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SHIFT-INVARIANT SUBSPACES INVARIANT FOR COMPOSITION OPERATORS ON THE HARDY-HILBERT SPACE

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ABSTRACT. If φ is an analytic map of the unit disk $\mathbb{D}$ into itself, the composition operator C_φ on a Hardy space H^2 is defined by $C_\varphi(f) = f \circ \varphi$. The unilateral shift on H^2 is the operator of multiplication by z. Beurling (1949) characterized the invariant subspaces for the shift. In this paper, we consider the shift-invariant subspaces that are invariant for composition operators. More specifically, necessary and sufficient conditions are provided for an atomic inner function with a single atom to be invariant for a composition operator, and the Blaschke product invariant subspaces for a composition operator are described. We show that if φ has Denjoy-Wolff point a on the unit circle, the atomic inner function subspaces with a single atom at a are invariant subspaces for the composition operator C_φ.

1. Introduction

Composition operators can be defined on any Hilbert space of analytic functions. If $\mathcal{H}$ is a Hilbert space of analytic functions on a domain in the plane, a closed subspace, M, of $\mathcal{H}$ will be called *shift-invariant* if f in M implies that zf is also in M. The goal of this paper is to describe (some of) the shift-invariant subspaces that are also invariant under composition operators.

Here we consider composition operators on the classical Hardy-Hilbert space $H^2(\mathbb{D})$, that we denote H^2, the set of functions f analytic on the unit disk $\mathbb{D}$ satisfying

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \,\frac{d\theta}{2\pi} < \infty.$$

When the above inequality is satisfied, the left-hand side is the square of the norm of f. In 1949, Beurling [1] characterized the shift-invariant subspaces of H^2 as being JH^2 for some inner function J.

If φ is an analytic map of the unit disk $\mathbb{D}$ into itself, the composition operator C_φ is defined on a Hardy space by $C_\varphi(f) = f \circ \varphi$, where f is in the Hardy space. The operator C_φ is bounded on H^2 for all such φ; see [4]. Typically, results about composition operators are related to the fixed point(s) of φ in the closed unit disk. We show in Section 3 that the mapping properties of φ and its derivatives at its fixed points are also closely related to the C_φ-invariant, shift-invariant subspaces of the Hardy space.

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In [6], Jones considers some subspaces of H^2 that are invariant for C_φ when φ is inner. In particular, he determined, in this case, some shift-invariant subspaces of the form $S_\mu H^2$, where S_μ is a singular inner function, and BH^2 for certain types of Blaschke products B are also C_φ-invariant. In [7], Mahvidi considers the question of common invariant subspaces for two composition operators, and the question of lattice containment for two composition operators. Those papers and this one clearly have overlapping goals.

We restrict our attention to invariant subspaces generated by atomic singular inner functions with a single atom, that is, $e^{\alpha\frac{z+a}{z-a}}H^2$ with $|a| = 1$ and $\alpha > 0$, and those of the form BH^2 for a Blaschke product B and for these two types of shift-invariant subspaces we largely determine the C_φ-invariant subspaces in general, for the various cases of model type for the iteration of φ. Specifically, in Section 3, we determine the composition operators having singular inner function invariant subspaces of the form given above. The proofs of Theorems 5 and 6 combine to prove the following main result of Section 3:

Corollary 1. *Let $|b| = 1$ and let φ be an analytic map of the unit disk into itself. If $\varphi(b) = b$ and $\varphi'(b) \leq 1$, then $e^{\alpha\frac{z+b}{z-b}}H^2$ is an invariant subspace for C_φ whenever $\alpha > 0$. Conversely, if $\alpha > 0$ and the subspace $e^{\alpha\frac{z+b}{z-b}}H^2$ is invariant for C_φ, then $\varphi(b) = b$ and $\varphi'(b) \leq 1$.*

In Section 4, we establish properties of Blaschke products B and maps φ that permit shift-invariant subspaces BH^2 to be also C_φ-invariant.

2. Preliminaries

In this section, we present the necessary background and notation for what follows.

It is known that in H^2, if J is the atomic singular inner function with a single atom $J(z) = e^{\alpha\frac{z+a}{z-a}}$ for $|a| = 1$ and $\alpha > 0$, then (JH^2) is a shift-invariant subspace. If Z is a subset of the disk, the set $M_Z = \{f \in H^2 : f(z) = 0 \text{ for } z \in Z\}$ is a shift-invariant subspace of H^2 and it is the zero subspace if Z is too big. For a non-constant function f in H^2, we will denote by Z_f the Blaschke sequence that is the zero sequence of f, that is, $Z_f = \{z \in \mathbb{D} : f(z) = 0\}$ written as a (possibly empty) sequence. It is known that Z_f is a finite or countably infinite sequence, (z_j), with $\sum_{Z_f}(1 - |z_j|) < \infty$. Furthermore, we note that if z_j is in Z_f, then the multiplicity of the zero of f at z_j is the number of integers k such that $z_k = z_j$. In other words, if w is in Z_f, then the multiplicity of w as a zero of f, that is, $\text{mult}_f(w)$, is the non-negative integer m so that $(z-w)^m$ divides f but $(z-w)^{m+1}$ does not.

In H^2, of course, the non-zero subspaces M_Z are just BH^2 where B is the Blaschke product whose zero sequence is $Z = Z_B$. Of course, if $f = BJg$ where B is a Blaschke product, J is a singular inner function and g is an outer function, then $Z_f = Z_B$ because neither J nor g vanishes in the disk. If B is a Blaschke product, there is a complex number λ with $|\lambda| = 1$ and a zero sequence Z_B so that

$$B(z) = \lambda \prod_{Z_B} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j}z}.$$

If $z_j = 0$, we will take the j^{th} term in this product to be z. We will occasionally refer to Z_f as the zero set of f even though it is more properly called the zero sequence.

This paper is largely about composition operators that have JH^2 or BH^2 as invariant subspaces and the relationship between φ and J or B that must exist.

If a is a point of the open disk, we say *a is a fixed point of φ* if $\varphi(a) = a$. We give the following definition to extend the meaning of 'fixed point' to include points on the boundary of the disk as well.

Definition. If φ is an analytic mapping of the unit disk into itself and a is a point of the closed unit disk, we say that *a is a fixed point of φ* if

$$\lim_{r \to 1^-} \varphi(ra) = a.$$

Of course, by the continuity of φ in the open disk, this agrees with the usual definition for $|a| < 1$, but it extends the definition to the case in which $|a| = 1$, where φ may not be defined.

It follows from Julia's Lemma and the Julia-Carathéodory Theorem (see, for example, [4, Sec. 2.3]) that if a is a fixed point of φ on the boundary of the disk that $\lim_{r\to 1^-} \varphi'(ra)$ exists as a positive real number or $+\infty$; we will abuse the notation and write $\varphi'(a)$ for this limit.

Definition. For $k > 0$ and ζ in the unit circle let

$$E(k,\zeta) = \{z \in \mathbb{D} : |\zeta - z|^2 \leq k(1 - |z|^2)\}. \tag{1}$$

The set $E(k,\zeta)$ is a relatively closed disk internally tangent to the circle at ζ with center $\frac{1}{1+k}\zeta$ and radius $\frac{k}{k+1}$.

The Julia-Carathéodory Theorem [4, Sec. 2.3] shows that when ζ is a fixed point of the unit circle we have $\varphi'(\zeta) = d(\zeta) > 0$. So if φ has fixed point ζ with $\varphi'(\zeta) \leq 1$, we have

$$\varphi\left(E(k,\zeta)\right) \subset E(k\varphi'(\zeta),\zeta)$$

for all $k > 0$. This means that φ maps disks in $\mathbb{D}$ internally tangent to ζ into smaller disks in $\mathbb{D}$ internally tangent to ζ.

Theorem 2 (Denjoy-Wolff Theorem)**.** *If φ, not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the disk into itself, then there is a point a in $\overline{\mathbb{D}}$ so that the iterates φ_n of φ converge to a uniformly on compact subsets of $\mathbb{D}$. Moreover, the point a (called the* Denjoy-Wolff point of φ*) is the unique fixed point of $\overline{\mathbb{D}}$ such that $|\varphi'(a)| \leq 1$.*

Experience has shown that the location and behavior of φ near the Denjoy-Wolff point has a dramatic effect on the operator theoretic properties of C_φ.

Suppose ω is an automorphism of the disk; that is, ω is a one-to-one map of $\mathbb{D}$ onto itself. Then C_ω is a bounded and invertible composition operator and $C_\omega^{-1} = C_{\omega^{-1}}$. Thus, for φ a map of the disk into itself, we see that

$$C_\omega^{-1} C_\varphi C_\omega = C_{\omega\circ\varphi\circ\omega^{-1}}$$

so that the composition operator C_φ is similar to the composition operator with symbol $\omega \circ \varphi \circ \omega^{-1}$.

Finally, because ω is an inner function and compositions of inner functions are again inner, C_ω maps shift-invariant subspaces to shift-invariant subspaces. Indeed, if J is an inner function, C_ω carries the shift-invariant subspace JH^2 to the shift-invariant subspace $(J \circ \omega)H^2$. This means that if JH^2 is invariant for C_φ, then $(J \circ \omega)H^2$ is invariant for $C_{\omega\circ\varphi\circ\omega^{-1}}$. This will allow us to concentrate on special types of maps φ without significant loss of generality. In particular, when showing that certain shift-invariant subspaces are invariant for some composition operators, we will often assume that if φ is a map of the disk into itself with Denjoy-Wolff point a, then either $a = 0$ (when φ has a fixed point in the open disk $\mathbb{D}$) or $a = 1$ (when φ has no fixed point in the open disk $\mathbb{D}$), and this assumption will not result in loss of generality. Moreover, if J is a Blaschke product with zeros $\{z_j\}$, then $J \circ \omega$ is also a Blaschke product but with zeros $\{\omega^{-1}(z_j)\}$ and similarly with other zero-sets. If J is a singular inner function, then $J \circ \omega$ is also a singular inner function; and if J has an atom at b on the circle, then $J \circ \omega$ has an atom at $\omega^{-1}(b)$ on the circle.

It is well known that analytic self maps of the disk can be classified in ways related to the locations of their Denjoy-Wolff points and their derivatives there (see [2] or [4, Section 2.4]). Although the extra structure that allows the classification to be proved unique up to automorphism has been omitted from this version, the main theorem is paraphrased below as 'The Linear Fractional Model'.

Theorem 3 (The Linear Fractional Model)**.** *If φ, non-constant and not an elliptic automorphism, is an analytic map of the unit disk into itself with Denjoy-Wolff point a and $\varphi'(a) \neq 0$, then the iteration of φ can be described by a linear fractional model as follows. There is a domain Ω, either the plane $\mathbb{C}$, the right half-plane RHP, or the upper half plane UHP, and an automorphism Φ of Ω such that*

$$\Phi \circ \sigma = \sigma \circ \varphi$$

where $\sigma : \mathbb{D} \to \Omega$ is analytic. This classifies φ into one of the four cases:

(1) *Plane/Dilation:* $\Omega = \mathbb{C}$, $\sigma(a) = 0$, $\Phi(z) = sz$, $0 < |s| < 1$.
(2) *Plane/Translation:* $\Omega = \mathbb{C}$, $\sigma(a) = \infty$, $\Phi(z) = z + 1$.
(3) *Half-Plane/Dilation:* $\Omega = RHP$, $\sigma(a) = 0$, $\Phi(z) = sz$, $0 < s < 1$.
(4) *Half-Plane/Translation:* $\Omega = UHP$, $\sigma(a) = \infty$, $\Phi(z) = z \pm 1$.

The map φ is in the Plane/Dilation case if and only if φ has Denjoy-Wolff point a in the disk and $0 < |\varphi'(a)| < 1$. If φ has Denjoy-Wolff point a with $|a| = 1$ and $\varphi'(a) < 1$, then it is in the Half-Plane/Dilation case and $\{\varphi_n(w)\}$ is an interpolating sequence for every w in $\mathbb{D}$. If φ has Denjoy-Wolff point a with $|a| = 1$ and $\varphi'(a) = 1$ and $\{\varphi_n(w)\}$ is an interpolating sequence for every (any) w in $\mathbb{D}$, then φ is in the Half-Plane/Translation case. If φ has Denjoy-Wolff point a with $|a| = 1$ and $\varphi'(a) = 1$ and $\{\varphi_n(w)\}$ is *NOT* an interpolating sequence for any (every) w in $\mathbb{D}$, then φ is in the Plane/Translation case.

3. Singular inner function invariant subspaces

The shift-invariant subspaces generated by atomic singular inner functions are those of the form $e^{\alpha\frac{z+a}{z-a}}H^2$, for $|a| = 1$ and $\alpha > 0$. In the main result of this section, we show that these subspaces are also C_φ-invariant exactly when φ has Denjoy-Wolff point a on the boundary, which, by the considerations of the previous section, we may take as $a = 1$.

An application of Julia's Lemma yields the following lemma.

Lemma 4. *Let φ be an analytic map of the unit disk into itself such that $\varphi(1) = 1$ and $\varphi'(1) \leq 1$. Then, for z in $\mathbb{D}$,*

$$\Re\left(\frac{\varphi(z)+1}{\varphi(z)-1} - \frac{z+1}{z-1}\right) \leq 0.$$

Proof. An easy calculation shows that

$$\Re\left(\frac{z+1}{z-1}\right) = \frac{|z|^2-1}{|z-1|^2} \tag{2}$$

and

$$\Re\left(\frac{\varphi(z)+1}{\varphi(z)-1}\right) = \frac{|\varphi(z)|^2-1}{|\varphi(z)-1|^2}. \tag{3}$$

Additionally, by Julia's Lemma [4, Section 2.3] with $\eta = 1$, $\zeta = 1$ and $d(1) = \varphi'(1) \leq 1$, we have for all $z \in \mathbb{D}$,

$$\frac{|1-\varphi(z)|^2}{1-|\varphi(z)|^2} \leq \varphi'(1)\frac{|1-z|^2}{1-|z|^2}.$$

Equivalently,

$$\frac{|\varphi(z)|^2-1}{|\varphi(z)-1|^2} \leq \frac{1}{\varphi'(1)}\frac{|z|^2-1}{|z-1|^2}.$$

Writing this in terms of the real parts, we have

$$\Re\left(\frac{\varphi(z)+1}{\varphi(z)-1}\right) \leq \frac{1}{\varphi'(1)}\Re\left(\frac{z+1}{z-1}\right).$$

Finally, recalling that $\varphi'(1) \leq 1$ and that $\frac{z+1}{z-1}$ maps the unit disk into the left half-plane,

$$\begin{aligned}\Re\left(\frac{\varphi(z)+1}{\varphi(z)-1} - \frac{z+1}{z-1}\right) &= \Re\left(\frac{\varphi(z)+1}{\varphi(z)-1}\right) - \Re\left(\frac{z+1}{z-1}\right)\\ &\leq \frac{1}{\varphi'(1)}\Re\left(\frac{z+1}{z-1}\right) - \Re\left(\frac{z+1}{z-1}\right)\\ &= \frac{1-\varphi'(1)}{\varphi'(1)}\,\Re\left(\frac{z+1}{z-1}\right) \leq 0. \qquad \square\end{aligned}$$

Next, we use this fact to prove one of the main results of this section.

Theorem 5. *If φ is an analytic map of the unit disk into itself with $\varphi(1) = 1$ and $\varphi'(1) \leq 1$, then $e^{\alpha\frac{z+1}{z-1}}H^2$ is an invariant subspace for C_φ whenever $\alpha > 0$.*

Proof. To see that $e^{\alpha\frac{z+1}{z-1}}H^2$ is C_φ-invariant, recall that the boundedness of C_φ implies that $h = g \circ \varphi$ is in H^2 for all $z \in \mathbb{D}$. Letting $F(z) = e^{\alpha\left(\frac{\varphi+1}{\varphi-1} - \frac{z+1}{z-1}\right)}$, we see for all $g \in H^2$ that

$$\begin{aligned}C_\varphi(e^{\alpha\frac{z+1}{z-1}}g(z)) &= e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}\,(g\circ\varphi)(z) = e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}h(z)\\ &= e^{\alpha\frac{z+1}{z-1}}e^{\alpha\left(\frac{\varphi(z)+1}{\varphi(z)-1} - \frac{z+1}{z-1}\right)}h(z) = e^{\alpha\frac{z+1}{z-1}}F(z)h(z).\end{aligned}$$

Since $h \in H^2$ and the product of an H^∞ function and an H^2 function is in H^2, we need only show that $F \in H^\infty$ to see that $Fh \in H^2$. To this end, let $\Re(z)$ denote the real part of z and recall that $|e^z| = e^{\Re(z)}$. Then we have $|F(z)| = e^{\Re(F(z))} \leq 1$ since $\Re(F(z)) \leq 0$ by the previous lemma. So $F \in H^\infty$, $Fh \in H^2$, and C_φ maps $e^{\alpha\frac{z+1}{z-1}}H^2$ into itself. $\square$

Recall that H^2 functions have radial limits a.e. on the unit circle given by

$$\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta}).$$

The previous fact allows us to characterize the composition operators that have shift-invariant subspaces associated with atomic singular inner functions.

Theorem 6. *Let φ be an analytic map of the unit disk $\mathbb{D}$ into itself. If there is $\alpha > 0$ such that $e^{\alpha\frac{z+1}{z-1}}H^2$ is an invariant subspace for C_φ, then $\varphi(1) = 1$ and $\varphi'(1) \leq 1$; that is, 1 is the Denjoy-Wolff point of φ.*

Proof. If $\varphi(z) \equiv z$, we clearly have $\varphi(1) = \varphi'(1) = 1$, so we assume φ is not the identity map.

Since $e^{\alpha\frac{z+1}{z-1}}$ is analytic on the disk with H^∞-norm 1, the function

$$C_\varphi\left(e^{\alpha\frac{z+1}{z-1}}\right) = e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}$$

is analytic on the disk with H^∞-norm at most 1. We have assumed the shift-invariant subspace $e^{\alpha\frac{z+1}{z-1}}H^2(\mathbb{D})$ is also invariant for C_φ, so $C_\varphi\left(e^{\alpha\frac{z+1}{z-1}}\right)$ is in $e^{\alpha\frac{z+1}{z-1}}H^2$. This implies that $e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}-\alpha\frac{z+1}{z-1}}$ is in H^2.

Now $e^{\alpha\frac{z+1}{z-1}}$ is an inner function, so the function $e^{-\alpha\frac{z+1}{z-1}}$ is in $L^\infty(\partial\mathbb{D})$ and has modulus 1 almost everywhere on the unit circle. It follows that the H^2 function $e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}-\alpha\frac{z+1}{z-1}}$ satisfies

$$\left|e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}-\alpha\frac{z+1}{z-1}}\right| = \left|e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}e^{-\alpha\frac{z+1}{z-1}}\right| = \left|e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}\right| \leq 1$$

almost everywhere on the unit circle. This means that $e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}-\alpha\frac{z+1}{z-1}}$ is actually an H^∞ function with H^∞-norm at most 1.

The properties of the exponential function and the fact that $\alpha > 0$ imply

$$\Re\left(\frac{\varphi(z)+1}{\varphi(z)-1}\right) < \Re\left(\frac{z+1}{z-1}\right).$$

Using equations (2) and (3), we get

$$\frac{|\varphi(z)|^2-1}{|\varphi(z)-1|^2} < \frac{|z|^2-1}{|z-1|^2}$$

or

$$\frac{|1-\varphi(z)|^2}{1-|\varphi(z)|^2} < \frac{|1-z|^2}{1-|z|^2}. \tag{4}$$

In particular, since the inequality is strict, φ cannot have a fixed point in the open disk $\mathbb{D}$ so the Denjoy-Wolff point must be on the unit circle. Replacing z by r where $0 < r < 1$ and taking the limit as r tends to 1, we see that $\lim_{r\to 1^-}\varphi(r) = 1$, so 1 is a boundary fixed point of φ.

Finally, if the Denjoy-Wolff point of φ is a point ζ on the unit circle with $\zeta \neq 1$, then Julia's Lemma [4, Section 2.4] implies

$$\frac{|\zeta-\varphi(z)|^2}{1-|\varphi(z)|^2} \leq \varphi'(\zeta)\frac{|\zeta-z|^2}{1-|z|^2} \leq \frac{|\zeta-z|^2}{1-|z|^2}. \tag{5}$$

Now let $\Delta_1 = E(1,1)$ be the disk as in equation (1). Since $\zeta \neq 1$, there is $\tau > 0$ so that the disk $\Delta_2 = E(\tau,\zeta)$ intersects Δ_1 in exactly one point, z_0, in the unit disk that satisfies

$$\frac{|\zeta - z_0|^2}{1-|z_0|^2} = \tau \quad \text{and} \quad \frac{|1-z_0|^2}{1-|z_0|^2} = 1$$

and the disks Δ_1 and Δ_2 are tangent at z_0. Inequality (4) says $\frac{|1-\varphi(z_0)|^2}{1-|\varphi(z_0)|^2} < 1$, so $\varphi(z_0)$ is in the interior of the disk Δ_1. On the other hand, inequality (5) says $\frac{|\zeta-\varphi(z_0)|^2}{1-|\varphi(z_0)|^2} \leq \tau$, so $\varphi(z_0)$ is in the closed disk Δ_2. Since $\Delta_1 \cap \Delta_2 = \{z_0\}$, the open disk with boundary Γ_1 and the closed disk with boundary Γ_2 are disjoint, so the inequalities (4) and (5) are inconsistent. Thus, the Denjoy-Wolff point of φ must be 1 and the theorem is proved. $\square$

The following corollary combines the results of Theorems 5 and 6 and generalizes them using a simple change of variable to give a complete description of the composition operators that have invariant subspaces JH^2 when J is a singular inner function whose singular measure is a point mass.

Corollary 7. *Let $|b| = 1$ and let φ be an analytic map of the unit disk into itself. If $\varphi(b) = b$ and $\varphi'(b) \leq 1$, then $e^{\alpha\frac{z+b}{z-b}}H^2$ is an invariant subspace for C_φ whenever $\alpha > 0$. Conversely, if $\alpha > 0$ and the subspace $e^{\alpha\frac{z+b}{z-b}}H^2$ is invariant for C_φ, then $\varphi(b) = b$ and $\varphi'(b) \leq 1$.*

The exploration of our question for shift-invariant subspaces associated with more complicated singular inner functions quickly becomes much more difficult and is not considered in this paper. As an easy first example,

$$\psi(z) = e^{\frac{z^2+1}{z^2-1}} = e^{\frac{1}{2}\frac{z+1}{z-1}}e^{\frac{1}{2}\frac{z-1}{z+1}}$$

is a singular inner function whose singular measure is an atomic measure with atoms at 1 and -1; that is, it is a product of two inner functions of the sort considered above. It is easy to see that the shift-invariant subspace ψH^2 is invariant for the composition operator C_φ where $\varphi(z) = -z$. This composition operator does not resemble those in the conclusions of the results above, and this φ has 0 as its only fixed point in the closed unit disk.

4. Blaschke product invariant subspaces

Recalling the notation and definitions established in Section 2 we determine in this section the shift-invariant subspaces BH^2 that are also C_φ-invariant where B is a Blaschke product.

The following lemma is the key to the results of this section.

Lemma 8. *Let B be a Blaschke product and let φ, non-constant and not an elliptic automorphism, be an analytic map of the unit disk into itself. The subspace BH^2 is C_φ-invariant if and only if $\mathrm{mult}_B(w) \leq \mathrm{mult}_{B\circ\varphi}(w)$ for each w in Z_B.*

Proof. If BH^2 is C_φ-invariant, for each f in BH^2, then $C_\varphi f$ is also in BH^2. Since B is in BH^2, we must have $C_\varphi(B) = B\circ\varphi = Bg$ where g is in H^2. In particular, this means that if w is in Z_B and $\mathrm{mult}_B(w) = m$, then $(z-w)^m$ divides B, so $(z-w)^m$ divides $B\circ\varphi$, and $\mathrm{mult}_{B\circ\varphi}(w) \geq m$ also.

Suppose $\mathrm{mult}_{B\circ\varphi}(w) \geq \mathrm{mult}_B(w)$ for all w in Z_B. Then B divides $B\circ\varphi$ and there is an analytic function g on the disk so that $B\circ\varphi = Bg$. In fact, because

B is in H^∞, we see $B \circ \varphi$ is in H^∞; and since $|B(z)| = 1$ for almost all z on the unit circle, we see that $\|B \circ \varphi\|_\infty = \|gB\|_\infty = \|g\|_\infty$, which means g is in H^∞ also. Now, if f is in BH^2, say $f = Bh$ for h in H^2, then

$$C_\varphi f = f \circ \varphi = (B \circ \varphi)(h \circ \varphi) = (Bg)(h \circ \varphi) = B(g \cdot h \circ \varphi).$$

Because C_φ bounded means $h \circ \varphi$ is in H^2 and g in H^∞ implies $g \cdot h \circ \varphi$ is also in H^2, we see that $C_\varphi f = B(g \cdot h \circ \varphi)$ is in BH^2. $\square$

The following example gives an illustration of the condition in Lemma 8.

Example 1. Consider $B(z) = z\left((z + 1/2)/(1 + z/2)\right)^2$ and $\varphi(z) = (2z^2 + z)/4$. In this case, $Z_B = \{0, -1/2, -1/2\}$, $\varphi(0) = 0$, and $\varphi(-1/2) = 0$, so for each w in Z_B, we have $\varphi(w)$ is also in Z_B, but since the multiplicity of 0 is 1 and the multiplicity of $\varphi(0) = 0$ is 1, and the multiplicity of $-1/2$ is 2 and the multiplicity of $\varphi(-1/2) = 0$ is 1, the multiplicity condition is *NOT* met. In this example, we have

$$\begin{aligned}(B \circ \varphi)(z) &= \frac{2z^2 + z}{4}\left(\frac{(2z^2 + z)/4 + 1/2}{1 + (2z^2 + z)/8}\right)^2 = (z(z + 1/2))\,\frac{1}{2}\left(\frac{4z^2 + 2z + 4}{2z^2 + z + 8}\right)^2 \\ &= \left(z\left(\frac{z + 1/2}{1 + z/2}\right)\right)\left((2 + z)\left(\frac{2z^2 + z + 2}{2z^2 + z + 8}\right)^2\right).\end{aligned}$$

The first factor in the final expression is an inner function, and the second factor is an outer function, so B does *NOT* divide the composition $B \circ \varphi$.

It is worth noting that B may have zeros other than $\varphi(w)$ for w in Z_B.

Example 2. Consider $B(z) = z^2(z + 1/2)/(1 + z/2)$ and $\varphi(z) = (2z^2 + z)/4$. In this case, $Z_B = \{0, 0, -1/2\}$, $\varphi(0) = 0$, and $\varphi(-1/2) = 0$, so for each w in Z_B, we have $\varphi(w)$ is also in Z_B, and since the multiplicity of 0 is 2 and the multiplicity of $\varphi(0) = 0$ is 2, and the multiplicity of $-1/2$ is 1 and the multiplicity of $\varphi(-1/2) = 0$ is 2, the multiplicity condition is met. On the other hand, there is no z in $\mathbb{D}$ for which $\varphi(z) = -1/2$ and certainly no w in Z_B with $\varphi(w) = -1/2$. As a confirmation of Lemma 8, we have

$$\begin{aligned}(B \circ \varphi)(z) &= \left(\frac{2z^2 + z}{4}\right)^2\left(\frac{(2z^2 + z)/4 + 1/2}{1 + (2z^2 + z)/8}\right) = (z(z + 1/2))^2\left(\frac{4z^2 + 2z + 4}{4(2z^2 + z + 8)}\right) \\ &= \left(z\left(\frac{z + 1/2}{1 + z/2}\right)\right)^2\left(\frac{(1 + z/2)^2(2z^2 + z + 2)}{2(2z^2 + z + 8)}\right) \\ &= B(z)\left(\frac{(z + 1/2)(1 + z/2)(2z^2 + z + 2)}{2(2z^2 + z + 8)}\right).\end{aligned}$$

We can use Lemma 8 to understand the relationship between the location and character of the fixed points of φ and the kinds of Blaschke product invariant subspaces that are possible for C_φ. For example, we find that Blaschke products vanishing at the Denjoy-Wolff point are the only Blaschke type C_φ-invariant subspaces when φ has a fixed point in the disk. We note that the third part of the next theorem follows from a result of Mahvidi [7, p. 465].

Theorem 9. *Let φ, non-constant and not an elliptic automorphism, be an analytic map of the disk into itself with Denjoy-Wolff point a. If B is a Blaschke product for which BH^2 is C_φ-invariant, then*

(i) *for each w in Z_B,*

$$mult_{\varphi-\varphi(w)}(w) \cdot mult_B(\varphi(w)) \geq mult_B(w);$$

(ii) *$\varphi_n(w)$ is in Z_B for every w in Z_B and for every positive integer n;*
(iii) *if a is in $\mathbb{D}$, the point a is in Z_B; and*
(iv) *if a is in $\mathbb{D}$, for every w in Z_B, there is an integer n_w such that $\varphi_{n_w}(w) = a$.*

Proof. For any function g in H^2, we can use the inner-outer factorization to factor the function g into a product of factors, f_j of the form $\frac{|z_j|}{z_j}\frac{z_j - z}{1-\overline{z_j}z}$ and a function f_0 that is never zero. Now, the zeros of $g \circ \varphi$ are going to arise from the factors $f_j \circ \varphi$ because $f_0 \circ \varphi$ is also a non-zero function. If w is a zero of $g \circ \varphi$, then it must be a zero of $f_j \circ \varphi$ for at least one f_j. On the other hand, w is a zero of $f_j \circ \varphi$ if and only if $(z_j - \varphi(w)) = 0$, which means $\varphi(w) = z_j$. But φ is a function, so w is a zero of $f_j \circ \varphi$ and $f_k \circ \varphi$ if and only if $z_j = z_k$. We conclude that for each w in the disk,

$$\text{mult}_{g\circ\varphi}(w) = \text{mult}_{\varphi-\varphi(w)}(w) \cdot \text{mult}_g(\varphi(w)).$$

Suppose that BH^2 is C_φ-invariant. By Lemma 8 above, then

$$\text{mult}_{\varphi-\varphi(w)}(w) \cdot \text{mult}_B(\varphi(w)) = \text{mult}_{B\circ\varphi}(w) \geq \text{mult}_B(w).$$

This proves (i).

In particular, for each w in Z_B, we see that $\varphi(w)$ is also in Z_B, since $\text{mult}_{\varphi-\varphi(w)}(w)$ is non-zero. Using this observation repeatedly, we see that (ii) holds.

Suppose a is in the open unit disk $\mathbb{D}$. Since the iterates of φ converge to the Denjoy-Wolff point a in the disk, we have $\varphi_n(z) \to a$ as n tends to infinity for all z in $\mathbb{D}$. By (ii), for w in Z_B, the set $\{\varphi_n(w)\}_{n=1}^{\infty}$ consists of zeros of B, and we see that this set must be finite since; otherwise, it provides an infinite set with a limit point in $\mathbb{D}$ on which B is zero. But since $B \not\equiv 0$, this is impossible. Hence, there exist integers k and ℓ with $\varphi_k(w) = \varphi_\ell(w)$. If, without loss of generality, $k > \ell$, then we have $\varphi_{k-\ell}(\varphi_\ell(w)) = \varphi_\ell(w)$ and note that $\varphi_\ell(w)$ is a fixed point of $\varphi_{k-\ell}$. But the iterates of φ cannot have a Denjoy-Wolff point different from that of φ, so we conclude that $\varphi_\ell(w) = a$. Then the fourth result follows since w was arbitrary (but our choice of ℓ was not). Recalling the fact that every iterate of w is in Z_B, we see that $\varphi_\ell(w) = a$ is in Z_B, so a is in Z_B and the result is proved. □

When φ is univalent, we can say more about the C_φ-invariant subspaces of the Blaschke type.

Corollary 10. *If φ, non-constant and not an elliptic automorphism, is a univalent analytic map of the unit disk into itself with Denjoy-Wolff point a in $\mathbb{D}$ and B is a Blaschke product with BH^2 invariant for C_φ, then $B(z) = \lambda\left(\frac{z-a}{1-\overline{a}z}\right)^m$ for some positive integer m.*

Proof. If z is a zero of B, then by (iv) of Theorem 9, $\varphi_m(z) = a$ for some positive integer m. But φ univalent implies φ_m is also univalent, and $\varphi_m(z) = a = \varphi_m(a)$ means $z = a$. □

After a few definitions, we consider the remaining cases of the model for iteration, Theorem 3.

Definition. An *interpolating sequence* is a sequence $\{z_j\}$ in the disk such that for any bounded sequence $\{c_j\}$ of complex numbers there is a bounded analytic function f on $\mathbb{D}$ with $f(z_j) = c_j$.

A characterization of interpolating sequences $\{z_j\}$ in H^∞ given by Carleson (see [5], [8], or [9]) depends on the relative closeness of the points of the sequence to each other in the hyperbolic metric.

Definition. A non-constant sequence $\{z_k\}_{k=q}^{\infty}$, where q is an integer or $-\infty$, is called a *forward iteration sequence* for φ, an analytic map of the unit disk into itself, if $\varphi(z_k) = z_{k+1}$ for all $k \geq q$. Of course, except when φ is an elliptic automorphism of the disk onto itself, a forward iteration sequence for φ converges to the Denjoy-Wolff point of φ.

Cowen determined [3, Prop. 4.2, 4.9] that when φ is in cases (3) or (4) of the model as described in Theorem 3 (or see [2, p. 80]), any forward iteration sequence for φ is an interpolating sequence and if φ is case (2), none are. Since interpolating sequences are Blaschke sequences, we find that in these cases there are many Blaschke product invariant subspaces that are also invariant for C_φ.

For convenience, we consider the following definition.

Definition. If $S_1,\ S_2,\ S_3,\ \cdots$ is a finite or countable collection of sequences, $S_k = \{s_{j,k}\}_j$, a *combination of the sequences* $\{S_k\}$ is a sequence $T = \{t_\ell\}_{\ell=1}^{\infty}$ so that for each $s_{j,k}$ in one of the sequences S_k, there is t_ℓ in T such that $t_\ell = s_{j,k}$, and if $t_{\ell_1} = s_{j_{\ell_1},k_{\ell_1}}$ and $t_{\ell_2} = s_{j_{\ell_2},k_{\ell_2}}$ where either $s_{j_{\ell_1},k_{\ell_1}}$ and $s_{j_{\ell_2},k_{\ell_2}}$ are from different sequences ($k_{\ell_1} \neq k_{\ell_2}$) or they are different terms from the same sequence ($k_{\ell_1} = k_{\ell_2}$ and $j_{\ell_1} \neq j_{\ell_2}$), then $\ell_1 \neq \ell_2$.

In other words, the combination is a sequence T whose terms are the union of the terms of the sequences S_k, but the number of times a particular number w occurs in T is the sum of the number of times it occurs in each of the sequences S_k.

Theorem 11. *Let φ be an analytic map of the disk into itself with Denjoy-Wolff point on the unit circle such that φ is in case* (3) *or* (4) *of Theorem* 3. *If B is a Blaschke product for which BH^2 is C_φ-invariant, then the zero set Z_B is the union of finitely many, or countably infinitely many, forward iteration sequences for φ. Conversely, if Z is a sequence in the unit disk that is the combination of finitely many forward iteration sequences of φ, then the Blaschke product B with zero set $Z_B = Z$ gives the shift-invariant subspace BH^2 which is also C_φ-invariant.*

Proof. If w is in Z_B and $\{z_k\}_{k=q}^{\infty}$ is a forward iteration sequence that includes w, say $w = z_j$, then the sequence $\{z_k\}_{k=j}^{\infty}$ is a forward iteration sequence that starts with w and is a subsequence of the given sequence. For each w in Z_B, there is a forward iteration sequence for φ starting with w, namely, w, $\varphi(w)$, $\varphi_2(w)$, $\cdots$, and, indeed, this is the unique forward iteration sequence for φ that starts with w. The second conclusion follows from the fact that each iteration sequence is an interpolating sequence, which means it is a Blaschke sequence. Since we have only finitely many of these, their combination is also a Blaschke sequence. Since

$\text{mult}_{\varphi-\varphi(w)}(w)$ is at least 1, and we are taking the combination of the Blaschke sequences, we have

$$\text{mult}_{B\circ\varphi}(w) = \text{mult}_{\varphi-\varphi(w)}(w) \cdot \text{mult}_B(\varphi(w)) \geq \text{mult}_B(\varphi(w)) \geq \text{mult}_B(w)$$

for each w in Z_B, so BH^2 is an invariant subspace for C_φ by Lemma 8. $\square$

If $\varphi(1) = 1$ and $\varphi'(1) = 1$ and φ is in case (2) of the model [2, p. 80], Cowen [3, Prop. 4.9] determined that the iterates under φ are not an interpolating sequence. This allows a partial understanding of the Blaschke product invariant subspaces for functions in this case.

Theorem 12. *If φ is an analytic map of the unit disk into itself with Denjoy-Wolff point a with $|a| = 1$, $\varphi'(a) = 1$ and $\varphi_n(0)$ converges non-tangentially to a, then there are no Blaschke product invariant subspaces.*

Proof. If φ is an analytic map of the disk into itself that is in case (4) of the model, then the sequence $\varphi_n(w)$ converges tangentially to the Denjoy-Wolff point a. It follows that the map φ in the hypothesis must be in case (2). Thus, the sequences $\{\varphi_n(w)\}$ all converge non-tangentially to the Denjoy-Wolff point a and $\varphi'(a) = 1$ implies the sequences are *not* Blaschke sequences. In particular, if B were a Blaschke product such that BH^2 is C_φ-invariant, then w in Z_B would imply $\varphi_n(z)$ is also in Z_B for each n, but this is impossible because these points are not a Blaschke sequence. $\square$

Corollary 13. *If $\varphi(1) = \varphi'(1) = 1$ and φ is real on real axes, then there are no Blaschke product invariant subspaces for C_φ.*

It is possible for an analytic map φ of the disk into itself to be in case (2) but have $\varphi_n(w)$ converge tangentially to a for each w in the disk. We cannot say if such maps can have Blaschke product subspaces that are invariant for C_φ or not.

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References

[1] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), no. 1, 239–255, DOI 10.1007/BF02395019. MR0027954 (10,381e)

[2] Carl C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. **265** (1981), no. 1, 69–95, DOI 10.2307/1998482. MR607108 (82i:30036)

[3] Carl C. Cowen, *Composition operators on H^2*, J. Operator Theory **9** (1983), no. 1, 77–106. MR695941 (84d:47038)

[4] Carl C. Cowen and Barbara D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397026 (97i:47056)

[5] John B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, vol. 96, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981. MR628971 (83g:30037)

[6] M. M. Jones, *Shift invariant subspaces of composition operators on H^p*, Arch. Math. (Basel) **84** (2005), no. 3, 258–267, DOI 10.1007/s00013-004-1017-z. MR2134140 (2005m:47046)

[7] Ali Mahvidi, *Invariant subspaces of composition operators*, J. Operator Theory **46** (2001), no. 3, suppl., 453–476. MR1897149 (2003c:47044)

[8] D. J. Newman, *Interpolation in H^∞*, Trans. Amer. Math. Soc. **92** (1959), 501–507. MR0117350 (22 #8130)

[9] H. S. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513–532. MR0133446 (24 #A3280)

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