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TOPOLOGICAL COMPLEXITY OF WEDGES AND COVERING MAPS

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Abstract. We present some results supporting the Iwase-Sakai conjecture about coincidence of the topological complexity $\mathrm{TC}(X)$ and monoidal topological complexity $\mathrm{TC}^M(X)$. Using these results we provide lower and upper bounds for the topological complexity of the wedge $X \vee Y$. We use these bounds to give a counterexample to the conjecture asserting that $\mathrm{TC}(X') \leq \mathrm{TC}(X)$ for any covering map $p : X' \to X$.

Also we discuss a possible reduction of the monoidal topological complexity to the Lusternik-Schnirelmann category.

1. Introduction

Let $PX = X^{[0,1]}$ denote the space of all paths in a path connected space X. Let $i_X : X \to PX$ be the inclusion of X into PX as a subspace of constant paths. There is a natural fibration $\pi : PX \to X \times X$ defined as $\pi(f) = (f(0), f(1))$ for $f \in PX$, $f : [0,1] \to X$.

Let X be an ENR. A piecewise continuous section $s : X \times X \to PX$ of π is called a *motion planning algorithm*. We say that a motion planning algorithm s has *complexity* k if $X \times X$ can be presented as a disjoint union $F_0 \cup \dots \cup F_k$ of $k+1$ ENRs such that s is continuous on each F_i. The normalized *topological complexity* $\mathrm{TC}(X)$ of a space X was defined by Farber as the minimum k such that there is a motion planning algorithm of complexity k [F1]. Equivalently, $\mathrm{TC}(X)$ is the minimal number k such that $X \times X$ admits an open cover $U_0, \dots, U_k$ by $k+1$ sets such that over each U_i there is a continuous section of π.

We say that a motion planning algorithm $s : X \times X \to PX$ is *reserved* if $s|_{\Delta X} = i_X$, where $\Delta X \subset X \times X$ is the diagonal. In other words, if the initial position of a robot in the configuration space X coincides with the terminal position, then the algorithm keeps the robot still. This condition on the motion planning algorithms seems to be very natural. The corresponding complexity of a space X was denoted by Iwase and Sakai as $\mathrm{TC}^M(X)$ and was called the *monoidal topological complexity* of X [IS1]. In the original definition they additionally assumed that all sets U_i contain the diagonal. Their definition agrees with the above since their condition can always be achieved by reduction of an open cover $U_0, \dots, U_k$ with reserved sections s_i to a closed cover $F_0, \dots, F_k$, $F_i \subset U_i$, then by adding the

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diagonal to each F_i with the natural extension of the sections $\bar{s}_i$, and then by taking an open enlargement V_i of the sets $F_i \cup \Delta X$ that admit extensions of the sections $\bar{s}_i$.

Iwase and Saki conjectured that $\mathrm{TC}^M(X) = \mathrm{TC}(X)$. In fact, first they gave a proof to the conjecture in [IS1] and then withdrew it in [IS2]. We prove this conjecture under the assumption that $\mathrm{TC}(X) > \dim X$. Also, using the Weinberger Lemma from [F3] we show that the conjecture holds true when X is a Lie group.

The topological complexity is closely related to the Lusternik-Schnirelmann category $\mathrm{cat}_{\mathrm{LS}}(X)$ of a space which is defined as the minimal number k such that X can be covered by $k+1$ open sets $U_0, U_1, \ldots, U_k$, all contractible to a point in X. Throughout the paper we use the normalized LS-category (as well as the normalized topological complexity), since they improve several important inequalities in the theory:

$$\text{cup-length}(X) \leq \mathrm{cat}_{\mathrm{LS}}(X) \leq \dim(X) \text{ and } \mathrm{cat}_{\mathrm{LS}}(X \times Y) \leq \mathrm{cat}_{\mathrm{LS}}(X) + \mathrm{cat}_{\mathrm{LS}}(Y),$$

where in the first inequality the cup-length is taken for any reduced cohomology (possibly twisted) [CLOT].

Some of the formulas for $\mathrm{cat}_{\mathrm{LS}}$ translate to similar statements for TC. For example, for TC there is an inequality similar to the above for the product of two spaces [F4]. Also, there are analogous estimates of TC in terms of the cup product and dimension [F4]. On the other hand, the simple $\mathrm{cat}_{\mathrm{LS}}$ formula for the wedge $\mathrm{cat}_{\mathrm{LS}}(X \vee Y) = \max\{\mathrm{cat}_{\mathrm{LS}} X, \mathrm{cat}_{\mathrm{LS}} Y\}$ does not hold for TC. So far there is no good analog of it for TC. The best that we can prove here is Theorem 3.6 from this paper. Another example is the formula $\mathrm{cat}_{\mathrm{LS}}(Y) \geq \mathrm{cat}_{\mathrm{LS}}(X)$ for a covering map $p : X \to Y$ which supports an intuitive idea that a covering space is always simpler than the base. So it was natural to assume that the same holds true for TC. We learned about this problem from Yuli Rudyak. In this paper Theorem 3.8 gives a negative answer to this question.

There have been several attempts to reformulate the topological complexity of X as some modified category of a related space. Thus, Garcia Calcines and Vandembroucq defined the weak topological complexity $\mathrm{wTC}(X)$ [GV1] in the spirit of Berstein-Hilton's [BH] definition of the weak category $\mathrm{wcat}(X)$. They proved the equality $\mathrm{wTC}(X) = \mathrm{wcat}(X \times X)/\Delta X$ and found some sufficient conditions for the equality $\mathrm{wTC}(X) = \mathrm{TC}(X)$. Then in [GV2] they established the equality $\mathrm{TC}(X) = \mathrm{cat}_{\mathrm{LS}}(X \times X)/\Delta X$ for large classes of spaces. In this paper we discuss a possibility of such equality for the monoidal topological complexity: $\mathrm{TC}^M(X) = \mathrm{cat}_{\mathrm{LS}}(X \times X)/\Delta X$. We define a *rel* ∞ category $\infty\text{-}\mathrm{cat}_{\mathrm{LS}}(Y)$ of non-compact spaces Y and discuss the problem of coincidence between $\mathrm{cat}_{\mathrm{LS}}(X/A)$ and $\infty\text{-}\mathrm{cat}_{\mathrm{LS}}(X \setminus A)$ for a subcomplex $A \subset X$ of a finite complex X. Then we show that $\mathrm{TC}^M(X)$ is bounded between $\mathrm{cat}_{\mathrm{LS}}(X \times X)/\Delta X$ and $\infty\text{-}\mathrm{cat}_{\mathrm{LS}}(X \times X \setminus \Delta X)$.

Note that for the topological complexity Farber proved the inequality [F2]

$$\mathrm{cat}_{\mathrm{LS}}(X \times X)/\Delta X - 1 \leq \mathrm{TC}(X).$$

Thus, in view of the Iwase-Sakai conjecture, examples of spaces for which Farber's inequality is sharp are of great interest.

We recall that both $\mathrm{cat}_{\mathrm{LS}}(X)$ and $\mathrm{TC}(X)$ are special cases of the Schwarz genus [Sch]: $\mathrm{cat}_{\mathrm{LS}}(X) = sg(\pi_0 : P_0X \to X)$ and $\mathrm{TC}(X) = sg(\pi : PX \to X \times X)$, where $P_0X \subset PX$ is the subspace of paths $f : [0,1] \to X$ that start in a base point $x_0 \in X$, $f(0) = x_0$, and $\pi_0(f) = f(1)$. We recall that the normalized *Schwarz* *genus* [Sch] (also known as the *sectional category* [J]) of a fibration $p : X \to Y$ is the minimal number of open sets $U_0, \dots, U_k$ that cover Y and admit sections $s_i : U_i \to X$ of p. In this paper we estimate the Schwarz genus [Sch] of arbitrary fibration $p : X \to Y$ in terms of the category of its mapping cone C_p,

$$\mathrm{cat}_{\mathrm{LS}}(C_p) - 1 \le sg(p) \le \infty\text{-}\,\mathrm{cat}_{\mathrm{LS}}(C_p \setminus \{*\}).$$

Similar estimates for the weak sectional category in terms of $\mathrm{wcat}(C_p)$ were made in [GV1]:

$$\mathrm{wcat}(C_p) - 1 \le wsecat(p) \le \mathrm{wcat}(C_p).$$

The similarity would be complete if the problem about the equality $\mathrm{cat}_{\mathrm{LS}} X/A = \infty\text{-}\,\mathrm{cat}_{\mathrm{LS}}(X \setminus A)$ could be solved affirmatively.

2. Monoidal topological complexity

Theorem 2.1. *For ENR spaces,*

$$\mathrm{TC}(X) \le \mathrm{TC}^M(X) \le \mathrm{TC}(X) + 1.$$

This theorem was proved in [IS2]. Since the proof there is too technical, we give an alternative proof.

Proof. The first inequality is obvious. Since X is ANR, there is an open neighborhood W of the diagonal ΔX in $X \times X$ and a continuous map $\phi : W \times [0,1] \to X$ such that $\phi(x, x', 0) = x$, $\phi(x, x', 1) = x'$, and $\phi(x, x, t) = x$ for all $t \in [0,1]$, $(x, x') \in W$. Let $\mathrm{TC}(X) = n$ and let $U_0, U_1, \dots, U_n$ be an open cover of $X \times X$ by sets that admit sections $s_i : U_i \to PX$ of π. Let F be a closed neighborhood of ΔX that lies in W. Then all sets in the open cover $U_0 \setminus F, \dots, U_n \setminus F, W$ of $X \times X$ admit reserved sections. Hence $\mathrm{TC}^M(X) \le n + 1$. $\square$

Note that the path fibration $\pi : PX \to X \times X$ restricted over the diagonal defines the free loop fibration $p : LX \to X$. A *canonical* section $\bar{s} : \Delta X \to LX$ of p is defined as $\bar{s}(x) = c_x$, where $c_x : I \to X$ is the constant map to x.

We use the standard convention to denote the elements of the iterated join product $X_0 * \dots * X_n$ as formal linear combinations $t_0 x_0 + t_1 x_1 + \dots + t_n x_n$, $\sum t_i = 1$, $t_i \ge 0$, $x_i \in X_i$, where all summands of the type $0x_i$ are dropped. We use the notation $*^n X$ for the iterated join product of n copies of X with itself.

We recall that a fiberwise join of maps $f_i : X_i \to Y$, $i = 0, \dots, n$, is the map

$$f_0 \tilde{*} \cdots \tilde{*} f_n : X_0 \tilde{*}_Y \cdots \tilde{*}_Y X_n \to Y,$$

where

$$X_0 \tilde{*}_Y \cdots \tilde{*}_Y X_n = \{t_0 x_0 + \dots + t_n x_n \in X_0 * \dots * X_n \mid f_0(x_0) = \dots = f_n(x_n)\}$$

is the fiberwise join of spaces $X_0, \dots, X_n$ and

$$(f_0 \tilde{*} \dots \tilde{*} f_n)(t_0 x_0 + \dots + t_n x_n) = f_i(x_i).$$

Thus, the preimage $(f_0 \tilde{*} \cdots \tilde{*} f_n)^{-1}(y)$ of a point $y \in Y$ is the join product of the preimages $f_0^{-1}(y) * \dots * f_n^{-1}(y)$.

We define $P_n X = PX \tilde{*}_{X \times X} \cdots \tilde{*}_{X \times X} PX$ and

$$\pi_n = \pi \tilde{*} \cdots \tilde{*} \pi : P_n X \to X \times X$$

to be the fiberwise join product of $n + 1$ copies of π. Note that there are imbeddings $PX = P_0 X \subset P_1 X \subset \dots \subset P_n X$ such that $\pi_i|_{P_{i-1}(X)} = \pi_{i-1}$. Then the section $\bar{s} : X \times X \to PX$ of π can be regarded as a section of π_n. Also we define

$p_0 = p : LX \to X$, $L_nX = L_{n-1}X\tilde{*}_X LX$, and $p_n = p_{n-1}\tilde{*}p : L_nX \to X$. Note that $\pi_n^{-1}(\Delta X) \cong L_nX$ and p_n is the restriction of π_n to $\pi_n^{-1}(\Delta X)$. Note also that the canonical section $\bar{s}$ defines a trivial subbundle $p'_n : E \to X$ of p_n with the fiber n-simplex Δ^n.

We recall that a map $p : E \to B$ satisfies the *Homotopy Lifting Property for a pair* (X, A) if for any homotopy $H : X \times I \to B$ with a lift $H' : A \times I \to E$ of the restriction $H|_{A\times I}$ and a lift H_0 of $H|_{X\times 0}$ which agrees with H', there is a lift $\bar{H} : X \times I \to E$ of H which agrees with H_0 and H'. We recall that a pair of spaces (X, A) is called an NDR pair if A is a deformation retract of a neighborhood in X. A pair (X, A) is NDR if and only if the inclusion $A \to X$ is a closed cofibration [TD]. The following is well-known; see [TD], Corollary 5.5.3:

Theorem 2.2. *Any Hurewicz fibration $p : E \to B$ satisfies the Homotopy Lifting Property for NDR pairs (X, A).*

Corollary 2.3. *Let $p : E \to X$ be a Hurewicz fibration with a section $s : X \to E$. A fiberwise homotopy $G : A\times I \to E$ of the restriction $s|_A$ to a closed subset $A \subset X$ can be extended to a fiberwise homotopy $\bar{G} : X \times I \to E$ of s provided (X, A) is an NDR pair.*

Proposition 2.4. *For CW complexes X,*

(1) $\mathrm{TC}(X) \le n \;\Leftrightarrow\; \pi_n : P_nX \to X \times X$ *admits a section.*

(2) $\mathrm{TC}^M(X) \le n \;\Leftrightarrow\; \pi_n : P_nX \to X \times X$ *admits a section s which agrees with the canonical section over the diagonal, $s|_{\Delta X} = \bar{s}$.*

Proof. Statement (1) is a part of a general theorem proven by Schwarz [Sch] for fibrations $q : X \to Y$, $sg(q) \le n$ if and only if the fiberwise join product $\tilde{*}^{n+1}q : \tilde{*}_Y^{n+1}X \to Y$ of $n + 1$ copies of q admits a section.

The implication $\Leftarrow$ in (2) is obvious. For the other direction we note that $n+1$ reserved sections $s_i : U_i \to PX$ defined for an open cover $U_0, \dots, U_n$ of $X\times X$ define a section s of π_n with the image $s(\Delta X)$ lying in E. Therefore over ΔX it could be fiberwise deformed to $\bar{s}$. By Theorem 2.2 that deformation can be extended to a fiberwise deformation of s. □

Theorem 2.5. *The equality*

$$\mathrm{TC}(X) = \mathrm{TC}^M(X)$$

holds true for k-connected simplicial complexes X such that

$$(k + 1)(\mathrm{TC}(X) + 1) > \dim X + 1.$$

Proof. Let $\mathrm{TC}(X) = n$. Note that the fiber $\pi^{-1}(x, x')$ is homotopy equivalent to the loop space $\Omega(X)$. Since $\Omega(X)$ is $(k - 1)$-connected, the iterated join product $*^{n+1}\Omega(X)$ is $((k + 1)(n + 1) - 2)$-connected. We show that any section $s : \Delta X \to L_nX$ can be joined by a fiberwise homotopy to the canonical section $\bar{s} : \Delta X \to L_nX$. By induction on i we construct a section $s_i : X \to L_nX$, that coincides with $\bar{s}$ on the i-skeleton $X^{(i)}$, together with a fiberwise homotopy joining s and s_i. Here we use the identification $\Delta X = X$. For $i = 0$ we take paths in the fibers $p_n^{-1}(v)$ joining $s(v)$ and $\bar{s}(v)$ for all $v \in X^{(0)}$. Then we extend them to a fiberwise homotopy of s to a section s_0. Assume that s_{i-1} is already constructed and $i \le \dim X \le (k + 1)(n + 1) - 2$. Independently for every i-simplex $\sigma \subset X$ we consider the problem of joining s_{i-1} with $\bar{s}$ over σ by a fiberwise homotopy. Since the fiber bundle p_n is trivial over σ with an i-connected fiber, the identity homotopy on the boundary $\partial\sigma$ can be extended to a homotopy between $\bar{s}|_\sigma$ and $s_{i-1}|_\sigma$. This extension can be deformed to a fiberwise homotopy. All these homotopies together define a fiberwise homotopy between s_{i-1} and $\bar{s}$ over $X^{(i)}$. Since $(X, X^{(i)})$ is an NDR pair, by Corollary 2.3 we can extend it to a fiberwise homotopy over X.

Let $s : X \times X \to P_nX$ be a section. We proved that on ΔX it can be deformed to a canonical section $\bar{s}$. Since $(X \times X, \Delta X)$ is an NDR pair, by Corollary 2.3 there is a fiberwise homotopy of s to a section s' that coincides with $\bar{s}$ on ΔX. Therefore, $\mathrm{TC}^M(X) \leq n$. ☐

Corollary 2.6. $\mathrm{TC}(S^n) = \mathrm{TC}^M(S^n)$ *for* $n > 1$.

The case of $n = 1$ is taken care of by Lemma 2.7.

The following is an extension of Weinberger's Lemma from [F3] to the case of monoidal topological complexity.

Lemma 2.7. *For a connected Lie group* G*,*

$$\mathrm{TC}(G) = \mathrm{TC}^M(G) = \mathrm{cat}_{\mathrm{LS}}(G).$$

Proof. In view of what is already known [F3], it suffices to show the inequality $\mathrm{TC}^M(G) \leq \mathrm{cat}_{\mathrm{LS}}(G)$. Let $\mathrm{cat}_{\mathrm{LS}}(G) = n$ and let $U_0, U_1, \dots, U_n$ be an open cover of G together with homotopies $H_i : U_i \times [0,1] \to G$ contracting U_i to the unit $e \in G$. Clearly, we may assume that $e \notin U_i$ for $i > 0$. Since the inclusion $e \in G$ is a cofibration, we may assume that $H_0(e,t) = e$ for all t (see Lemma 1.25 of [CLOT]). Then for the open cover of $G \times G$ as defined in [F3],

$$W_i = \{(a,b) \in G \times G \mid b^{-1}a \in U_i\},$$

the sections $s_i : W_i \to PG$ defined by the formula

$$s_i(a,b)(t) = bH_i(b^{-1}a, t) \in G, \quad (a,b) \in W_i,$$

are reserved. Indeed, $\Delta G \cap W_i = \varnothing$ for $i > 0$ and

$$s_0(a,a)(t) = aH_0(a^{-1}a, t) = aH_0(e,t) = ae = a$$

for all $(a,a) \in \Delta G$. ☐

3. Topological complexity of wedge and covering maps

A *deformation* of $U \subset Z$ in Z to a subset $A \subset Z$ is a continuous map $D : U \times I \to Z$ such that $D(u,0) = u$, $D(u,1) \in A$ for all $u \in U$. A *strict deformation* of $U \subset Z$ in Z to $A \subset Z$ is a deformation $D : U \times I \to Z$ such that $D(u,t) = u$ for all $t \in I$ whenever $u \in A$.

Proposition 3.1. *Let* X *be a metric space. For an open set* $U \subset X \times X$ *the following are equivalent:*

(1) There is a reserved section $s : U \to PX$ *over* U *of the fibration* $\pi : PX \to X \times X$.

(2) There is a strict deformation $D : U \times I \to X \times X$ *to the diagonal* $\Delta X = \{(x,x) \in X \times X \mid x \in X\}$.

(3) For any choice of a base point $x_0 \in X$ *there is a strict deformation* D *of* U *to* ΔX *which preserves faces* $X \times x_0$ *and* $x_0 \times X$*; i.e., for all* $t \in I$*,*

$$D((x,x_0),t) \in X \times x_0 \quad \text{and} \quad D((x_0,x),t) \in x_0 \times X.$$

Proof. (1) $\Rightarrow$ (3). Let $\|x\| = d(x, x_0)$. We define

$$D((x,y),t) = s(x,y)(\frac{\|x\|}{\|x\|+\|y\|}t), s(x,y)(1 - \frac{\|y\|}{\|x\|+\|y\|}t)$$

if $(x,y) \neq (x_0,x_0)$ and define $D((x_0,x_0),t) = (x_0,x_0)$. Since $s(x,y)(0) = x$ and $s(x,y)(1) = y$, we obtain that $D((x,y),0) = (x,y)$. Note that

$$D((x,y),1) = s(x,y)(\frac{\|x\|}{\|x\|+\|y\|}), s(x,y)(\frac{\|x\|}{\|x\|+\|y\|}) \in \Delta X.$$

Since the section s is reserved, $D((x,x),t) = (s(x,x)(t/2), s(x,x)(t/2)) = (x,x)$. Note that

$$D((x,x_0),t) = (s(x,x_0)(t), s(x,x_0)(1)) = (s(x,x_0)(t), x_0) \in X \times x_0$$

and

$$D((x_0,y),t) = (s(x_0,y)(0), s(x_0,y)(1-t)) = (x_0, s(x_0,y)(1-t)) \in x_0 \times X.$$

The deformation D is continuous at (x_0,x_0) since the section $s(x_0,x_0)$ is stationary at (x_0,x_0).

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let $pr_1 : X \times X \to X$ denote the projection to the first factor and $pr_2 : X \times X \to X$ to the second. Given a strict deformation $D : U \times I \to X \times X$ we define a section $s : U \to PX$ as follows:

$$s(x,y)(t) = \begin{cases} pr_1 D((x,y),2t) & \text{if } t \leq 1/2, \\ pr_2 D((x,y),2-2t) & \text{if } t \geq 1/2. \end{cases}$$

This path is well-defined since $D((x,y),1) \in \Delta X$. Clearly it is a path from x to y. If $x = y$, the path is stationary. Thus s is a reserved section. ☐

The following lemma was proven in [F4], Lemma 4.25.

Lemma 3.2. *Let A be a retract of an ENR space X. Then* $\mathrm{TC}(X) \geq \mathrm{TC}(A)$.

We recall that a family $\mathcal{U}$ of subsets of X is called a *k-cover*, $k \in \mathbb{N}$, if every subfamily of $\mathcal{U}$ that consists of k elements forms a cover of X. We use the following theorem [Dr].

Theorem 3.3. *Let $\{U_0', \dots, U_n'\}$ be an open cover of a normal topological space X. Then for any $m = n, n+1, \dots, \infty$ there is an open $(n+1)$-cover of X, $\{U_k\}_{k=0}^m$ such that $U_k = U_k'$ for $k \leq n$ and $U_k = \bigcup_{i=0}^n V_i$ is a disjoint union with $V_i \subset U_i$ for $k > n$.*

Corollary 3.4. *Suppose that all sets U_i', $i = 0, \dots, n$, in the theorem are (strictly) deformable in X to a subspace $A \subset X$. Then the sets U_k for all k are (strictly) deformable in X to A.*

The following proposition is well-known. It can be traced back to the work of Kolmogorov on Hilbert's 13th problem [Os].

Proposition 3.5. *Let $\mathcal{U} = \{U_0, \dots, U_{n+m}\}$ be an $(n+1)$-cover of X and let $\mathcal{V} = \{V_0, \dots, V_{m+n}\}$ be an $(m+1)$-cover of Y. Then the sets $W_k = U_k \times V_k$, $k = 0, \dots, n+m$, cover $X \times Y$.*

Proof. Let $(x,y) \in X \times Y$. A point x is covered by at least $m+1$ elements of $\mathcal{U}$. Otherwise $n+1$ elements that do not cover x would not form a cover of the space X. That would give a contradiction with the assumption that $\mathcal{U}$ is an $(n+1)$-cover of X. Let $x \in U_{i_0} \cap \cdots \cap U_{i_m}$. By the assumption, the family $V_{i_0}, \dots, V_{i_m}$ covers Y. Hence $y \in V_{i_s}$ for some s. Then $(x,y) \in W_{i_s}$. □

Theorem 3.6. *For all ANR spaces X and Y,*

$$\max\{\mathrm{TC}(X), \mathrm{TC}(Y), \mathrm{cat}_{\mathrm{LS}}(X \times Y)\} \leq \mathrm{TC}(X \vee Y)$$

$$\leq \mathrm{TC}^M(X \vee Y) \leq \mathrm{TC}^M(X) + \mathrm{TC}^M(Y).$$

Proof. Note that $\mathrm{TC}(X \vee Y) \geq \mathrm{TC}(X), \mathrm{TC}(Y)$ by Lemma 3.2. Let $r_X : X \vee Y \to X$ and $r_Y : X \vee Y \to Y$ be the retractions collapsing the wedge onto X and Y respectively. We may assume that the subset

$$X \times Y \subset (X \vee Y) \times (X \vee Y)$$

is covered by $\leq \mathrm{TC}(X \vee Y) + 1$ open sets U supplied with homotopies

$$H_U : U \times I \to X \vee Y$$

such that $H_U(x,y,0) = x$ and $H_U(x,y,1) = y$. For each U we define a homotopy $G_U : U \times I \to X \times Y$ by the formula

$$G_U(x,y,t) = (r_X H_U(x,y,t), r_Y H_U(x,y,1-t)).$$

Then

$$G_U(x,y,0) = (r_X H_U(x,y,0), r_Y H_U(x,y,1)) = (r_X(x), r_Y(y)) = (x,y)$$

and

$$G_U(x,y,1) = (r_X H_U(x,y,1), r_Y H_U(x,y,0)) = (r_X(y), r_Y(x)) = (v_0, v_0),$$

where v_0 is the wedge point in $X \vee Y$. Thus, G_U contracts U to a point in $X \times Y$.

Let $\mathrm{TC}^M(X) = n$ and $\mathrm{TC}^M(Y) = m$. Then there is an open cover $\tilde{U}_0, \dots, \tilde{U}_n$ of $X \times X$ with reserved sections $s_i : \tilde{U}_i \to PX$, $i = 0, \dots, n$. Similarly, let $\tilde{V}_0, \dots, \tilde{V}_m$ be an open covering of $Y \times Y$ with reserved sections $\sigma_j : \tilde{V}_j \to PY$, $j = 0, \dots, m$. By Proposition 3.1 all these sets are strictly deformable to the diagonal in $X \times X$ and $Y \times Y$ respectively. By Corollary 3.4 there is an open $(n+1)$-cover $\tilde{U}_0, \dots, \tilde{U}_n, \dots, \tilde{U}_{n+m}$ of $X \times X$ by sets strictly deformable to the diagonal. By Proposition 3.1 there are strict deformations

$$D_X^k : \tilde{U}_k \times I \to X \times X$$

of $\tilde{U}_k$ to ΔX that preserve faces $X \times v_0$ and $v_0 \times X$. Similarly, there is an open $(m+1)$-cover $\tilde{V}_0, \dots, \tilde{V}_m, \dots, \tilde{V}_{m+n}$ of $Y \times Y$ and there are strict deformations D_Y^k of $\tilde{V}_k$ in $Y \times Y$ to the diagonal ΔY that preserve faces.

We use the notation

$$U_k = \tilde{U}_k \cap (X \times v_0) \quad \text{and} \quad V_k = \tilde{V}_k \cap (v_0 \times Y), \quad k = 0, \dots, m+n.$$

Note that $U_0, \dots, U_{m+n}$ is an $(n+1)$-cover of $X \times v_0 = X$ and $V_0, \dots, V_{m+n}$ is an $(m+1)$-cover of $v_0 \times Y = Y$. Let $W_k = U_k \times V_k$. By Proposition 3.5, $W_0, \dots, W_{m+n}$ is an open cover of $X \times Y$.

The deformations D_X^k define the deformations $H_k : U_k \times I \to X \times v_0$ to the point $v_0 \in X$ and the deformations D_Y^k define the deformations $G_k : V_k \times I \to v_0 \times Y$ to the point $v_0 \in Y$. These deformations define the deformations

$$T_k : W_k \times I \to X \times Y$$

to the point (v_0, v_0) such that if $W_k \cap (X \times v_0) \neq \varnothing$, then $W_k \cap (X \times v_0) = U_k$ and $T_k|_{U_k \times I} = H_k$, and if $W_k \cap (v_0 \times Y) \neq \varnothing$, then $W_k \cap (v_0 \times Y) = V_k$ and $T_k|_{V_k \times I} = G_k$ for $k = 0, \dots, m+n$.

Symmetrically, define

$$U'_k = \tilde{U}_k \cap (v_0 \times X) \quad \text{and} \quad V'_k = \tilde{V}_k \cap (Y \times v_0), \quad k = 0, \dots, m+n,$$

and the corresponding deformations

$$H'_k : U'_k \times I \to X \quad \text{and} \quad G'_k : V'_k \times I \to Y$$

to the base points. Define $W'_k = U'_k \times V'_k$. By Proposition 3.5, the family $W'_0, \dots, W'_{n+m}$ is an open cover of $Y \times X$. As before there are deformations

$$T'_k : W'_k \times I \to Y \times X$$

to the point (v_0, v_0) such that if $W'_k \cap (v_0 \times X) \neq \varnothing$, then $W'_k \cap (v_0 \times X) = U'_k$ and $T'_k|_{U'_k \times I} = H'_k$, and if $W'_k \cap (Y \times v_0) \neq \varnothing$, then $W'_k \cap (Y \times v_0) = V_k$, $T'_k|_{V'_k \times I} = G'_k$ for $k = 0, \dots, m+n$.

We define the open sets

$$O_k = W_k \cup W'_k \cup \tilde{U}_k \cup \tilde{V}_k \subset (X \vee Y) \times (X \vee Y), \quad k = 0, \dots, n+m,$$

and note that $\mathcal{O} = \{O_k\}$ covers $(X \vee Y) \times (X \vee Y)$. Note that the set

$$C = (X \vee Y) \times v_0 \bigcup v_0 \times (X \vee Y)$$

defines a partition of $(X \vee Y) \times (X \vee Y)$ into four pieces: $X \times X$, $X \times Y$, $Y \times X$, and $Y \times Y$. Also note the inclusion $O_k \cap C \subset U_k \cup V_k \cup U'_k \cup V'_k$. By the construction the deformations D_X^k, D_Y^k, T_k, and T'_k all agree on $O_k \cap C$. Therefore the union of deformations

$$T_k \cup T'_k \cup D_X^k \cup D_Y^k : O_k \times I \to (X \vee Y) \times (X \vee Y)$$

is a well-defined deformation Q_k of O_k to the diagonal $\Delta(X \vee Y)$. Note that for all k, Q_k are strict deformations. By Proposition 3.1 each Q_k defines a reserved section $\alpha_k : O_k \to P(X \vee Y)$. Therefore,

$$\mathrm{TC}^M(X \vee Y) \leq n + m = \mathrm{TC}^M(X) + \mathrm{TC}^M(Y). \qquad \square$$

Remark 3.7. A stronger version of the upper bound of Theorem 3.6 was proposed in [F2], Theorem 19.1:

$$\mathrm{TC}(X \vee Y) \leq \max\{\mathrm{TC}(X), \mathrm{TC}(Y), \mathrm{cat}_{\mathrm{LS}}(X) + \mathrm{cat}_{\mathrm{LS}}(Y)\}.$$

Since the proof in [F2] is incomplete, we call this inequality *Farber's Conjecture*. Note that Farber's inequality in view of Theorem 3.6 would turn into an equality for spaces X and Y satisfying the product law: $\mathrm{cat}_{\mathrm{LS}}(X \times Y) = \mathrm{cat}_{\mathrm{LS}}(X) + \mathrm{cat}_{\mathrm{LS}}(Y)$.

Theorem 3.8. *(1) There is a* 2*-to-*1 *covering map* $p : E \to B$ *with* $\mathrm{TC}(E) > \mathrm{TC}(B)$.

(2) There is a finite complex X *with* $\mathrm{TC}(X) < \mathrm{TC}(\tilde{X})$*, where* $\tilde{X}$ *is the universal covering of* X.

Proof. (1) We take $B = T \vee S^1$, where $T = S^1 \times S^1$ is a 2-torus. Let E be the covering space defined by the 2-fold covering of S^1. Note that E is homeomorphic to the circle with two tori T attached at antipodal points. Thus, E is homotopy equivalent to $T \vee T \vee S^1$. By Theorem 3.6 and Lemma 2.7,

$$\mathrm{TC}(B) \leq \mathrm{TC}^M(T) + \mathrm{TC}^M(S^1) = \mathrm{cat_{LS}}(T) + \mathrm{cat_{LS}}(S^1) = 2 + 1 = 3.$$

On the other hand, by Theorem 3.6,

$$\mathrm{TC}(E) \geq \mathrm{cat_{LS}}((T \vee S^1) \times T)) \geq \text{cup-length}((T \vee S^1) \times T) = 4.$$

(2) Consider $X = (S^3 \times S^3) \vee S^1$. Since $S^3 \times S^3$ is a connected Lie group, by Lemma 2.7, $\mathrm{TC}^M(S^3 \times S^3) = \mathrm{cat_{LS}}(S^3 \times S^3) = 3$. By Theorem 3.6 and Lemma 2.7,

$$\mathrm{TC}(X) \leq \mathrm{TC}^M(S^3 \times S^3) + \mathrm{TC}^M(S^1) = \mathrm{cat_{LS}}(S^3 \times S^3) + \mathrm{cat_{LS}}(S^1) = 2 + 1 = 3.$$

Note that the universal cover $\tilde{X}$ is homotopy equivalent to an infinite wedge

$$Y = \bigvee^{\infty} (S^3 \times S^3).$$

Then Y admits a retraction onto $(S^3 \times S^3) \vee (S^3 \times S^3)$. By Lemma 3.2, Theorem 3.6, and the cup-length lower bound on $\mathrm{cat_{LS}}$,

$$\mathrm{TC}(\tilde{X}) \geq \mathrm{TC}((S^3 \times S^3) \vee (S^3 \times S^3)) \geq \mathrm{cat_{LS}}(S^3 \times S^3 \times S^3 \times S^3) \geq 4. \qquad \square$$

4. Topological complexity, LS-category, and Schwarz genus

We say a subset $A \subset X$ can be *rel* ∞ *contracted to infinity* if for every compact subset $F \subset X$ there is a larger compact set $F \subset C$ and a homotopy $h_t : A \to X$ with $h_0 = incl_A \subset X$, $h_1(A) \cap F = \varnothing$, and $h_t(a) = a$ for $a \in A \setminus C$.

Definition 4.1. We define the normalized *rel* ∞ category $\infty\text{-}\mathrm{cat_{LS}}(X)$ of a locally compact space X as the minimal k such that there is a cover $X = V_0 \cup V_1 \cup \cdots \cup V_k$ by $k+1$ closed subsets where each V_i can be *rel* ∞ contracted to infinity.

It follows from the definition that for every locally compact space X, $\mathrm{cat_{LS}}(\alpha X) \leq \infty\text{-}\mathrm{cat_{LS}}(X)$, where αX is the one-point compactification of X.

Question 4.2. Does the equality $\mathrm{cat_{LS}}(\alpha X) = \infty\text{-}\mathrm{cat_{LS}}(X)$ hold for all locally finite complexes with tame ends?

We recall that X has a tame end if there is a compactum $C \subset X$ such that $X \setminus \mathrm{Int}(C)$ is homeomorphic to $\partial C \times [0, 1)$.

Since the one-point compactification of $X \times X$ with the diagonal ΔX removed is the quotient space $(X \times X)/\Delta X$, the following theorem shows that Question 4.2 is closely related to characterization of the topological complexity TC^M by means of the LS-category.

Theorem 4.3. *For any compact ENR X,*

$$\mathrm{cat_{LS}}(X \times X)/\Delta X \leq \mathrm{TC}^M(X) \leq \infty\text{-}\mathrm{cat_{LS}}((X \times X) \setminus \Delta X).$$

Proof. The proof of the first inequality was presented informally in [GV2] after Corollary 9. Suppose that $\mathrm{TC}^M(X) = k$. Then by the definition there is an open cover $U_0, U_1, \ldots, U_k$ of $X \times X$ with continuous reserved sections $s_i : U_i \to PX$ of $\pi : PX \to X \times X$. By Proposition 3.1 there are strict deformations of U_i in $X \times X$ to the diagonal ΔX. They define the deformations of $U_i/(U_i \cap \Delta X)$ to the point $\{\Delta X\}$ in $(X \times X)/\Delta X$. Thus, $\mathrm{cat_{LS}}(X \times X)/\Delta X \leq k$.

Let $\infty\text{-}\operatorname{cat}_{\mathrm{LS}}((X\times X)\setminus\Delta X)=k$ and let $(X\times X)\setminus\Delta X=F_0\cup F_1\cup\cdots\cup F_k$ be the union of $k+1$ closed sets *rel* ∞ contractible to infinity. Let W be a neighborhood of the diagonal ΔX in $X\times X$ that admits a deformation retraction r_t to ΔX. Let h^i_t be a deformation of F_i into W. Then the concatenation of h^i_t and r_t defines a deformation H_i of F_i to the diagonal. Let $\bar{F}_i=F_i\cup\Delta X$. Note that H_i together with identity on ΔX defines a strict deformation of $\bar{F}_i$ to the diagonal. □

Theorem 4.3 together with Theorem 2.5 recover Theorem 10(2) of [GV2]:

Corollary 4.4. *For a k-connected CW complex X,*

$$\operatorname{cat}_{\mathrm{LS}}((X\times X)/\Delta X)\le \mathrm{TC}(X),$$

provided $(k+1)(\mathrm{TC}(X)+1)>\dim X+1$.

Proof. By Theorem 2.5, $\mathrm{TC}(X)=\mathrm{TC}^M(X)$ in this case. □

The relations between $\mathrm{TC}(X)$, $TC^M(X)$, and $\operatorname{cat}_{\mathrm{LS}}(X\times X)/\Delta X$ are still unknown. It could be that all three invariants coincide. A combination of results of [GV2] and this paper shows that they do coincide for large classes of spaces. In view of Theorem 4.3 we believe the following might be true:

Conjecture 4.5. *For CW complexes X,* $\mathrm{TC}^M(X)=\operatorname{cat}_{\mathrm{LS}}(X\times X)/\Delta X$.

It seems rather surprising that Theorem 10(2) of [GV2] and Theorem 2.5 have the same conditions. Perhaps this is another indication in favour of this conjecture.

Remark 4.6. For the topological complexity $\mathrm{TC}(X)$ a weaker version of the first inequality from Theorem 4.3 was proven in [F2], Lemma 18.3:

$$\operatorname{cat}_{\mathrm{LS}}(X\times X)/\Delta X-1\le \mathrm{TC}(X).$$

Thus Theorem 4.3 suggests an approach to a possible counterexample to the Iwase-Sakai conjecture. Namely, if one finds a space X where the above Farber's inequality is sharp, by Theorem 4.3 we would obtain $\mathrm{TC}^M(X)>\mathrm{TC}(X)$. In view of Theorem 2.5 such space in the non-simply connected case should satisfy the inequality $\mathrm{TC}(X)\le\dim X$. The computation of the topological complexity of real projective spaces ([FTY]) produces an example of such $X=\mathbb{R}P^3$. But $\mathbb{R}P^3$ cannot be a counterexample to the Iwase-Sakai conjecture, since it is a Lie group (see Lemma 2.7). Moreover, projective spaces could not give a counterexample to the Iwase-Sakai conjecture in this approach in view of the equality $\mathrm{TC}(\mathbb{R}P^n)=\operatorname{cat}_{\mathrm{LS}}((\mathbb{R}P^n\times\mathbb{R}P^n)/\Delta\mathbb{R}P^n)$ proven for all n in [GV2], Theorem 11. Perhaps the simplest space for which it's unknown whether $\mathrm{TC}=\mathrm{TC}^M$ is the Heisenberg manifold H^3. We recall that H^3 can be viewed as the pull-back of the Hopf bundle $S^3\to S^2$ by means of a degree one map $f: S^1\times S^1\to S^2$.

Question 4.7. What is $\operatorname{cat}_{\mathrm{LS}}(H^3\times H^3)/\Delta H^3$?

As was mentioned, the topological complexity of X equals the Schwarz genus of a certain fibration. It turns out that for general fibrations we still have the inequalities similar to Theorem 4.3.

Theorem 4.8. *For any fibration of compact spaces $p: X\to Y$,*

$$\operatorname{cat}_{\mathrm{LS}}(C_p)-1\le sg(p)\le\infty\text{-}\operatorname{cat}_{\mathrm{LS}}(C_p\setminus\{*\}).$$

Proof. We claim that if a subset $U \subset Y$ admits a section $s : U \to X$, then U is contractible in C_p. Indeed, it can be moved to X in the mapping cylinder M_p. Since the cone $\mathrm{Con}(X)$ is contained in C_p, it could be further contracted to a point. Moreover, the mapping cylinder $\hat{U} = M_{p|_{p^{-1}(U)}}$ of the restriction of p to the preimage $p^{-1}(U)$ is contractible in C_p, since it can be pushed to U first. If Y is covered by $n+1$ open sets $U_0, \dots, U_n$, each of which admits a section of p, then the mapping cylinder M_p can be covered by $n+1$ sets $\hat{U}_0, \dots, \hat{U}_n$, all contractible in the mapping cylinder C_p. Since $C_p = M_p \cup \mathrm{Con}(X)$, the open enlargements of the sets $\hat{U}_0, \dots, \hat{U}_n$, and $\mathrm{Con}(X)$ define an open cover of C_p by $n+1$ elements, all contractible in C_p. Hence $\mathrm{cat}_{\mathrm{LS}}(C_p) - 1 \le sg(p)$.

Suppose that $\infty\text{-}\,\mathrm{cat}_{\mathrm{LS}}(C_p \setminus \{*\}) \le n$. Let $V_0, \dots, V_n$ be a closed cover of $C_p \setminus \{*\}$ by sets that can be *rel* ∞ contracted to infinity. Let

$$H_i : V_i \times I \to C_p \setminus \{*\}$$

be a contraction such that

$$H_i(V_i \times 1) \subset \mathrm{Con}(X) \setminus \{*\} \subset C_p \setminus \{*\}.$$

We define $F_i = V_i \cap Y \subset C_p$. Let $\pi : \mathrm{Con}(X) \setminus \{*\} \to X$ be the projection. By the Homotopy Lifting Property, the homotopy $p \circ H_i|_{F_i \times [0,1]} : F_i \times [0,1] \to Y$ has a lift $H_i' : F_i \times [0,1] \to X$ which coincides with $\pi \circ H_i$ on $F_i \times 1$. Then H_i' restricted to $F_i \times 0$ is a section of p over F_i. Thus, $sg(p) \le \infty\text{-}\,\mathrm{cat}_{\mathrm{LS}}(C_p \setminus \{*\})$. $\square$

The following example shows that neither of the two inequalities of Theorem 4.8 can be improved.

Example 4.9. (1) For the identity map $1_X : X \to X$ in view of the equality $C_{1_X} = \mathrm{Con}(X)$ we obtain

$$\mathrm{cat}_{\mathrm{LS}}(C_{1_X}) - 1 = -1 < sg(1_X) = 0 = \mathrm{cat}_{\mathrm{LS}}(\mathrm{Con}(X)) = \infty\text{-}\,\mathrm{cat}_{\mathrm{LS}}(C_{1_X} \setminus \{*\}).$$

For the square map $p : S^1 \to S^1$, $p(z) = z^2$,

$$\mathrm{cat}_{\mathrm{LS}}(C_p) - 1 = 1 = sg(p) < 2 = \mathrm{cat}_{\mathrm{LS}}(C_p) \le \infty\text{-}\,\mathrm{cat}_{\mathrm{LS}}(C_p \setminus \{*\}),$$

since $C_p = \mathbb{R}P^2$ and $\mathrm{cat}_{\mathrm{LS}}(\mathbb{R}P^2) = 2$.

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