EXACT DECAY RATE OF A NONLINEAR ELLIPTIC EQUATION RELATED TO THE YAMABE FLOW

SHU-YU HSU

(Communicated by Walter Craig)

Abstract. Let 0 < m < \frac{n-2}{n}, n \geq 3, \alpha = \frac{2\beta + \rho}{1-m}, and \beta > \frac{m\rho}{n-2-mn} for some constant \rho > 0. Suppose v is a radially symmetric solution of \frac{n-1}{m} \Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, v > 0, in \mathbb{R}^n. When m = \frac{n-2}{n+2}, the metric g = v^{\frac{n-2}{n+2}} dx^2 corresponds to a locally conformally flat Yamabe shrinking gradient soliton with positive sectional curvature. We prove that the solution v of the above nonlinear elliptic equation has the exact decay rate \lim_{r \to \infty} r^2 v(r)^{1-m} = \frac{2(n-1)(n(1-m)-2)}{(1-m)(\alpha(1-m)-2\beta)}.

1. Introduction

Recently, there has been a lot of study of the equation

\begin{equation}
\frac{n-1}{m} \Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, \quad v > 0, \quad \text{in } \mathbb{R}^n
\end{equation}

where

\begin{equation}
0 < m < \frac{n-2}{n}, \quad n \geq 3,
\end{equation}

and

\begin{equation}
\alpha = \frac{2\beta + \rho}{1-m}
\end{equation}

for some constant \rho \in \mathbb{R} by P. Daskalopoulos and N. Sesum [DS2]; S.Y. Hsu [H1], [H2]; M.A. Peletier and H. Zhang [PZ]; and J.L. Vázquez [V1]. In the paper [DS2] P. Daskalopoulos and N. Sesum (cf. [CSZ], [CMM]) proved the important result that any locally conformally flat non-compact gradient Yamabe soliton g with positive sectional curvature on an n-dimensional manifold, n \geq 3, must be radially symmetric and have the form g = v^{\frac{n-2}{n+2}} dx^2, where dx^2 is the Euclidean metric on \mathbb{R}^n and v is a radially symmetric solution of (1.1) with m = \frac{n-2}{n+2}, and \alpha, \beta satisfy (1.3) for some constant \rho > 0, \rho = 0 or \rho < 0, depending on whether g is a shrinking, steady, or expanding Yamabe soliton.

On the other hand, as observed by B.H. Gilding, M.A. Peletier and H. Zhang [GP], [PZ], and others ([DS1], [DS2], [V1], [V2]), (1.1) also arises in the study of the self-similar solutions of the degenerate diffusion equation

\begin{equation}
u_t = \frac{n-1}{m} \Delta u^m \quad \text{in } \mathbb{R}^n \times (0,T).
\end{equation}
For example (cf. [H1], [V1]) if \( v \) is a radially symmetric solution of \((1.1)\) with 
\[
\alpha = \frac{2\beta + 1}{1 - m} > 0,
\]
than for any \( T > 0 \) the function 
\[
(1.5) \quad u(x,t) = (T - t)^\alpha v(x(T - t)^\beta)
\]
is a solution of \((1.4)\) in \( \mathbb{R}^n \times (-\infty, T) \). We refer the reader to the book [V1] and the paper [H1] for the relation between solutions of \((1.1)\) and the other self-similar solutions of \((1.4)\) for the other parameter ranges of \( \alpha, \beta \).

Note that when \( v \) is a radially symmetric solution of \((1.1)\), then \( v \) satisfies 
\[
(1.6) \quad \frac{n - 1}{m} \left((v^m)'' + \frac{n - 1}{r} (v^m)'\right) + \alpha v + \beta rv' = 0, \quad v > 0, \quad \text{in } (0, \infty)
\]
and 
\[
(1.7) \quad \begin{cases} 
  v(0) = \eta, \\
  v'(0) = 0,
\end{cases}
\]
for some constant \( \eta > 0 \). Existence of solutions of \((1.6), (1.7)\), for the case \( n \geq 3, 0 < m \leq (n - 2)/n, \beta > 0 \) and \( \alpha \leq \beta(n - 2)/m \) is proved by S.Y. Hsu in [H1]. On the other hand, by the result of [PZ] and Theorem 7.4 of [V1] if \((1.2)\) holds, then there exists a constant \( \overline{\beta} \) with \( \beta = 0 \) when \( m = \frac{n - 2}{n + 2} \) such that for any \( \alpha = \frac{2\beta + 1}{1 - m} \) and \( \beta > \overline{\beta} \), there exists a unique solution of \((1.6), (1.7)\). Moreover, if \( 0 < \alpha = \frac{2\beta + 1}{1 - m} \) and \( \beta < \overline{\beta} \), then \((1.6), (1.7)\) have no global solution.

Since the asymptotic behavior of solutions of \((1.4)\) is usually similar to the behavior of the radially symmetric self-similar solutions of \((1.4)\), in order to understand the asymptotic behavior of solutions of \((1.4)\) and the asymptotic behavior of locally conformally flat non-compact gradient Yamabe solitons, it is important to study the asymptotic behavior of the solutions of \((1.6), (1.7)\).

Exact decay rate of the solutions of \((1.6), (1.7)\) for the case 
\[
\alpha = \frac{2\beta}{1 - m} > 0
\]
and the case 
\[
\frac{2\beta}{1 - m} > \max(\alpha, 0),
\]
with \( m, n \) satisfying \((1.2)\), was obtained by S.Y. Hsu in [H1]. When \((1.2)\) and \((1.3)\) hold for some constant \( \rho > 0 \), although it is known ([DS2], [V1]) that solution \( v \) of \((1.6), (1.7)\) satisfies \( v(r) = O(r^{-\frac{3\beta}{2(n - 2)}}) \) as \( r \to \infty \), nothing is known about the exact decay rate of \( v \). In [H2] S.Y. Hsu proved, by using estimates for the scalar curvature of the metric \( g = v^{\frac{4}{n + 2}} dx^2 \) where \( v \) is a radially symmetric solution of \((1.1)\), that when \( m = \frac{n - 2}{n + 2}, \beta > \frac{\rho}{n - 2} > 0, \)
\[
(1.8) \quad \lim_{r \to \infty} r^2 v(r) = \frac{(n - 1)(n - 2)}{\rho}.
\]
In this paper we will extend the above result and prove the exact decay rate of radially symmetric solution \( v \) of \((1.1)\) when \((1.2)\) and \((1.3)\) hold for some constant \( \rho > 0 \). More precisely we will prove the following theorem.
**Theorem 1.1.** Let $\eta > 0$, $\rho > 0$, $m$, $n$, $\alpha$, $\beta$, satisfy (1.2), (1.3), and

\begin{equation}
\beta > \frac{m \rho}{n - 2 - mn}.
\end{equation}

Suppose $v$ is a solution of (1.6), (1.7). Then

\begin{equation}
\lim_{r \to \infty} r^2 v(r)^{1 - m} = \frac{2(n - 1)(1 - m - 2)}{(1 - m)(\alpha(1 - m) - 2 \beta)}.
\end{equation}

**Remark 1.2.** The function

\begin{equation}
v_0(x) = \left(\frac{2(n - 1)(1 - m - 2)}{(1 - m)(\alpha(1 - m) - 2 \beta)}\right)^{\frac{1}{1 - m}}
\end{equation}

is a singular solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. If $v$ is a solution of (1.1), then for any $\lambda > 0$ the function

\begin{equation}
v_\lambda(x) = \lambda^{\frac{2}{1 - m}} v(\lambda x)
\end{equation}

is also a solution of (1.1).

**Corollary 1.3.** Let $\rho$, $m$, $n$, $\alpha$, $\beta$ satisfy (1.2), (1.3), (1.9). Suppose $v$ is a radially symmetric solution of (1.1), and $v_0$, $v_\lambda$ are given by (1.11) and (1.12), respectively. Then $v_\lambda(x)$ converges uniformly on $\mathbb{R}^n \setminus B_R(0)$ to $v_0(x)$ for any $R > 0$ as $\lambda \to \infty$.

**Corollary 1.4** (cf. [H2]). The metric $g_{ij} = v^{\frac{4}{n - 2}} dx^2$, $n \geq 3$, of a locally conformally flat non-compact gradient shrinking Yamabe soliton where $v$ is radially symmetric and satisfies (1.1) with $m = \frac{n - 2}{n + 2}$, and $\beta > \frac{\rho}{2} > 0$, $\alpha$, satisfying (1.3) has the exact decay rate (1.8).

Since the scalar curvature of the metric $g_{ij} = v^{\frac{4}{n - 2}} dx^2$, $n \geq 3$, where $v$ is a radially symmetric solution of (1.1) with $m = \frac{n - 2}{n + 2}$ is given by (DS2, H2)

\begin{equation}
R(r) = (1 - m) \left(\alpha + \beta \frac{rv'(r)}{v(r)}\right),
\end{equation}

by Corollary 1.4 and an argument similar to the proof of Lemma 3.4 and Theorem 1.3 of [H2], we obtain the following extensions of Theorem 1.2 and Theorem 1.3 of [H2].

**Theorem 1.5.** Let $m = \frac{n - 2}{n + 2}$, $n \geq 3$, $\beta > \frac{\rho}{2} > 0$, $\alpha$, satisfy (1.3). Let $v$ be a radially symmetric solution of (1.1). Then

\begin{equation}
\lim_{r \to \infty} \frac{rv'(r)}{v(r)} = -\frac{2}{1 - m}
\end{equation}

and the scalar curvature $R(r)$ of the metric $g_{ij} = v^{\frac{4}{n - 2}} dx^2$ satisfies

\begin{equation}
\lim_{r \to \infty} R(r) = \rho.
\end{equation}

If $K_0$ and $K_1$ are the sectional curvatures of the 2-planes perpendicular to and tangent to the spheres $\{x\} \times S^{n-1}$, respectively, then

\begin{equation}
\lim_{r \to \infty} K_0(r) = 0
\end{equation}

and

\begin{equation}
\lim_{r \to \infty} K_1(r) = \frac{\rho}{(n - 1)(n - 2)}.
\end{equation}
Corollary 1.6. Let $\eta > 0$, $\rho > 0$, $m$, $n$, $\alpha$, $\beta$ satisfy (1.2), (1.3), and (1.9). Suppose $v$ is a solution of (1.6), (1.7). Then (1.13) holds.

The plan of the paper is as follows. We will prove the boundedness of the function (1.14) 
\[ w(r) = r^{2}v(r)^{1-m} \]
where $v$ is the solution of (1.1) in section two. We will also find the lower bound of $w$ in section two. In section three we will prove Theorem 1.1 and Corollary 1.3. We will assume that (1.2), (1.3) hold for some constant $\rho > 0$ and let $v$ be a radially symmetric solution of (1.1) or equivalently the solution of (1.6), (1.7), for some $\eta > 0$, and
\[ w_{\infty} = \frac{2(n-1)(n(1-m) - 2)}{(1-m)(\alpha(1-m) - 2\beta)} \]
for the rest of the paper. Note that when $\alpha = n\beta$ and $\alpha = \frac{2\beta + 1}{1-m}$, the solution of (1.1) is given explicitly by (cf. [DS2])
\[ v_{\lambda}(x) = \left( \frac{2(n-1)(n-2-\lambda)}{(1-m)(\lambda^{2} + |x|^{2})} \right)^{\frac{1}{1-m}}, \quad \lambda > 0, \]
which satisfies (1.10).

2. $L^{\infty}$ Estimate of $w$

Lemma 2.1. Let $\rho > 0$, $m$, $n$, $\alpha$, $\beta$ satisfy (1.2) and (1.3) and let $v$ be a radially symmetric solution of (1.1). Let $w$ be given by (1.14). Suppose there exists a constant $C_1 > 0$ such that
\[ w(r) \leq C_1 \quad \forall r \geq 1. \]
Then any sequence $\{w(r_{i})\}_{i=1}^{\infty}$, $r_{i} \to \infty$ as $i \to \infty$, has a subsequence $\{w(r'_{i})\}_{i=1}^{\infty}$ such that
\[ \lim_{r \to \infty} w(r'_{i}) = \begin{cases} 
0 & \text{or} \quad w_{\infty} \quad \text{if} \quad v \not\in L^{1}(\mathbb{R}^{n}),
0 & \text{or} \quad w_{1} \quad \text{if} \quad v \in L^{1}(\mathbb{R}^{n}) \quad \text{and} \quad \beta > 0,
0 & \text{if} \quad v \in L^{1}(\mathbb{R}^{n}) \quad \text{and} \quad \beta \leq 0,
\end{cases} \]
where
\[ w_{1} = \frac{2(n-1)}{(1-m)\beta} \quad \text{if} \quad \beta > 0. \]

Proof. Let $\{r_{i}\}_{i=1}^{\infty}$ be a sequence such that $r_{i} \to \infty$ as $i \to \infty$. By (2.1) the sequence $\{w(r_{i})\}_{i=1}^{\infty}$ has a subsequence that we may assume, without loss of generality, to be the sequence itself that converges to some constant $a \in [0,C_{1}]$ as $i \to \infty$. Integrating (1.6) over $(0,r)$ and simplifying,
\[ -\frac{n-1}{m}(v^{m})'(r) = \beta rv(r) + \frac{\alpha - n\beta}{r^{n-1}} \int_{0}^{r} z^{n-1}v(z) dz \quad \forall r > 0. \]
Integrating (2.4) over $(r, \infty)$, by (2.1) we get
\[ \frac{n-1}{m}v(r)^{m} = \beta \int_{r}^{\infty} sv(s) ds + \int_{r}^{\infty} \frac{\alpha - n\beta}{s^{n-1}} \left( \int_{0}^{s} z^{n-1}v(z) dz \right) ds \quad \forall r > 0. \]
Let \( b = a^{\frac{1}{m}} = \lim_{i \to \infty} r_i^{\frac{2}{m}} v(r_i) \). Then by (2.1), (2.5), and the l'Hospital rule,

\[
\frac{(n - 1)}{m} b^m = \frac{(n - 1)}{m} \lim_{i \to \infty} (r_i^{\frac{2}{m}} v(r))^m
\]

\[
= \beta \lim_{i \to \infty} \frac{\int_{r_i}^\infty s v(s) \, ds}{r_i^{\frac{2m}{m - 2}}} + \lim_{i \to \infty} \frac{\int_{r_i}^\infty \alpha - n \beta}{r_i^{\frac{2m}{m - 2}}} \left( \int_0^{r_i} z^{n - 1} v(z) \, dz \right) \, ds
\]

\[
= (1 - m) \left( \beta \lim_{i \to \infty} \frac{r_i v(r_i)}{r_i^{\frac{2m}{m - 2} - 2}} + (\alpha - n \beta) \lim_{i \to \infty} \frac{1}{r_i^{\frac{2m}{m - 2} - 2}} \int_0^{r_i} z^{n - 1} v(z) \, dz \right)
\]

(2.6)

\[
= (1 - m) \frac{\beta b + (\alpha - n \beta) \lim_{i \to \infty} \int_0^{r_i} z^{n - 1} v(z) \, dz}{r_i^{\frac{2m}{m - 2} - 2}}.
\]

We now divide the proof into two cases.

Case 1: \( v \not\in L^1(\mathbb{R}^n) \). By (2.6) and the l'Hospital rule,

\[
\frac{(n - 1)}{m} b^m = (1 - m) \frac{\beta b + (\alpha - n \beta)}{2m} \lim_{i \to \infty} r_i^{n - 1} v(r_i)
\]

\[
= (1 - m) \frac{\beta b + (\alpha - n \beta)}{2m}
\]

\[
= (1 - m) \frac{\alpha(1 - m) - 2\beta}{2m \alpha(n - 1)} b
\]

(2.7) \( \Rightarrow \) \( a = b = 0 \) or \( a = b^{1 - m} = w_\infty \).

Case 2: \( v \in L^1(\mathbb{R}^n) \). By (2.6),

(2.8)

\[
\frac{(n - 1)}{m} b^m = (1 - m) \frac{\beta b}{2m} \Rightarrow \begin{cases} a = b = 0 & \text{or } a = b^{1 - m} = w_1 \text{ if } \beta > 0, \\ a = b = 0 & \text{if } \beta \leq 0, \end{cases}
\]

By (2.7) and (2.8) the lemma follows. \( \square \)

Remark 2.2. When \( \beta > 0, w_1 > w_\infty \) if and only if \( \alpha > n \beta \).

Corollary 2.3. Suppose there exist constants \( C_1 > C_2 > 0 \) such that

\[ C_2 \leq w(r) \leq C_1 \quad \forall r \geq 1. \]

Then (1.10) holds.

Lemma 2.4. Let \( \eta > 0, \rho > 0, \beta > 0, m, n, \alpha \leq n \beta \) satisfy (1.2) and (1.3). Then

(2.9)

\[
v(r) \geq \left( \eta^{m - 1} + \frac{(1 - m)\beta}{2(n - 1)} r^2 \right)^{-\frac{1}{1 - m}} \quad \forall r \geq 0.
\]

Hence, there exists a constant \( C_2 > 0 \) such that

(2.10)

\[ w(r) \geq C_2 \quad \forall r \geq 1. \]
Proof. (2.9) is proved on page 22 of [DS2]. For the sake of completeness, we will give a simple different proof here. By (2.4),

\[- \frac{n-1}{m} (v^m)'(r) \leq \beta rv(r) \quad \forall r > 0\]

\[\Rightarrow (n-1)v^{m-2}(r) \leq \beta r \quad \forall r > 0\]

\[\Rightarrow \frac{n-1}{1-m} (v(r)^{m-1} - \eta^{m-1}) \leq \frac{\beta}{2} r^2 \quad \forall r > 0\]

and (2.9) follows. By (2.9), we get (2.10) and the lemma follows. □

We now recall a result of [H2]. Let \( \eta > 0, \ \rho > 0, \ m, \ n, \ \alpha \geq n \beta > 0 \) satisfy (1.2) and (1.3). Then there exists a constant \( C_1 > 0 \) such that (2.11) holds.

**Lemma 2.5** (cf. Lemma 2.3 of [H2]). Let \( \eta > 0, \ \rho > 0, \ m, \ n, \ \alpha \geq n \beta > 0 \) satisfy (1.2) and (1.3). Then there exists a constant \( C_1 > 0 \) such that (2.11) holds.

Proof. This result is proved in [H2]. For the sake of completeness, we will repeat the proof here. By (2.4), \( v'(r) < 0 \) for all \( r > 0 \). Then by (2.4),

\[\frac{n-1}{m} r^{n-1}(v^m)'(r) \leq -\beta r^n v(r) - (\alpha - n \beta) \int_0^r z^{n-1} v(z) \, dz\]

\[= -\frac{\alpha}{n} r^n v(r) \quad \forall r > 0\]

\[\Rightarrow v^{m-2}(r) v'(r) \leq -\frac{\alpha}{n(n-1)} r \quad \forall r > 0\]

\[\Rightarrow v(r) \leq \left( \frac{\eta^{m-1} + \alpha(1-m)}{2(n-1)} \right)^{-\frac{1}{m-1}} \leq \left( \frac{2n(n-1)}{\alpha(1-m)} r^2 \right)^{\frac{1}{m-1}} \quad \forall r > 0\]

Hence, (2.11) holds with \( C_1 = \frac{2n(n-1)}{\alpha(1-m)} \) and the lemma follows. □

**Lemma 2.6.** Let \( \eta > 0, \ \rho > 0, \ m, \ n, \ 0 < \alpha \leq n \beta \) satisfy (1.2) and (1.3). Then there exists a constant \( C_1 > 0 \) such that (2.11) holds.

Proof. Let \( A = \{ r \in [1, \infty) : w'(r) \geq 0 \} \). We now divide the proof into two cases.

**Case 1:** \( A \cap [R_0, \infty) \neq \emptyset \ \forall R_0 > 1 \). We will use a modification of the proof of Lemma 3.2 of [H2] to prove this case. By Lemma 2.4 there exists a constant \( C_2 > 0 \) such that (2.10) holds. Hence, by (2.10),

\[r^n v(r) = r^{n-\frac{2}{1-m}} w(r) \frac{1}{1-m} \geq C_2 r^{n-\frac{2}{1-m}} \quad \forall r \geq 1\]

(2.11) \( \Rightarrow r^n v(r) \to \infty \) as \( r \to \infty \).

We now claim that

\[ \limsup_{r \to \infty} \frac{\int_0^r z^{n-1} v(z) \, dz}{r^n v(r)} \leq \frac{1-m}{n(1-m) - 2}. \]

We divide the proof of the above claim into two cases.

**Case (1a):** \( \int_0^\infty z^{n-1} v(z) \, dz < \infty \). By (2.11) we get (2.12).

**Case (1b):** \( \int_0^\infty z^{n-1} v(z) \, dz = \infty \). Since

\[\frac{d}{dr} (r^n v(r)) = \left( n - \frac{2}{1-m} \right) r^{n-1} v(r) + \frac{1}{1-m} r^{n-\frac{2}{1-m}} w^{\frac{m}{1-m}}(r) w'(r)\]

\[\geq \left( n - \frac{2}{1-m} \right) r^{n-1} v(r) \quad \forall r \in A,\]
by (2.11) and the l'Hospital rule,
\[
\limsup_{r \to \infty} \frac{\int_0^r z^{n-1}v(z) \, dz}{r^n v(r)} = \limsup_{r \to \infty} \frac{r^{n-1}v(r)}{(n-\frac{2}{1-m})r^{n-1}v(r) + \frac{1}{1-m}r^{n-\frac{2}{1-m}}w^m(r)w'(r)} \leq \left(n - \frac{2}{1-m}\right)^{-1}
\]
and (2.12) follows. Let 0 < \delta < \frac{\rho}{n(1-m)-2}. By (2.12) there exists a constant \( R_1 > 1 \) such that
\[
\int_0^r z^{n-1}v(z) \, dz < \frac{(1-m)}{n(1-m)-2} + \frac{\delta}{1+n\beta-\alpha} \quad \forall r \geq R_1, r \in A,
\]
(2.13)
\[\Rightarrow \int_0^r z^{n-1}v(z) \, dz \leq \left(\frac{(1-m)}{n(1-m)-2} + \frac{\delta}{1+n\beta-\alpha}\right)r^n v(r) \quad \forall r \geq R_1, r \in A.
\]
By (2.11) and (2.13),
\[
\frac{n-1}{m}r^{n-1}(v^m)'(r) \leq -\beta r^n v(r) + \left(\frac{(n\beta - \alpha)(1-m)}{n(1-m)-2} + \delta\right)r^n v(r) \leq -\left(\frac{\rho}{n(1-m)-2} - \delta\right)r^n v(r) \quad \forall r \geq R_1, r \in A,
\]
\[\Rightarrow (n-1)v^{n-2}v'(r) \leq -\left(\frac{\rho}{n(1-m)-2} - \delta\right)r \quad \forall r \geq R_1, r \in A.
\]
Hence, there exists a constant \( C_3 > 0 \) such that
\[
\frac{rv'(r)}{v(r)} \leq -C_3 r^2 v(r)^{1-m} = -C_3 w(r) \quad \forall r \geq R_1, r \in A,
\]
(2.14)
\[\Rightarrow 0 \leq w'(r) = \frac{2w(r)}{r} \left(1 + \frac{1-m}{2} \cdot \frac{rv'(r)}{v(r)}\right) \leq \frac{2w(r)}{r} \left(1 - \frac{(1-m)C_3}{2} w(r)\right) \quad \forall r \geq R_1, r \in A,
\]
\[\Rightarrow w(r) \leq \frac{2}{(1-m)C_3} \quad \forall r \geq R_1, r \in A.
\]
Let \( r_1 \in A \cap [R_1, \infty) \). Then for any \( r' \in (r_1, \infty) \setminus A \), there exists \( r_2 \in A \cap [r_1, \infty) \) such that
\[
w'(r) < 0 \quad \forall r_2 < r \leq r' \quad \text{and} \quad w'(r_2) = 0
\]
(2.15)
\[\Rightarrow w(r') \leq w(r_2) \leq \frac{2}{(1-m)C_3} \quad \forall r' > r_1, r' \notin A \quad (\text{by (2.14)}).
\]
By (2.14) and (2.15),
\[
w(r) \leq \frac{2}{(1-m)C_3} \quad \forall r \geq r_1
\]
and (2.1) holds with \( C_1 = \max \left(\frac{2}{(1-m)C_3}, \max_{1 \leq r \leq r_1} w(r)\right) \).
Case 2: There exists a constant $R_0 > 1$ such that $A \cap [R_0, \infty) = \emptyset$. Then $w'(r) < 0$ for all $r \geq R_0$. Hence, (2.1) holds with $C_1 = \max_{1 \leq r \leq R_0} w(r)$ and the lemma follows.

3. PROOF OF THEOREM 1.1

We first recall a result of [H1]:

**Lemma 3.1** (cf. Lemma 2.1 of [H1]). Let $\eta > 0$, $m$, $n$, $\alpha > 0$, $\beta \neq 0$ satisfy (1.2) and

$$\frac{m \alpha}{\beta} \leq n - 2.$$ 

Let $v$ be the solution of (1.6), (1.7). Then

(3.1) $v(r) + \frac{\beta}{\alpha} r v'(r) > 0 \ \forall r \geq 0$

and

(3.2) $v'(r) < 0 \ \forall r > 0$.

**Lemma 3.2.** Let $\rho > 0$, $m$, $n$, $\alpha > n \beta$ satisfy (1.2), (1.3) and (1.9). Then

(3.3) $\lim_{r \to \infty} r^{n-2} v^m(r) = \infty.$

Proof. Suppose (3.3) does not hold. Then there exists a sequence $\{r_i\}_{i=1}^\infty$, $r_i \to \infty$ as $i \to \infty$, such that $r_i^{n-2} v^m(r_i) \to a_1$ as $i \to \infty$ for some constant $a_1 \geq 0$. By Lemma 2.1, the sequence $\{r_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself such that $w(r_i) \to a_2$ as $i \to \infty$, where $a_2 = 0$, $w_\infty$, or $w_1$ with $w_1$ being given by (2.3). By (2.5), Lemma 2.6 and the l’Hospital rule,

$$\frac{(n-1)}{m} a_1 = \frac{(n-1)}{m} \lim_{i \to \infty} r_i^{n-2} v(r_i)^m$$

$$= \lim_{i \to \infty} \int_{r_i}^\infty \frac{sv(s)}{r_i^{2-n}} ds + \lim_{i \to \infty} \int_{r_i}^\infty \frac{\alpha - n \beta}{z^{n-2}} \left(\int_0^s z^{n-1} v(z) dz\right) ds$$

$$= \frac{\beta}{n-2} \lim_{i \to \infty} r_i^n v(r_i) + \frac{\alpha - n \beta}{n-2} \lim_{i \to \infty} \int_0^{r_i} z^{n-1} v(z) dz$$

$$= \frac{\beta}{n-2} \lim_{i \to \infty} r_i^{n-2} v(r_i)^m \lim_{i \to \infty} r_i^{2} v(r_i)^{1-m} + \frac{\alpha - n \beta}{n-2} \int_0^\infty z^{n-1} v(z) dz$$

$$= \frac{\beta}{n-2} a_1 a_2 + \frac{\alpha - n \beta}{n-2} \int_0^\infty z^{n-1} v(z) dz.$$ 

Hence,

(3.4) $\frac{\alpha - n \beta}{a_1} \int_0^\infty z^{n-1} v(z) dz = \frac{(n-1)(n-2)}{m} - \beta a_2.$

By (2.4) and (3.4),

(3.5) $-(n-1) \lim_{i \to \infty} r_i v'(r_i) \leq \frac{\beta}{n-2} a_1 a_2 + \frac{\alpha - n \beta}{n-2} \int_0^\infty z^{n-1} v(z) dz$
Hence,
\[(3.6) \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = -\frac{(n - 2)}{m}.\]

By (1.2), (1.3) and (1.9),
\[
\frac{m \alpha}{\beta} < n - 2
\]
holds. Hence, there exists a constant \(\varepsilon > 0\) such that
\[(3.7) \frac{m \alpha}{\beta} < n - 2 - \varepsilon.\]

By (3.7) and Lemma 3.1, (3.8) and (3.2) hold. Then by (3.1), (3.2) and (3.7),
\[(3.8) 0 > \frac{rv'(r)}{v(r)} > -\frac{\alpha}{\beta} > -\frac{n - 2}{m} + \frac{\varepsilon}{m} \quad \forall r > 0\]
\[
\Rightarrow \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} \geq -\frac{n - 2}{m} + \frac{\varepsilon}{m},
\]
which contradicts (3.6). Hence, no such sequence \(\{r_i\}_{i=1}^\infty\) exists, and the lemma follows.

\[\square\]

**Lemma 3.3.** Let \(\rho > 0\), \(m\), \(n\), \(\alpha > n \beta\) satisfy (1.2), (1.3) and (1.9). Then there exists a constant \(\varepsilon \in (0, \min(1, w_\infty/2))\) such that for any \(R_0 > 1\) there exists \(r' > R_0\) such that
\[w(r') \geq \varepsilon.\]

**Proof.** Suppose the lemma is false. Then
\[(3.9) \lim_{r \to \infty} w(r) = 0.\]

We claim that
\[(3.10) \lim_{r \to \infty} \frac{r v'(r)}{v(r)} = 0.\]

By the proof of Lemma 3.2 there exists a constant \(\varepsilon > 0\) such that (3.8) holds. Suppose (3.10) does not hold. Then by (3.8) and (3.9) there exists a sequence \(\{r_i\}_{i=1}^\infty\), \(r_i \to \infty\) as \(i \to \infty\), such that \(r_i v'(r_i)/v(r_i) \to a_3\) as \(i \to \infty\) for some constant \(a_3\) satisfying
\[\frac{-n - 2}{m} + \frac{\varepsilon}{m} \leq a_3 < 0 \quad (3.11)\]
and (3.5) holds. By Lemma 3.2 (3.5), (3.9) and (3.11), we get
\[-(n - 1) \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = 0 \quad \text{if } v \in L^1(\mathbb{R}^n),\]
and if \(v \not\in L^1(\mathbb{R}^n)\), then by the l'Hospital rule,
\[\begin{align*}
-(n - 1) \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} &= (\alpha - n \beta) \lim_{i \to \infty} \frac{r_i^{n-1} v(r_i)}{(n - 2)r_i^{n-3}v(r_i)^m + mr_i^{n-2}v(r_i)^{m-1}v'(r_i)} \\
&= (\alpha - n \beta) \lim_{i \to \infty} \frac{r_i^{n-1}v(r_i)}{n - 2 + m(r_i v'(r_i)/v(r_i))} \\
&= \frac{\alpha - n \beta}{n - 2 + ma_3} \cdot \lim_{i \to \infty} r_i^2 v(r_i)^{1-m} \\
&= 0.
\end{align*}\]
Hence,
\[ a_3 = \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = 0, \]
which contradicts (3.11). Thus, no such sequence \( \{r_i\}_{i=1}^{\infty} \) exists and (3.10) follows. Since
\[ w'(r) = \frac{2w(r)}{r} \left( 1 + \frac{1 - m}{2} \cdot \frac{rv'(r)}{v(r)} \right), \]
by (3.10) there exists a constant \( R_0 > 0 \) such that
\[ w'(r) > 0 \quad \forall r \geq R_0, \]
which contradicts (3.9) and the lemma follows. \( \square \)

We are now ready for the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We divide the proof into two cases.

**Case 1:** \( \alpha \leq n\beta \). By Corollary 2.3, Lemma 2.4 and Lemma 2.6, we get (1.10).

**Case 2:** \( \alpha > n\beta \). By Lemma 2.5 there exists a constant \( C_1 > 0 \) such that (2.1) holds. Let \( 0 < \varepsilon < \min(1, w_\infty/2) \) be as in Lemma 3.3. Suppose there exists a sequence \( \{r_i\}_{i=1}^{\infty} \) such that \( w(r_i) < \varepsilon \) for all \( r_i \geq R_1 \). Then by Lemma 3.3 there exists a subsequence of \( \{r_i\}_{i=1}^{\infty} \) which we may assume without loss of generality to be the sequence itself and a sequence \( \{r'_i\}_{i=1}^{\infty} \) such that
\[ r_i < r'_i < r_{i+1} \quad \text{for all} \quad i = 1, 2, \ldots \]
and
\[ w(r_i) < \varepsilon < w(r'_i) \quad \forall i = 1, 2, \ldots. \]

By (3.12) and the intermediate value theorem, for any \( i = 1, 2, \ldots \), there exists \( a_i \in (r_i, r'_i) \) such that
\[ w(a_i) = \varepsilon \quad \forall i = 1, 2, \ldots. \]
Hence, \( a_i \to \infty \) as \( i \to \infty \) and
\[ \lim_{i \to \infty} w(a_i) = \varepsilon. \]
This contradicts Lemma 2.1 and Remark 2.2. Hence no such sequence \( \{r_i\}_{i=1}^{\infty} \) exists. Thus there exists a constant \( R_1 > 1 \) such that \( w(r) \geq \varepsilon \) for all \( r \geq R_1 \). Hence (2.10) holds with \( C_2 = \min(\varepsilon, \min_{1 \leq r \leq R_1} w(r)) > 0 \). By Corollary 2.3 we get (1.10) and the theorem follows. \( \square \)

**Proof of Corollary 1.3.** By Theorem 1.1,
\[ |x|^2 v_\lambda(x)^{1-m} = (\lambda|x|)^2 v(\lambda x)^{1-m} \]
\[ \to \frac{2(n-1)(n(1-m) - 2)}{(1-m)(\alpha(1-m) - 2\beta)} \quad \text{uniformly on} \quad \mathbb{R}^n \setminus B_R(0) \]
as \( \lambda \to \infty \) for any \( R > 0 \) and the corollary follows. \( \square \)
REFERENCES


Department of Mathematics, National Chung Cheng University, 168 University Road, Min-Hsiung, Chia-Yi 621, Taiwan, Republic of China

E-mail address: syhsu@math.ccu.edu.tw