv) Suppose μ is as in Corollary 1 and is supported on a compact set of Hausdorff dimension 1/2. By Theorem 1, the restriction estimate (4) holds for $1 \leq p \leq 4/3$, $1 \leq q \leq p'/2$. By dimensionality considerations (see Proposition 2 and Proposition 4), these are all the possible exponents $1 \leq p, q \leq \infty$ for which (4) holds.

Proof of Theorem 1. The proof proceeds in a similar spirit as in [11], [4]. Fix a nonnegative function $\phi \in C_c^\infty(\mathbb{R}^d)$ that satisfies $\int_{\mathbb{R}^d} \phi(\xi) d\xi = 1$. Let $\phi_\epsilon(\xi) = \epsilon^{-d}\phi(\xi/\epsilon)$ and $\mu_\epsilon(\xi) = \phi_\epsilon * \mu(\xi) = \int_{\mathbb{R}^d} \phi_\epsilon(\xi - \eta) d\mu(\eta)$. Since μ_ϵ converges weakly to μ, we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^q \mu_\epsilon(\xi) d\xi = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^q d\mu(\xi)$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. Thus it suffices to show that

$$\|\hat{f}\|_{L^q(\mu_\epsilon)} \leq C \|f\|_{L^p(\mathbb{R}^d)},$$

where C is a constant independent of f and ϵ. By Hölder's inequality, we may assume $q = \frac{p'}{nr'}$. Set $s = p'/n$. Note that by our assumption on the range of p, $s \geq 2, q \geq 1$. By duality, we need to prove that

$$\Big(\int_{\mathbb{R}^d} |\widehat{g\mu_\epsilon}(x)|^{ns} dx \Big)^{1/ns} \leq C \Big(\int_{\mathbb{R}^d} |g(\xi)|^{q'} \mu_\epsilon(\xi) d\xi \Big)^{1/q'} \tag{5}$$

for all bounded Borel functions g. By the Hausdorff-Young inequality,

$$\begin{aligned}\Big(\int_{\mathbb{R}^d}|\widehat{g\mu_\epsilon}(x)|^{ns}dx\Big)^{1/s} &= \Big(\int_{\mathbb{R}^d}|\widehat{g\mu_\epsilon}^n(x)|^{s}dx\Big)^{1/s}\\ &\le \Big(\int_{\mathbb{R}^d}|g\mu_\epsilon*\cdots*g\mu_\epsilon(\xi)|^{s'}d\xi\Big)^{1/s'}\\ &= \Big(\int_{\mathbb{R}^d}|\int_{\mathbb{R}^{(n-1)d}}G(\xi,\eta)M_\epsilon(\xi,\eta)d\eta|^{s'}d\xi\Big)^{1/s'},\end{aligned}$$

where $\eta=(\eta_1,\ldots,\eta_{d-1})$, $\eta_0\equiv\xi$,

$$G(\xi,\eta)=g(\eta_{n-1})\prod_{j=1}^{n-1}g(\eta_{j-1}-\eta_j),$$

$$M_\epsilon(\xi,\eta)=\mu_\epsilon(\eta_{n-1})\prod_{j=1}^{n-1}\mu_\epsilon(\eta_{j-1}-\eta_j).$$

Now by Hölder's inequality for the inner integral,

$$\begin{aligned}&\Big(\int_{\mathbb{R}^d}|\int_{\mathbb{R}^{(n-1)d}}G(\xi,\eta)M_\epsilon(\xi,\eta)d\eta|^{s'}d\xi\Big)^{1/s'}\\ &\le\Big(\int_{\mathbb{R}^d}\Big(\mu_\epsilon^{*n}(\xi)\Big)^{s'/q}\Big(\int_{\mathbb{R}^{(n-1)d}}|G(\xi,\eta)|^{q'}M_\epsilon(\xi,\eta)d\eta\Big)^{s'/q'}d\xi\Big)^{1/s'}.\end{aligned}$$

Applying Hölder's inequality again, this is bounded by

$$\begin{aligned}&\|\mu_\epsilon^{*n}\|_r^{1/q}\Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^{(n-1)d}}|G(\xi,\eta)|^{q'}M_\epsilon(\xi,\eta)d\eta d\xi\Big)^{\frac{1}{s'}-\frac{1}{qr}}\\ &=\|\mu_\epsilon^{*n}\|_r^{1/q}\Big(\int_{\mathbb{R}^d}|g(\xi)|^{q'}\mu_\epsilon(\xi)d\xi\Big)^{n(\frac{1}{s'}-\frac{1}{qr})}\\ &\le\|\mu^{*n}\|_r^{1/q}\Big(\int_{\mathbb{R}^d}|g(\xi)|^{q'}\mu_\epsilon(\xi)d\xi\Big)^{n(\frac{1}{s'}-\frac{1}{qr})},\end{aligned}$$

where we have used Young's inequality in the last line. Since $\frac{1}{s'}-\frac{1}{qr}=\frac{1}{q'}$, we obtain (5) after taking the nth root. □

Appendix

For the sake of completeness, we include the proofs of the claims made in the remarks. Similar results can be found in [10] and [9].

Proposition 1. *Let μ be a Borel probability measure on $\mathbb{R}^d$. If μ^{*n} satisfies* (3) *with $0\le\alpha\le d$, then μ satisfies* (3) *with exponent α/n.*

Proof. Assume to the contrary that given k, $\mu(B_{r_k})\ge kr_k^{\alpha/n}$ for some ball B_{r_k} with radius $r_k>0$. Let $B^*_{nr_k}=B_{r_k}+\cdots+B_{r_k}$ be the n-fold Minkowski sum; then

$$\mu^{*n}(B^*_{nr_k})\ge\mu(B_{r_k})^n\ge k^nr_k^\alpha.$$

On the other hand, since μ^{*n} satisfies (3),

$$\mu^{*n}(B^*_{nr_k})\lesssim(nr_k)^\alpha\lesssim r_k^\alpha,\forall k.$$

Letting $k\to\infty$, we obtain a contradiction. □

Proposition 2. *Let μ be a Borel probability measure on $\mathbb{R}^d$ supported on a compact set of Hausdorff dimension $0 \le \gamma < d$; then*

$$\|\hat{\mu}\|_s = \infty, \forall 0 < s < \frac{2d}{\gamma}.$$

Proof. Assume to the contrary that $\|\hat{\mu}\|_s < \infty$ for some $2 < s < 2d/\gamma$. Then

$$\fint_{B(0,R)} |\hat{\mu}(\xi)|^2 d\xi \le \Big(\fint_{B(0,R)} |\hat{\mu}(\xi)|^s d\xi\Big)^{2/s} \lesssim R^{-2d/s}.$$

This decay in $R \to \infty$ implies $\gamma \ge 2d/s$ (cf. [14], Corollary 8.7). Since $2d/s > \gamma$, we obtain a contradiction. ☐

For the endpoint $s = \frac{2d}{\gamma}$ we have

Proposition 3. *Let μ be a Borel probability measure on $\mathbb{R}^d$ supported on a compact set K. Suppose $d/2 \le \gamma < d$ and there exists $C \ge 1$ so that*

$$C^{-1} r^\gamma \le \mu(B(x,r)) \le C r^\gamma \tag{6}$$

for all $x \in K$ and $0 < r < 1$. Then $\|\hat{\mu}\|_{\frac{2d}{\gamma}} = \infty$.

Proof. Assume to the contrary that $\|\hat{\mu}\|_{2d/\gamma} < \infty$. Let $\tilde{\mu}$ be the reflection of μ, i.e. $\tilde{\mu}(A) = \mu(-A)$ for Borel sets A. Then $\widehat{\mu * \tilde{\mu}} = |\hat{\mu}|^2 \in L^{d/\gamma}$. By the Hausdorff-Young inequality, this implies $\mu * \tilde{\mu} \in L^{(d/\gamma)'}$, and hence

$$\mu * \tilde{\mu}(B(0,\epsilon)) \lesssim \|\mu * \tilde{\mu}\|_{L^{(d/\gamma)'}(B(0,\epsilon))} \epsilon^\gamma. \tag{7}$$

On the other hand, by the upper regularity assumption in (6) we can find N_ϵ many disjoint balls B_j of radius $\epsilon/2$ centered in K with $N_\epsilon \gtrsim \epsilon^{-\gamma}$. Since the difference set $B_j - B_j \subset B(0,\epsilon)$, we have

$$\mu * \tilde{\mu}(B(0,\epsilon)) \gtrsim \sum_{j=1}^{N_\epsilon} \mu(B_j)^2 \gtrsim N_\epsilon \epsilon^{2\gamma} \gtrsim \epsilon^\gamma,$$

where we have used the lower regularity assumption in (6) in the second inequality. Comparing this with (7) and noticing that $\|\mu * \tilde{\mu}\|_{L^{(d/\gamma)'}(B(0,\epsilon))} \to 0$ as $\epsilon \to 0$, we obtain a contradiction. ☐

Proposition 4. *Let μ be a Borel probability measure on $\mathbb{R}^d$ supported on a compact set of Hausdorff dimension $0 < \gamma \le d$. If (4) holds with $1 \le p, q \le \infty$, then $q \le \frac{\gamma}{d} p'$.*

Proof. Given $\epsilon > 0$, by Billingsley's lemma (cf. [3], Proposition 4.9) there exist $x_0 \in \mathbb{R}^d$ and $r_k \to 0$ such that $\mu(B(x_0, r_k)) \gtrsim r_k^{\gamma+\epsilon}, \forall k$. For our purpose, we may assume $x_0 = 0$. Pick a bump function ϕ at 0 and let $\hat{f} = \phi(\cdot/r_k)$ in (4); we obtain $r_k^{(\gamma+\epsilon)/q} \lesssim r_k^{d/p'}, \forall k$. Comparing the powers then gives the desired result. ☐

Additional remarks

(i) After the submission of this paper, Hambrook and Łaba posted a preprint [5] in which they provide examples of Cantor-type measures for which the range obtained from [2] is sharp.

(ii) If μ is as in Corollary 1 with compact support, then by Proposition 3 it cannot have lower regularity as in (6) of degree 1/2.

(iii) As pointed out by the referee, Corollary 1 also follows from

$$\|f\mu * g\mu\|_{L^p(\mathbb{R}^d)} \le \|\mu * \mu\|_\infty^{1/p'} \|f\|_{L^p(\mu)} \|g\|_{L^p(\mu)},$$

which can be obtained by interpolating the cases $p = 1, p = \infty$. See also Bak and McMichael [1], and Iosevich and Roudenko [7].

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