TOPOLOGICAL COMPLEXITY OF WEDGES
AND COVERING MAPS

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Abstract. We present some results supporting the Iwase-Sakai conjecture about coincidence of the topological complexity $\text{TC}(X)$ and monoidal topological complexity $\text{TC}^M(X)$. Using these results we provide lower and upper bounds for the topological complexity of the wedge $X \vee Y$. We use these bounds to give a counterexample to the conjecture asserting that $\text{TC}(X') \leq \text{TC}(X)$ for any covering map $p: X' \rightarrow X$.

Also we discuss a possible reduction of the monoidal topological complexity to the Lusternik-Schnirelmann category.

1. Introduction

Let $PX = X^{[0,1]}$ denote the space of all paths in a path connected space $X$. Let $i_X : X \rightarrow PX$ be the inclusion of $X$ into $PX$ as a subspace of constant paths. There is a natural fibration $\pi : PX \rightarrow X \times X$ defined as $\pi(f) = (f(0), f(1))$ for $f \in PX, f : [0,1] \rightarrow X$.

Let $X$ be an ENR. A piecewise continuous section $s : X \times X \rightarrow PX$ of $\pi$ is called a motion planning algorithm. We say that a motion planning algorithm $s$ has complexity $k$ if $X \times X$ can be presented as a disjoint union $F_0 \cup \cdots \cup F_k$ of $k + 1$ ENRs such that $s$ is continuous on each $F_i$. The normalized topological complexity $\text{TC}(X)$ of a space $X$ was defined by Farber as the minimum $k$ such that there is a motion planning algorithm of complexity $k$ [F1]. Equivalently, $\text{TC}(X)$ is the minimal number $k$ such that $X \times X$ admits an open cover $U_0, \ldots, U_k$ by $k + 1$ sets such that over each $U_i$ there is a continuous section of $\pi$.

We say that a motion planning algorithm $s : X \times X \rightarrow PX$ is reserved if $s|_{\Delta X} = i_X$, where $\Delta X \subset X \times X$ is the diagonal. In other words, if the initial position of a robot in the configuration space $X$ coincides with the terminal position, then the algorithm keeps the robot still. This condition on the motion planning algorithms seems to be very natural. The corresponding complexity of a space $X$ was denoted by Iwase and Sakai as $\text{TC}^M(X)$ and was called the monoidal topological complexity of $X$ [IS1]. In the original definition they additionally assumed that all sets $U_i$ contain the diagonal. Their definition agrees with the above since their condition can always be achieved by reduction of an open cover $U_0, \ldots, U_k$ with reserved sections $s_i$ to a closed cover $F_0, \ldots, F_k$, $F_i \subset U_i$, then by adding the
diagonal to each $F_i$ with the natural extension of the sections $\bar{s}_i$, and then by
taking an open enlargement $V_i$ of the sets $F_i \cup \Delta X$ that admit extensions of the
sections $\bar{s}_i$.

Iwase and Saki conjectured that $TC^M(X) = TC(X)$. In fact, first they gave
a proof to the conjecture in [S1] and then withdrew it in [S2]. We prove this
conjecture under the assumption that $TC(X) > \dim X$. Also, using the Weinberger
Lemma from [F3] we show that the conjecture holds true when $X$ is a Lie group.

The topological complexity is closely related to the Lusternik-Schnirelmann cat-
egory $\text{cat}_{LS}(X)$ of a space which is defined as the minimal number $k$ such that $X$
can be covered by $k + 1$ open sets $U_0, U_1, \ldots, U_k$, all contractible to a point in $X$.
Throughout the paper we use the normalized LS-category (as well as the normal-
ized topological complexity), since they improve several important inequalities in
the theory:

\[ \text{cup-length}(X) \leq \text{cat}_{LS}(X) \leq \dim(X) \text{ and } \text{cat}_{LS}(X \times Y) \leq \text{cat}_{LS}(X) + \text{cat}_{LS}(Y), \]

where in the first inequality the cup-length is taken for any reduced cohomology
(possibly twisted) [CLOT].

Some of the formulas for $\text{cat}_{LS}$ translate to similar statements for $TC$. For
example, for $TC$ there is an inequality similar to the above for the product of two
spaces [F4]. Also, there are analogous estimates of $TC$ in terms of the cup product
and dimension [F4]. On the other hand, the simple $\text{cat}_{LS}$ formula for the wedge
$\text{cat}_{LS}(X \vee Y) = \max\{\text{cat}_{LS} X, \text{cat}_{LS} Y\}$ does not hold for $TC$. So far there is no
good analog of it for $TC$. The best that we can prove here is Theorem [3.6] from this
paper. Another example is the formula $\text{cat}_{LS}(Y) \geq \text{cat}_{LS}(X)$ for a covering map
$p: X \rightarrow Y$ which supports an intuitive idea that a covering space is always simpler
than the base. So it was natural to assume that the same holds true for $TC$. We
learned about this problem from Yuli Rudyak. In this paper Theorem [3.8] gives a
negative answer to this question.

There have been several attempts to reformulate the topological complexity
of $X$ as some modified category of a related space. Thus, Garcia Calcines and
Vandembroucq defined the weak topological complexity $\text{wTC}(X)$ [GV1] in the
spirit of Berstein-Hilton’s [BH] definition of the weak category $\text{wcat}(X)$. They
proved the equality $\text{wTC}(X) = \text{wcat}(X \times X)/\Delta X$ and found some sufficient
conditions for the equality $\text{wTC}(X) = \text{TC}(X)$. Then in [GV2] they established the
equality $\text{TC}(X) = \text{cat}_{LS}(X \times X)/\Delta X$ for large classes of spaces. In this paper
we discuss a possibility of such equality for the monoidal topological complexity:
$\text{TC}^M(X) = \text{cat}_{LS}(X \times X)/\Delta X$. We define a rel infinity category $\infty$-$\text{cat}_{LS}(Y)$
of non-compact spaces $Y$ and discuss the problem of coincidence between $\text{cat}_{LS}(X/A)$ and
$\infty$-$\text{cat}_{LS}(X \setminus A)$ for a subcomplex $A \subset X$ of a finite complex $X$. Then we show
that $\text{TC}^M(X)$ is bounded between $\text{cat}_{LS}(X \times X)/\Delta X$ and $\infty$-$\text{cat}_{LS}(X \times X \setminus \Delta X)$.

Note that for the topological complexity Farber proved the inequality [F2]

\[ \text{cat}_{LS}(X \times X)/\Delta X - 1 \leq \text{TC}(X). \]

Thus, in view of the Iwase-Sakai conjecture, examples of spaces for which Farber’s
inequality is sharp are of great interest.

We recall that both $\text{cat}_{LS}(X)$ and $\text{TC}(X)$ are special cases of the Schwarz
genus [Sch]: $\text{cat}_{LS}(X) = sg(\pi_0: P_0X \rightarrow X)$ and $\text{TC}(X) = sg(\pi: PX \rightarrow X \times X)$,
where $P_0X \subset PX$ is the subspace of paths $f: [0, 1] \rightarrow X$ that start in a base point
$x_0 \in X$, $f(0) = x_0$, and $\pi_0(f) = f(1)$. We recall that the normalized Schwarz
genus \([\text{Sch}]\) (also known as the sectional category [J]) of a fibration \(p : X \to Y\) is the minimal number of open sets \(U_0, \ldots, U_k\) that cover \(Y\) and admit sections \(s_i : U_i \to X\) of \(p\). In this paper we estimate the Schwarz genus \([\text{Sch}]\) of arbitrary fibration \(p : X \to Y\) in terms of the category of its mapping cone \(C_p\),
\[
\text{cat}_{\text{LS}}(C_p) - 1 \leq \text{sg}(p) \leq \infty \cdot \text{cat}_{\text{LS}}(C_p \setminus \{\ast\}).
\]
Similar estimates for the weak sectional category in terms of \(\text{wcat}(C_p)\) were made in [GV1]:
\[
\text{wcat}(C_p) - 1 \leq \text{wsecat}(p) \leq \text{wcat}(C_p).
\]
The similarity would be complete if the problem about the equality \(\text{cat}_{\text{LS}} X/A = \infty \cdot \text{cat}_{\text{LS}}(X \setminus A)\) could be solved affirmatively.

2. Monoidal topological complexity

Theorem 2.1. For ENR spaces,
\[
\text{TC}(X) \leq \text{TC}^M(X) \leq \text{TC}(X) + 1.
\]
This theorem was proved in [IS2]. Since the proof there is too technical, we give an alternative proof.

\textit{Proof.} The first inequality is obvious. Since \(X\) is ANR, there is an open neighborhood \(W\) of the diagonal \(\Delta X\) in \(X \times X\) and a continuous map \(\phi : W \times [0, 1] \to X\) such that \(\phi(x, x', 0) = x, \phi(x, x', 1) = x',\) and \(\phi(x, x, t) = x\) for all \(t \in [0, 1]\), \((x, x') \in W\).

Let \(\text{TC}(X) = n\) and let \(U_0, U_1, \ldots, U_n\) be an open cover of \(X \times X\) by sets that admit sections \(s_i : U_i \to PX\) of \(\pi\). Let \(F\) be a closed neighborhood of \(\Delta X\) that lies in \(W\). Then all sets in the open cover \(U_0 \setminus F, \ldots, U_n \setminus F, W\) of \(X \times X\) admit reserved sections. Hence \(\text{TC}^M(X) \leq n + 1\). \hfill \Box

Note that the path fibration \(PX \to X \times X\) restricted over the diagonal defines the free loop fibration \(p : LX \to X\). A canonical section \(\tilde{s} : \Delta X \to LX\) of \(p\) is defined as \(\tilde{s}(x) = c_x\), where \(c_x : I \to X\) is the constant map to \(x\).

We use the standard convention to denote the elements of the iterated join product \(X_0 \ast \cdots \ast X_n\) as formal linear combinations \(t_0x_0 + t_1x_1 + \cdots + t_nx_n, \sum t_i = 1, t_i \geq 0, x_i \in X_i\), where all summands of the type \(0x_i\) are dropped. We use the notation \(*^n X\) for the iterated join product of \(n\) copies of \(X\) with itself.

We recall that a fiberwise join of maps \(f_i : X_i \to Y, i = 0, \ldots, n,\) is the map
\[
f_0 \ast \cdots \ast f_n : X_0 \ast X \cdots \ast X_n \to Y,
\]
where
\[
X_0 \ast \cdots \ast X_n = \{t_0x_0 + \cdots + t_nx_n \in X_0 \ast \cdots \ast X_n \mid f_0(x_0) = \cdots = f_n(x_n)\}
\]
is the fiberwise join of spaces \(X_0, \ldots, X_n\) and
\[
(f_0 \ast \cdots \ast f_n)(t_0x_0 + \cdots + t_nx_n) = f_i(x_i).
\]
Thus, the preimage \((f_0 \ast \cdots \ast f_n)^{-1}(y)\) of a point \(y \in Y\) is the join product of the preimages \(f_0^{-1}(y) \ast \cdots \ast f_n^{-1}(y)\).

We define \(P_nX = PX \ast X \ast \cdots \ast X \ast PX\) and
\[
\pi_n = \ast \cdots \ast \pi : P_nX \to X \times X\]
to be the fiberwise join product of \(n + 1\) copies of \(\pi\). Note that there are imbeddings \(PX = P_0X \subset P_1X \subset \cdots \subset P_nX\) such that \(\pi_i|_{P_{i-1}(X)} = \pi_{i-1}\). Then the section \(\tilde{s} : X \times X \to PX\) of \(\pi\) can be regarded as a section of \(\pi_n\). Also we define
\[ p_0 = p : LX \to X, \quad L_n X = L_{n-1} X \star_X LX, \quad \text{and} \quad p_n = p_{n-1} \tilde{*} p : L_n X \to X. \]

Note that \( \pi_n^{-1}(\Delta X) \cong L_n X \) and \( p_n \) is the restriction of \( p_n \) to \( \pi_n^{-1}(\Delta X) \). Note also that the canonical section \( \tilde{s} \) defines a trivial subbundle \( p'_n : E \to X \) of \( p_n \) with the fiber \( n \)-simplex \( \Delta^n \).

We recall that a map \( p : E \to B \) satisfies the Homotopy Lifting Property for a pair \((X, A)\) if for any homotopy \( H : X \times I \to B \) with a lift \( H' : A \times I \to E \) of the restriction \( H|_{A \times I} \) and a lift \( H_0 \) of \( H|_{X \times 0} \) which agrees with \( H' \), there is a lift \( \tilde{H} : X \times I \to E \) of \( H \) which agrees with \( H_0 \) and \( H' \). We recall that a pair of spaces \((X, A)\) is called an NDR pair if \( A \) is a deformation retract of a neighborhood in \( X \). A pair \((X, A)\) is NDR if and only if the inclusion \( A \to X \) is a closed cofibration [TD]. The following is well-known; see [TD], Corollary 5.5.3:

**Theorem 2.2.** Any Hurewicz fibration \( p : E \to B \) satisfies the Homotopy Lifting Property for NDR pairs \((X, A)\).

**Corollary 2.3.** Let \( p : E \to X \) be a Hurewicz fibration with a section \( s : X \to E \). A fiberwise homotopy \( \tilde{G} : X \times I \to E \) of the restriction \( s|_A \) to a closed subset \( A \subset X \) can be extended to a fiberwise homotopy \( \tilde{G} : X \times I \to E \) of \( s \) provided \((X, A)\) is an NDR pair.

**Proposition 2.4.** For CW complexes \( X \),

1. \( TC(X) \leq n \iff \pi_n : p_n X \to X \times X \) admits a section.
2. \( TC^M(X) \leq n \iff \pi_n : p_n X \to X \times X \) admits a section \( s \) which agrees with the canonical section over the diagonal, \( s|_{\Delta X} = \tilde{s} \).

**Proof.** Statement (1) is a part of a general theorem proven by Schwarz [Sch] for fibrations \( q : X \to Y \), \( sg(q) \leq n \) if and only if the fiberwise join product \( \tilde{*}_Y^{n+1} q : \tilde{*}_Y^{n+1} X \to Y \) of \( n + 1 \) copies of \( q \) admits a section.

The implication \( \Leftarrow \) in (2) is obvious. For the other direction we note that \( n + 1 \) reserved sections \( s_i : U_i \to PX \) defined for an open cover \( U_0, \ldots, U_n \) of \( X \times X \) define a section \( s \) of \( \pi_n \) with the image \( s(\Delta X) \) lying in \( E \). Therefore over \( \Delta X \) it could be fiberwise deformed to \( \tilde{s} \). By Theorem 2.2 that deformation can be extended to a fiberwise deformation of \( s \).

**Theorem 2.5.** The equality

\[ TC(X) = TC^M(X) \]

holds true for \( k \)-connected simplicial complexes \( X \) such that

\[ (k+1)(TC(X) + 1) > \dim X + 1. \]

**Proof.** Let \( TC(X) = n \). Note that the fiber \( \pi^{-1}(x, x') \) is homotopy equivalent to the loop space \( \Omega(X) \). Since \( \Omega(X) \) is \((k-1)\)-connected, the iterated join product \( \tilde{*}_X^{n+1} \Omega(X) \) is \((k+1)(n+1) - 2\)-connected. We show that any section \( s : \Delta X \to L_n X \) can be joined by a fiberwise homotopy to the canonical section \( \tilde{s} : \Delta X \to L_n X \).

By induction on \( i \) we construct a section \( s_i : X \to L_n X \), that coincides with \( \tilde{s} \) on the \( i \)-skeleton \( X^{(i)} \), together with a fiberwise homotopy joining \( s \) and \( s_i \). Here we use the identification \( \Delta X = X \). For \( i = 0 \) we take paths in the fibers \( p_n^{-1}(v) \) joining \( s(v) \) and \( \tilde{s}(v) \) for all \( v \in X^{(0)} \). Then we extend them to a fiberwise homotopy of \( s \) to a section \( s_0 \). Assume that \( s_{i-1} \) is already constructed and \( i \leq \dim X \leq (k+1)(n+1) - 2 \). Independently for every \( i \)-simplex \( \sigma \subset X \) we consider the problem of joining \( s_{i-1} \) with \( \tilde{s} \) over \( \sigma \) by a fiberwise homotopy. Since
the fiber bundle $p_n$ is trivial over $\sigma$ with an $i$-connected fiber, the identity homotopy on the boundary $\partial \sigma$ can be extended to a homotopy between $s_i|_{\sigma}$ and $s_{i-1}|_{\sigma}$. This extension can be deformed to a fiberwise homotopy. All these homotopies together define a fiberwise homotopy between $s_{i-1}$ and $\bar{s}$ over $X^{(i)}$. Since $(X, X^{(i)})$ is an NDR pair, by Corollary 2.3 we can extend it to a fiberwise homotopy over $X$.

Let $s : X \times X \to P_n X$ be a section. We proved that on $\Delta X$ it can be deformed to a canonical section $\bar{s}$, $s_{\Delta X}=\bar{s}$. Since $(X \times X, \Delta X)$ is an NDR pair, by Corollary 2.3 there is a fiberwise homotopy of $s$ to a section $s'$ that coincides with $\bar{s}$ on $\Delta X$. Therefore, $\text{TC}^M(X) \leq n$. \hfill $\Box$

**Corollary 2.6.** $\text{TC}(S^n) = \text{TC}^M(S^n)$ for $n > 1$.

The case of $n = 1$ is taken care of by Lemma 2.7.

The following is an extension of Weinberger’s Lemma from [F3] to the case of monoidal topological complexity.

**Lemma 2.7.** For a connected Lie group $G$,

$$\text{TC}(G) = \text{TC}^M(G) = \text{cat}_{\text{LS}}(G).$$

**Proof.** In view of what is already known [F3], it suffices to show the inequality $\text{TC}^M(G) \leq \text{cat}_{\text{LS}}(G)$. Let $\text{cat}_{\text{LS}}(G) = n$ and let $U_0, U_1, \ldots, U_n$ be an open cover of $G$ together with homotopies $H_i : U_i \times [0, 1] \to G$ contracting $U_i$ to the unit $e \in G$. Clearly, we may assume that $e \notin U_i$ for $i > 0$. Since the inclusion $e \in G$ is a cofibration, we may assume that $H_0(e, t) = e$ for all $t$ (see Lemma 1.25 of [CLOT]). Then for the open cover of $G \times G$ as defined in [F3],

$$W_i = \{(a, b) \in G \times G \mid b^{-1}a \in U_i\},$$

the sections $s_i : W_i \to PG$ defined by the formula

$$s_i(a, b)(t) = bH_i(b^{-1}a, t) \in G, \quad (a, b) \in W_i,$$

are reserved. Indeed, $\Delta G \cap W_i = \emptyset$ for $i > 0$ and

$$s_0(a, a)(t) = aH_0(a^{-1}a, t) = aH_0(e, t) = ae = a$$

for all $(a, a) \in \Delta G$. \hfill $\Box$

### 3. Topological Complexity of Wedge and Covering Maps

A deformation of $U \subset Z$ in $Z$ to a subset $A \subset Z$ is a continuous map $D : U \times I \to Z$ such that $D(u, 0) = u$, $D(u, 1) \in A$ for all $u \in U$. A strict deformation of $U \subset Z$ in $Z$ to $A \subset Z$ is a deformation $D : U \times I \to Z$ such that $D(u, t) = u$ for all $t \in I$ whenever $u \in A$.

**Proposition 3.1.** Let $X$ be a metric space. For an open set $U \subset X \times X$ the following are equivalent:

1. There is a reserved section $s : U \to PX$ over $U$ of the fibration $\pi : PX \to X \times X$.

2. There is a strict deformation $D : U \times I \to X \times X$ to the diagonal $\Delta X = \{(x, x) \in X \times X \mid x \in X\}$.

3. For any choice of a base point $x_0 \in X$ there is a strict deformation $D$ of $U$ to $\Delta X$ which preserves faces $X \times x_0$ and $x_0 \times X$; i.e., for all $t \in I$,$$D((x, x_0), t) \in X \times x_0 \quad \text{and} \quad D((x_0, x), t) \in x_0 \times X.$$
Proof. (1) $\Rightarrow$ (3). Let $\|x\| = d(x, x_0)$. We define

$$D((x, y), t) = s(x, y)(\frac{\|x\|}{\|x\| + \|y\|} t), s(x, y)(1 - \frac{\|y\|}{\|x\| + \|y\|} t)$$

if $(x, y) \neq (x_0, x_0)$ and define $D((x_0, x_0), t) = (x_0, x_0)$. Since $s(x, y)(0) = x$ and $s(x, y)(1) = y$, we obtain that $D((x, y), 0) = (x, y)$. Note that

$$D((x, y), 1) = s(x, y)(\frac{\|x\|}{\|x\| + \|y\|}), s(x, y)(\frac{\|x\|}{\|x\| + \|y\|}) \in \Delta X.$$ 

Since the section $s$ is reserved, $D((x, x), t) = \mathcal{D}(x(t/2), s(x, x(t/2))) = (x, x)$. Note that

$$D((x, x_0), t) = (s(x, x_0)(t), s(x, x_0)(1)) = (s(x, x_0)(t), x_0) \in X \times x_0$$

and

$$D((x_0, y), t) = (s(x_0, y)(0), s(x_0, y)(1 - t)) = (x_0, s(x_0, y)(1 - t)) \in x_0 \times X.$$ 

The deformation $D$ is continuous at $(x_0, x_0)$ since the section $s(x_0, x_0)$ is stationary at $(x_0, x_0)$.

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let $pr_1 : X \times X \to X$ denote the projection to the first factor and $pr_2 : X \times X \to X$ to the second. Given a strict deformation $D : U \times I \to X \times X$ we define a section $s : U \to PX$ as follows:

$$s(x, y)(t) = \begin{cases} 
pr_1D((x, y), 2t) & \text{if } t \leq 1/2, \\
pr_2D((x, y), 2 - 2t) & \text{if } t \geq 1/2.
\end{cases}$$

This path is well-defined since $D((x, y), 1) \in \Delta X$. Clearly it is a path from $x$ to $y$. If $x = y$, the path is stationary. Thus $s$ is a reserved section. \qed

The following lemma was proven in [Dr], Lemma 4.25.

**Lemma 3.2.** Let $A$ be a retract of an ENR space $X$. Then $TC(X) \geq TC(A)$.

We recall that a family $\mathcal{U}$ of subsets of $X$ is called a $k$-cover, $k \in \mathbb{N}$, if every subfamily of $\mathcal{U}$ that consists of $k$ elements forms a cover of $X$. We use the following theorem [Dr].

**Theorem 3.3.** Let $\{U'_0, \ldots, U'_n\}$ be an open cover of a normal topological space $X$. Then for any $m = n, n + 1, \ldots, \infty$ there is an open $(n + 1)$-cover of $X$, $\{U_k\}_{k=0}^m$ such that $U_k = U'_k$ for $k \leq n$ and $U_k = \bigcup_{i=0}^n V_i$ is a disjoint union with $V_i \subset U_i$ for $k > n$.

**Corollary 3.4.** Suppose that all sets $U'_i$, $i = 0, \ldots, n$, in the theorem are (strictly) deformable in $X$ to a subspace $A \subset X$. Then the sets $U_k$ for all $k$ are (strictly) deformable in $X$ to $A$.

The following proposition is well-known. It can be traced back to the work of Kolmogorov on Hilbert’s 13th problem [Os].

**Proposition 3.5.** Let $\mathcal{U} = \{U_0, \ldots, U_{n+m}\}$ be an $(n + 1)$-cover of $X$ and let $\mathcal{V} = \{V_0, \ldots, V_{m+n}\}$ be an $(m + 1)$-cover of $Y$. Then the sets $W_k = U_k \times V_k$, $k = 0, \ldots, n + m$, cover $X \times Y$. 

Proof. Let \((x, y) \in X \times Y\). A point \(x\) is covered by at least \(m + 1\) elements of \(U\). Otherwise \(n + 1\) elements that do not cover \(x\) would not form a cover of the space \(X\). That would give a contradiction with the assumption that \(U\) is an \((n + 1)\)-cover of \(X\). Let \(x \in U_{i_0} \cap \cdots \cap U_{i_m}\). By the assumption, the family \(V_{i_0}, \ldots, V_{i_m}\) covers \(Y\). Hence \(y \in V_{i_s}\) for some \(s\). Then \((x, y) \in W_i\).

\[\text{Theorem 3.6. For all ANR spaces } X \text{ and } Y,\]
\[
\max\{\text{TC}(X), \text{TC}(Y), \text{cat}_{LS}(X \times Y)\} \leq \text{TC}(X \vee Y)
\]
\[
\leq \text{TC}^M(X \vee Y) \leq \text{TC}^M(X) + \text{TC}^M(Y).
\]

Proof. Note that \(\text{TC}(X \vee Y) \geq \text{TC}(X), \text{TC}(Y)\) by Lemma 3.2. Let \(r_X : X \vee Y \to X\) and \(r_Y : X \vee Y \to Y\) be the retractions collapsing the wedge onto \(X\) and \(Y\) respectively. We may assume that the subset
\[X \times Y \subset (X \vee Y) \times (X \vee Y)\]
is covered by \(\leq \text{TC}(X \vee Y) + 1\) open sets \(U\) supplied with homotopies
\[H_U : U \times I \to X \vee Y\]
such that \(H_U(x, y, 0) = x\) and \(H_U(x, y, 1) = y\). For each \(U\) we define a homotopy \(G_U : U \times I \to X \times Y\) by the formula
\[G_U(x, y, t) = (r_X H_U(x, y, t), r_Y H_U(x, y, 1 - t)).\]

Then
\[G_U(x, y, 0) = (r_X H_U(x, y, 0), r_Y H_U(x, y, 1)) = (r_X(x), r_Y(y)) = (x, y)\]
and
\[G_U(x, y, 1) = (r_X H_U(x, y, 1), r_Y H_U(x, y, 0)) = (r_X(y), r_Y(x)) = (v_0, v_0),\]
where \(v_0\) is the wedge point in \(X \vee Y\). Thus, \(G_U\) contracts \(U\) to a point in \(X \times Y\).

Let \(\text{TC}^M(X) = n\) and \(\text{TC}^M(Y) = m\). Then there is an open cover \(\tilde{U}_0, \ldots, \tilde{U}_n\)
of \(X \times X\) with reserved sections \(s_i : \tilde{U}_i \to PX, i = 0, \ldots, n\). Similarly, let \(\tilde{V}_0, \ldots, \tilde{V}_m\) be an open covering of \(Y \times Y\) with reserved sections \(\sigma_j : \tilde{V}_j \to PY, j = 0, \ldots, m\). By Proposition 3.1 all these sets are strictly deformable to the diagonal in \(X \times X\) and \(Y \times Y\) respectively. By Corollary 3.2 there is an open \((n + 1)\)-cover \(\tilde{U}_0, \ldots, \tilde{U}_n, \ldots, \tilde{U}_{n+m}\) of \(X \times X\) by sets strictly deformable to the diagonal. By Proposition 3.1 there are strict deformations
\[D^k_X : \tilde{U}_k \times I \to X \times X\]
of \(\tilde{U}_k\) to \(\Delta X\) that preserve faces \(X \times v_0\) and \(v_0 \times X\). Similarly, there is an open \((m + 1)\)-cover \(\tilde{V}_0, \ldots, \tilde{V}_m, \ldots, \tilde{V}_{m+n}\) of \(Y \times Y\) and there are strict deformations \(D^k_Y\)
of \(\tilde{V}_k\) in \(Y \times Y\) to the diagonal \(\Delta Y\) that preserve faces.

We use the notation
\[U_k = \tilde{U}_k \cap (X \times v_0) \text{ and } V_k = \tilde{V}_k \cap (v_0 \times Y),\]
\[k = 0, \ldots, m + n.\]

Note that \(U_0, \ldots, U_{m+n}\) is an \((n + 1)\)-cover of \(X \times v_0 = X\) and \(V_0, \ldots, V_{m+n}\) is an \((m + 1)\)-cover of \(v_0 \times Y = Y\). Let \(W_k = U_k \times V_k\). By Proposition 3.1 \(W_0, \ldots, W_{m+n}\)
is an open cover of \(X \times Y\).
The deformations $D_k^X$ define the deformations $H_k : U_k \times I \to X \times v_0$ to the point $v_0 \in X$ and the deformations $D_k^Y$ define the deformations $G_k : V_k \times I \to v_0 \times Y$ to the point $v_0 \in Y$. These deformations define the deformations
\[ T_k : W_k \times I \to X \times Y \]
to the point $(v_0, v_0)$ such that if $W_k \cap (X \times v_0) \neq \emptyset$, then $W_k \cap (X \times v_0) = U_k$ and $T_k|_{U_k \times I} = H_k$, and if $W_k \cap (v_0 \times Y) \neq \emptyset$, then $W_k \cap (v_0 \times Y) = V_k$ and $T_k|_{V_k \times I} = G_k$ for $k = 0, \ldots, m + n$.

Symmetrically, define
\[ U'_k = \tilde{U}_k \cap (v_0 \times X) \quad \text{and} \quad V'_k = \tilde{V}_k \cap (Y \times v_0), \quad k = 0, \ldots, m + n, \]
and the corresponding deformations
\[ H'_k : U'_k \times I \to X \quad \text{and} \quad G'_k : V'_k \times I \to Y \]
to the base points. Define $W'_k = U'_k \times V'_k$. By Proposition 3.5 the family $W'_0, \ldots, W'_{n+m}$ is an open cover of $Y \times X$. As before there are deformations
\[ T'_k : W'_k \times I \to Y \times X \]
to the point $(v_0, v_0)$ such that if $W'_k \cap (v_0 \times X) \neq \emptyset$, then $W'_k \cap (v_0 \times X) = U'_k$ and $T'_k|_{U'_k \times I} = H'_k$, and if $W'_k \cap (Y \times v_0) \neq \emptyset$, then $W'_k \cap (Y \times v_0) = V'_k$, $T'_k|_{V'_k \times I} = G'_k$ for $k = 0, \ldots, m + n$.

We define the open sets
\[ O_k = W_k \cup W'_k \cup \tilde{U}_k \cup \tilde{V}_k \subset (X \times Y) \times (X \times Y), \quad k = 0, \ldots, n + m, \]
and note that $O = \{O_k\}$ covers $(X \times Y) \times (X \times Y)$. Note that the set
\[ C = (X \times Y) \times v_0 \bigcup v_0 \times (X \times Y) \]
defines a partition of $(X \times Y) \times (X \times Y)$ into four pieces: $X \times X$, $X \times Y$, $Y \times X$, and $Y \times Y$. Also note the inclusion $O_k \cap C \subset U_k \cup V_k \cup U'_k \cup V'_k$. By the construction the deformations $D_k^X$, $D_k^Y$, $T_k$, and $T'_k$ all agree on $O_k \cap C$. Therefore the union of deformations
\[ T_k \cup T'_k \cup D_k^X \cup D_k^Y : O_k \times I \to (X \times Y) \times (X \times Y) \]
is a well-defined deformation $Q_k$ of $O_k$ to the diagonal $\Delta(X \times Y)$. Note that for all $k$, $Q_k$ are strict deformations. By Proposition 3.1 each $Q_k$ defines a reserved section $\alpha_k : O_k \to P(X \times Y)$. Therefore,
\[ TC^M(X \times Y) \leq n + m = TC^M(X) + TC^M(Y). \]

Remark 3.7. A stronger version of the upper bound of Theorem 3.6 was proposed in [F2], Theorem 19.1:
\[ TC(X \times Y) \leq \max\{TC(X), TC(Y), \text{cat}_{\text{LS}}(X) + \text{cat}_{\text{LS}}(Y)\}. \]

Since the proof in [F2] is incomplete, we call this inequality Farber’s Conjecture. Note that Farber’s inequality in view of Theorem 3.6 would turn into an equality for spaces $X$ and $Y$ satisfying the product law: $\text{cat}_{\text{LS}}(X \times Y) = \text{cat}_{\text{LS}}(X) + \text{cat}_{\text{LS}}(Y)$.

**Theorem 3.8.** (1) There is a 2-to-1 covering map $p : E \to B$ with $TC(E) > TC(B)$.

(2) There is a finite complex $X$ with $TC(X) < TC(\hat{X})$, where $\hat{X}$ is the universal covering of $X$. 

Proof. (1) We take \( B = T \lor S^1 \), where \( T = S^1 \times S^1 \) is a 2-torus. Let \( E \) be the covering space defined by the 2-fold covering of \( S^1 \). Note that \( E \) is homeomorphic to the circle with two tori \( T \) attached at antipodal points. Thus, \( E \) is homotopy equivalent to \( T \lor T \lor S^1 \). By Theorem 3.6 and Lemma 2.7
\[
\text{TC}(B) \leq \text{TC}^M(T) + \text{TC}^M(S^1) = \text{cat}_{\text{LS}}(T) + \text{cat}_{\text{LS}}(S^1) = 2 + 1 = 3.
\]
On the other hand, by Theorem 3.6
\[
\text{TC}(E) \geq \text{cat}_{\text{LS}}((T \lor S^1) \times T)) \geq \text{cup-length}((T \lor S^1) \times T) = 4.
\]
(2) Consider \( X = (S^3 \times S^3) \lor S^1 \). Since \( S^3 \times S^3 \) is a connected Lie group, by Lemma 2.7 \( \text{TC}^M(S^3 \times S^3) = \text{cat}_{\text{LS}}(S^3 \times S^3) = 3 \). By Theorem 3.6 and Lemma 2.7
\[
\text{TC}(X) \leq \text{TC}^M(S^3 \times S^3) + \text{TC}^M(S^1) = \text{cat}_{\text{LS}}(S^3 \times S^3) + \text{cat}_{\text{LS}}(S^1) = 2 + 1 = 3.
\]
Note that the universal cover \( \tilde{X} \) is homotopy equivalent to an infinite wedge
\[
Y = \left( S^3 \times S^3 \right) \lor \left( S^3 \times S^3 \right).
\]
Then \( Y \) admits a retraction onto \( (S^3 \times S^3) \lor (S^3 \times S^3) \). By Lemma 3.2 Theorem 3.6 and the cup-length lower bound on \( \text{cat}_{\text{LS}} \),
\[
\text{TC}(\tilde{X}) \geq \text{TC}((S^3 \times S^3) \lor (S^3 \times S^3)) \geq \text{cat}_{\text{LS}}(S^3 \times S^3 \times S^3 \times S^3) \geq 4. \quad \square
\]

4. TOPOLOGICAL COMPLEXITY, LS-CATEGORY, AND SCHWARTZ GENUS

We say a subset \( A \subset X \) can be rel \( \infty \) contracted to infinity if for every compact subset \( F \subset X \) there is a larger compact set \( C \subset F \) and a homotopy \( h_t : A \to X \) with \( h_0 = \text{incl}_A \subset X, h_1(A) \cap F = \emptyset \), and \( h_t(a) = a \) for \( a \in A \setminus C \).

**Definition 4.1.** We define the normalized rel \( \infty \) category \( \infty \)-\( \text{cat}_{\text{LS}}(X) \) of a locally compact space \( X \) as the minimal \( k \) such that there is a cover \( \alpha X = V_0 \cup V_1 \cup \cdots \cup V_k \) by \( k + 1 \) closed subsets where each \( V_i \) can be rel \( \infty \) contracted to infinity.

It follows from the definition that for every locally compact space \( X \), \( \text{cat}_{\text{LS}}(\alpha X) \leq \infty \)-\( \text{cat}_{\text{LS}}(X) \), where \( \alpha X \) is the one-point compactification of \( X \).

**Question 4.2.** Does the equality \( \text{cat}_{\text{LS}}(\alpha X) = \infty \)-\( \text{cat}_{\text{LS}}(X) \) hold for all locally finite complexes with tame ends?

We recall that \( X \) has a tame end if there is a compactum \( C \subset X \) such that \( X \setminus \text{Int}(C) \) is homeomorphic to \( \partial C \times [0,1) \).

Since the one-point compactification of \( X \times X \) with the diagonal \( \Delta X \) removed is the quotient space \( (X \times X)/\Delta X \), the following theorem shows that Question 4.2 is closely related to characterization of the topological complexity \( \text{TC}^M \) by means of the LS-category.

**Theorem 4.3.** For any compact ENR \( X \),
\[
\text{cat}_{\text{LS}}(X \times X)/\Delta X \leq \text{TC}^M(X) \leq \infty \)-\( \text{cat}_{\text{LS}}((X \times X) \setminus \Delta X) \).
\]

**Proof.** The proof of the first inequality was presented informally in [GV2] after Corollary 9. Suppose that \( \text{TC}^M(X) = k \). Then by the definition there is an open cover \( U_0, U_1, \ldots, U_k \) of \( X \times X \) with continuous reserved sections \( s_i : U_i \to PX \) of \( \pi : PX \to X \times X \). By Proposition 3.1 there are strict deformations of \( U_i \) in \( X \times X \) to the diagonal \( \Delta X \). They define the deformations of \( U_i / (U_i \cap \Delta X) \) to the point \( \{ \Delta X \} \) in \( (X \times X)/\Delta X \). Thus, \( \text{cat}_{\text{LS}}(X \times X)/\Delta X \leq k \).
Let $\infty \cdot \text{cat}_{LS}(\Delta X) = k$ and let $(X \times X) \setminus \Delta X = F_0 \cup F_1 \cup \cdots \cup F_k$ be the union of $k + 1$ closed sets rel $\infty$ contractible to infinity. Let $W$ be a neighborhood of the diagonal $\Delta X$ in $X \times X$ that admits a deformation retraction $r_i$ to $\Delta X$. Let $h^i_t$ be a deformation of $F_i$ into $W$. Then the concatenation of $h^i_t$ and $r_i$ defines a deformation $H_i$ of $F_i$ to the diagonal. Let $F_i = F_i \cup \Delta X$. Note that $H_i$ together with identity on $\Delta X$ defines a strict deformation of $F_i$ to the diagonal. \qed

Theorem 4.3 together with Theorem 2.5 recover Theorem 10(2) of \cite{GV2}.

**Corollary 4.4.** For a $k$-connected CW complex $X$,
\begin{equation}
\text{cat}_{LS}((X \times X) \setminus \Delta X) \leq \text{TC}(X),
\end{equation}
provided $(k + 1)(\text{TC}(X) + 1) > \dim X + 1$.

**Proof.** By Theorem 2.5 $\text{TC}(X) = \text{TC}^M(X)$ in this case. \qed

The relations between $\text{TC}(X)$, $\text{TC}^M(X)$, and $\text{cat}_{LS}(X \times X) \setminus \Delta X$ are still unknown. It could be that all three invariants coincide. A combination of results of \cite{GV2} and this paper shows that they do coincide for large classes of spaces. In view of Theorem 4.3 we believe the following might be true:

**Conjecture 4.5.** For CW complexes $X$, $\text{TC}^M(X) = \text{cat}_{LS}(X \times X) \setminus \Delta X$.

It seems rather surprising that Theorem 10(2) of \cite{GV2} and Theorem 2.5 have the same conditions. Perhaps this is another indication in favour of this conjecture.

**Remark 4.6.** For the topological complexity $\text{TC}(X)$ a weaker version of the first inequality from Theorem 4.3 was proven in \cite{F2}, Lemma 18.3:
\begin{equation}
\text{cat}_{LS}(X \times X) \setminus \Delta X - 1 \leq \text{TC}(X).
\end{equation}

Thus Theorem 4.3 suggests an approach to a possible counterexample to the Iwase-Sakai conjecture. Namely, if one finds a space $X$ where the above Farber’s inequality is sharp, by Theorem 4.3 we would obtain $\text{TC}^M(X) > \text{TC}(X)$. In view of Theorem 2.5 such space in the non-simply connected case should satisfy the inequality $\text{TC}(X) \leq \dim X$. The computation of the topological complexity of real projective spaces \cite{FTY} produces an example of such $X = \mathbb{R}P^{3}$. But $\mathbb{R}P^{3}$ cannot be a counterexample to the Iwase-Sakai conjecture, since it is a Lie group (see Lemma 2.7). Moreover, projective spaces could not give a counterexample to the Iwase-Sakai conjecture in this approach in view of the equality $\text{TC}(\mathbb{R}P^n) = \text{cat}_{LS}((\mathbb{R}P^n \times \mathbb{R}P^n) \setminus \Delta \mathbb{R}P^n)$ proven for all $n$ in \cite{GV2}, Theorem 11. Perhaps the simplest space for which it’s unknown whether $\text{TC} = \text{TC}^M$ is the Heisenberg manifold $H^3$. We recall that $H^3$ can be viewed as the pull-back of the Hopf bundle $S^3 \to S^2$ by means of a degree one map $f : S^1 \times S^1 \to S^2$.

**Question 4.7.** What is $\text{cat}_{LS}(H^3 \times H^3) \setminus \Delta H^3$?

As was mentioned, the topological complexity of $X$ equals the Schwarz genus of a certain fibration. It turns out that for general fibrations we still have the inequalities similar to Theorem 4.3.

**Theorem 4.8.** For any fibration of compact spaces $p : X \to Y$,
\begin{equation}
\text{cat}_{LS}(C_p) - 1 \leq \text{sg}(p) \leq \infty \cdot \text{cat}_{LS}(C_p \setminus \{\ast\}).
\end{equation}
Proof. We claim that if a subset \( U \subset Y \) admits a section \( s : U \to X \), then \( U \) is contractible in \( C_p \). Indeed, it can be moved to \( X \) in the mapping cylinder \( M_p \). Since the cone \( \text{Con}(X) \) is contained in \( C_p \), it could be further contracted to a point. Moreover, the mapping cylinder \( \hat{U} = M_{p|_{p^{-1}(U)}} \) of the restriction of \( p \) to the preimage \( p^{-1}(U) \) is contractible in \( C_p \), since it can be pushed to \( U \) first. If \( Y \) is covered by \( n+1 \) open sets \( U_0, \ldots, U_n \), each of which admits a section of \( p \), then the mapping cylinder \( M_p \) can be covered by \( n+1 \) sets \( \hat{U}_0, \ldots, \hat{U}_n \), all contractible in the mapping cylinder \( C_p \). Since \( C_p = M_p \cup \text{Con}(X) \), the open enlargements of the sets \( \hat{U}_0, \ldots, \hat{U}_n \), and \( \text{Con}(X) \) define an open cover of \( C_p \) by \( n+1 \) elements, all contractible in \( C_p \). Hence \( \text{cat}_{LS}(C_p) - 1 \leq \text{sg}(p) \).

Suppose that \( \infty - \text{cat}_{LS}(C_p \setminus \{\ast\}) \leq n \). Let \( V_0, \ldots, V_n \) be a closed cover of \( C_p \setminus \{\ast\} \) by sets that can be \( \text{rel} \infty \) contracted to infinity. Let

\[
H_i : V_i \times I \to C_p \setminus \{\ast\}
\]

be a contraction such that

\[
H_i((V_i \times 1) \subset \text{Con}(X) \setminus \{\ast\} \subset C_p \setminus \{\ast\}.
\]

We define \( F_i = V_i \cap Y \subset C_p \). Let \( \pi : \text{Con}(X) \setminus \{\ast\} \to X \) be the projection. By the Homotopy Lifting Property, the homotopy \( p \circ H_i|_{F_i \times [0,1]} : F_i \times [0,1] \to Y \) has a lift \( H'_i : F_i \times [0,1] \to X \) which coincides with \( \pi \circ H_i \) on \( F_i \times 1 \). Then \( H'_i \) restricted to \( F_i \times 0 \) is a section of \( p \) over \( F_i \). Thus, \( \text{sg}(p) \leq \infty - \text{cat}_{LS}(C_p \setminus \{\ast\}) \).

The following example shows that neither of the two inequalities of Theorem 4.8 can be improved.

Example 4.9. (1) For the identity map \( 1_X : X \to X \) in view of the equality \( C_{1_X} = \text{Con}(X) \) we obtain

\[
\text{cat}_{LS}(C_{1_X}) - 1 = -1 < \text{sg}(1_X) = 0 = \text{cat}_{LS}(\text{Con}(X)) = \infty - \text{cat}_{LS}(C_{1_X} \setminus \{\ast\}).
\]

For the square map \( p : S^1 \to S^1 \), \( p(z) = z^2 \),

\[
\text{cat}_{LS}(C_p) - 1 = 1 = \text{sg}(p) < 2 = \text{cat}_{LS}(C_p) \leq \infty - \text{cat}_{LS}(C_p \setminus \{\ast\}),
\]

since \( C_p = \mathbb{R} P^2 \) and \( \text{cat}_{LS}(\mathbb{R} P^2) = 2 \).

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References


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