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EXACT DECAY RATE OF A NONLINEAR ELLIPTIC EQUATION RELATED TO THE YAMABE FLOW

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Abstract. Let $0 < m < \frac{n-2}{n}$, $n \ge 3$, $\alpha = \frac{2\beta+\rho}{1-m}$ and $\beta > \frac{m\rho}{n-2-mn}$ for some constant $\rho > 0$. Suppose v is a radially symmetric solution of $\frac{n-1}{m}\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0$, $v > 0$, in $\mathbb{R}^n$. When $m = \frac{n-2}{n+2}$, the metric $g = v^{\frac{4}{n+2}} dx^2$ corresponds to a locally conformally flat Yamabe shrinking gradient soliton with positive sectional curvature. We prove that the solution v of the above nonlinear elliptic equation has the exact decay rate $\lim_{r\to\infty} r^2 v(r)^{1-m} = \frac{2(n-1)(n(1-m)-2)}{(1-m)(\alpha(1-m)-2\beta)}$.

1. Introduction

Recently, there has been a lot of study of the equation

$$\frac{n-1}{m}\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, \quad v > 0, \quad \text{in } \mathbb{R}^n \tag{1.1}$$

where

$$0 < m < \frac{n-2}{n}, \quad n \ge 3, \tag{1.2}$$

and

$$\alpha = \frac{2\beta + \rho}{1-m} \tag{1.3}$$

for some constant $\rho \in \mathbb{R}$ by P. Daskalopoulos and N. Sesum [DS2]; S.Y. Hsu [H1], [H2]; M.A. Peletier and H. Zhang [PZ]; and J.L. Vázquez [V1]. In the paper [DS2] P. Daskalopoulos and N. Sesum (cf. [CSZ], [CMM]) proved the important result that any locally conformally flat non-compact gradient Yamabe soliton g with positive sectional curvature on an n-dimensional manifold, $n \ge 3$, must be radially symmetric and have the form $g = v^{\frac{4}{n+2}} dx^2$, where dx^2 is the Euclidean metric on $\mathbb{R}^n$ and v is a radially symmetric solution of (1.1) with $m = \frac{n-2}{n+2}$, and α, β satisfy (1.3) for some constant $\rho > 0$, $\rho = 0$ or $\rho < 0$, depending on whether g is a shrinking, steady, or expanding Yamabe soliton.

On the other hand, as observed by B.H. Gilding, M.A. Peletier and H. Zhang [GP], [PZ], and others ([DS1], [DS2], [V1], [V2]), (1.1) also arises in the study of the self-similar solutions of the degenerate diffusion equation

$$u_t = \frac{n-1}{m}\Delta u^m \quad \text{in } \mathbb{R}^n \times (0,T). \tag{1.4}$$

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For example (cf. [H1], [V1]) if v is a radially symmetric solution of (1.1) with

$$\alpha = \frac{2\beta + 1}{1 - m} > 0,$$

then for any $T > 0$ the function

$$u(x,t) = (T-t)^{\alpha} v(x(T-t)^{\beta}) \tag{1.5}$$

is a solution of (1.4) in $\mathbb{R}^n \times (-\infty, T)$. We refer the reader to the book [V1] and the paper [H1] for the relation between solutions of (1.1) and the other self-similar solutions of (1.4) for the other parameter ranges of α, β.

Note that when v is a radially symmetric solution of (1.1), then v satisfies

$$\frac{n-1}{m}\left((v^m)'' + \frac{n-1}{r}(v^m)'\right) + \alpha v + \beta r v' = 0, \quad v > 0, \quad \text{in } (0, \infty) \tag{1.6}$$

and

$$\begin{cases} v(0) = \eta, \\ v'(0) = 0, \end{cases} \tag{1.7}$$

for some constant $\eta > 0$. Existence of solutions of (1.6), (1.7), for the case $n \ge 3$, $0 < m \le (n-2)/n$, $\beta > 0$ and $\alpha \le \beta(n-2)/m$ is proved by S.Y. Hsu in [H1]. On the other hand, by the result of [PZ] and Theorem 7.4 of [V1] if (1.2) holds, then there exists a constant $\overline{\beta}$ with $\overline{\beta} = 0$ when $m = \frac{n-2}{n+2}$ such that for any $\alpha = \frac{2\beta+1}{1-m}$ and $\beta > \overline{\beta}$, there exists a unique solution of (1.6), (1.7). Moreover, if $0 < \alpha = \frac{2\beta+1}{1-m}$ and $\beta < \overline{\beta}$, then (1.6), (1.7) have no global solution.

Since the asymptotic behavior of solutions of (1.4) is usually similar to the behavior of the radially symmetric self-similar solutions of (1.4), in order to understand the asymptotic behavior of solutions of (1.4) and the asymptotic behavior of locally conformally flat non-compact gradient Yamabe solitons, it is important to study the asymptotic behavior of the solutions of (1.6), (1.7).

Exact decay rate of the solutions of (1.6), (1.7) for the case

$$\alpha = \frac{2\beta}{1-m} > 0$$

and the case

$$\frac{2\beta}{1-m} > \max(\alpha, 0),$$

with m, n satisfying (1.2), was obtained by S.Y. Hsu in [H1]. When (1.2) and (1.3) hold for some constant $\rho > 0$, although it is known ([DS2], [V1]) that solution v of (1.6), (1.7) satisfies $v(r) = O(r^{-\frac{2}{1-m}})$ as $r \to \infty$, nothing is known about the exact decay rate of v. In [H2] S.Y. Hsu proved, by using estimates for the scalar curvature of the metric $g = v^{\frac{4}{n+2}} dx^2$ where v is a radially symmetric solution of (1.1), that when $m = \frac{n-2}{n+2}$, $\beta > \frac{\rho}{n-2} > 0$,

$$\lim_{r \to \infty} r^2 v(r) = \frac{(n-1)(n-2)}{\rho}. \tag{1.8}$$

In this paper we will extend the above result and prove the exact decay rate of radially symmetric solution v of (1.1) when (1.2) and (1.3) hold for some constant $\rho > 0$. More precisely we will prove the following theorem.

Theorem 1.1. *Let* $\eta > 0$, $\rho > 0$, m, n, α, β, *satisfy* (1.2), (1.3), *and*

$$\beta > \frac{m\rho}{n-2-mn}. \tag{1.9}$$

Suppose v *is a solution of* (1.6), (1.7). *Then*

$$\lim_{r\to\infty} r^2 v(r)^{1-m} = \frac{2(n-1)(n(1-m)-2)}{(1-m)(\alpha(1-m)-2\beta)}. \tag{1.10}$$

Remark 1.2. The function

$$v_0(x) = \left(\frac{2(n-1)(n(1-m)-2)}{(1-m)(\alpha(1-m)-2\beta)|x|^2}\right)^{\frac{1}{1-m}} \tag{1.11}$$

is a singular solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. If v is a solution of (1.1), then for any $\lambda > 0$ the function

$$v_\lambda(x) = \lambda^{\frac{2}{1-m}} v(\lambda x) \tag{1.12}$$

is also a solution of (1.1).

Corollary 1.3. *Let* ρ, m, n, α, β *satisfy* (1.2), (1.3), (1.9). *Suppose* v *is a radially symmetric solution of* (1.1), *and* v_0, v_λ *are given by* (1.11) *and* (1.12), *respectively. Then* $v_\lambda(x)$ *converges uniformly on* $\mathbb{R}^n \setminus B_R(0)$ *to* $v_0(x)$ *for any* $R > 0$ *as* $\lambda \to \infty$.

Corollary 1.4 (cf. [H2]). *The metric* $g_{ij} = v^{\frac{4}{n+2}} dx^2$, $n \ge 3$, *of a locally conformally flat non-compact gradient shrinking Yamabe soliton where* v *is radially symmetric and satisfies* (1.1) *with* $m = \frac{n-2}{n+2}$, *and* $\beta > \frac{\rho}{2} > 0$, α, *satisfying* (1.3) *has the exact decay rate* (1.8).

Since the scalar curvature of the metric $g_{ij} = v^{\frac{4}{n+2}} dx^2$, $n \ge 3$, where v is a radially symmetric solution of (1.1) with $m = \frac{n-2}{n+2}$ is given by ([DS2], [H2])

$$R(r) = (1-m)\left(\alpha + \beta \frac{rv'(r)}{v(r)}\right),$$

by Corollary 1.4 and an argument similar to the proof of Lemma 3.4 and Theorem 1.3 of [H2], we obtain the following extensions of Theorem 1.2 and Theorem 1.3 of [H2].

Theorem 1.5. *Let* $m = \frac{n-2}{n+2}$, $n \ge 3$, $\beta > \frac{\rho}{2} > 0$, α, *satisfy* (1.3). *Let* v *be a radially symmetric solution of* (1.1). *Then*

$$\lim_{r\to\infty} \frac{rv'(r)}{v(r)} = -\frac{2}{1-m} \tag{1.13}$$

and the scalar curvature $R(r)$ *of the metric* $g_{ij} = v^{\frac{4}{n+2}} dx^2$ *satisfies*

$$\lim_{r\to\infty} R(r) = \rho.$$

If K_0 *and* K_1 *are the sectional curvatures of the 2-planes perpendicular to and tangent to the spheres* $\{x\} \times S^{n-1}$, *respectively, then*

$$\lim_{r\to\infty} K_0(r) = 0$$

and

$$\lim_{r\to\infty} K_1(r) = \frac{\rho}{(n-1)(n-2)}.$$

Corollary 1.6. *Let* $\eta > 0$, $\rho > 0$, m, n, α, β *satisfy* (1.2), (1.3), *and* (1.9). *Suppose* v *is a solution of* (1.6), (1.7). *Then* (1.13) *holds.*

The plan of the paper is as follows. We will prove the boundedness of the function

$$w(r) = r^2 v(r)^{1-m} \tag{1.14}$$

where v is the solution of (1.1) in section two. We will also find the lower bound of w in section two. In section three we will prove Theorem 1.1 and Corollary 1.3. We will assume that (1.2), (1.3) hold for some constant $\rho > 0$ and let v be a radially symmetric solution of (1.1) or equivalently the solution of (1.6), (1.7), for some $\eta > 0$, and

$$w_\infty = \frac{2(n-1)(n(1-m)-2)}{(1-m)(\alpha(1-m)-2\beta)}$$

for the rest of the paper. Note that when $\alpha = n\beta$ and $\alpha = \frac{2\beta+1}{1-m}$, the solution of (1.1) is given explicitly by (cf. [DS2])

$$v_\lambda(x) = \left(\frac{2(n-1)(n-2-nm)}{(1-m)(\lambda^2+|x|^2)}\right)^{\frac{1}{1-m}}, \quad \lambda > 0,$$

which satisfies (1.10).

2. L^∞ ESTIMATE OF w

Lemma 2.1. *Let* $\rho > 0$, m, n, α, β *satisfy* (1.2) *and* (1.3) *and let* v *be a radially symmetric solution of* (1.1). *Let* w *be given by* (1.14). *Suppose there exists a constant* $C_1 > 0$ *such that*

$$w(r) \le C_1 \quad \forall r \ge 1. \tag{2.1}$$

Then any sequence $\{w(r_i)\}_{i=1}^\infty$, $r_i \to \infty$ *as* $i \to \infty$, *has a subsequence* $\{w(r_i')\}_{i=1}^\infty$ *such that*

$$\lim_{r\to\infty} w(r_i') = \begin{cases} 0 \quad \text{or} \quad w_\infty & \text{if } v \notin L^1(\mathbb{R}^n), \\ 0 \quad \text{or} \quad w_1 & \text{if } v \in L^1(\mathbb{R}^n) \quad \text{and} \quad \beta > 0, \\ 0 & \text{if } v \in L^1(\mathbb{R}^n) \quad \text{and} \quad \beta \le 0, \end{cases} \tag{2.2}$$

where

$$w_1 = \frac{2(n-1)}{(1-m)\beta} \quad \text{if } \beta > 0. \tag{2.3}$$

Proof. Let $\{r_i\}_{i=1}^\infty$ be a sequence such that $r_i \to \infty$ as $i \to \infty$. By (2.1) the sequence $\{w(r_i)\}_{i=1}^\infty$ has a subsequence that we may assume, without loss of generality, to be the sequence itself that converges to some constant $a \in [0, C_1]$ as $i \to \infty$. Integrating (1.6) over $(0, r)$ and simplifying,

$$-\frac{n-1}{m}(v^m)'(r) = \beta r v(r) + \frac{\alpha - n\beta}{r^{n-1}} \int_0^r z^{n-1} v(z)\, dz \quad \forall r > 0. \tag{2.4}$$

Integrating (2.4) over (r, ∞), by (2.1) we get

$$\frac{n-1}{m} v(r)^m = \beta \int_r^\infty s v(s)\, ds + \int_r^\infty \frac{\alpha - n\beta}{s^{n-1}} \left(\int_0^s z^{n-1} v(z)\, dz\right) ds \quad \forall r > 0. \tag{2.5}$$

Let $b = a^{\frac{1}{1-m}} = \lim_{i\to\infty} r_i^{\frac{2}{1-m}} v(r_i)$. Then by (2.1), (2.5), and the l'Hospital rule,

$$
\begin{aligned}
\frac{(n-1)}{m} b^m =& \frac{(n-1)}{m} \lim_{i\to\infty} (r_i^{\frac{2}{1-m}} v(r))^m \\
=& \beta \lim_{i\to\infty} \frac{\int_{r_i}^\infty s v(s)\, ds}{r_i^{-\frac{2m}{1-m}}} + \lim_{i\to\infty} \frac{\int_{r_i}^\infty \frac{\alpha - n\beta}{s^{n-1}} \left(\int_0^s z^{n-1} v(z)\, dz\right)\, ds}{r_i^{-\frac{2m}{1-m}}} \\
=& \frac{(1-m)}{2m} \left(\beta \lim_{i\to\infty} \frac{r_i v(r_i)}{r_i^{-\frac{2m}{1-m}-1}} + (\alpha - n\beta) \lim_{i\to\infty} \frac{\frac{1}{r_i^{n-1}} \int_0^{r_i} z^{n-1} v(z)\, dz}{r_i^{-\frac{2m}{1-m}-1}} \right) \\
=& \frac{(1-m)}{2m} \left(\beta b + (\alpha - n\beta) \lim_{i\to\infty} \frac{\int_0^{r_i} z^{n-1} v(z)\, dz}{r_i^{n-\frac{2}{1-m}}} \right). \qquad (2.6)
\end{aligned}
$$

We now divide the proof into two cases.

Case 1: $v \notin L^1(\mathbb{R}^n)$. By (2.6) and the l'Hospital rule,

$$
\begin{aligned}
\frac{(n-1)}{m} b^m =& \frac{(1-m)}{2m} \left(\beta b + \frac{\alpha - n\beta}{n - \frac{2}{1-m}} \cdot \lim_{i\to\infty} \frac{r_i^{n-1} v(r_i)}{r_i^{n-\frac{2}{1-m}-1}} \right) \\
=& \frac{(1-m)}{2m} \left(\beta b + \frac{\alpha - n\beta}{n - \frac{2}{1-m}} b \right) \\
=& \frac{(1-m)[\alpha(1-m) - 2\beta]}{2m[n(1-m) - 2]} b
\end{aligned}
$$

$$
\Rightarrow \qquad a = b = 0 \quad \text{or} \quad a = b^{1-m} = w_\infty. \qquad (2.7)
$$

Case 2: $v \in L^1(\mathbb{R}^n)$. By (2.6),

(2.8)

$$
\frac{(n-1)}{m} b^m = \frac{(1-m)\beta}{2m} b \quad \Rightarrow \quad \begin{cases} a = b = 0 \quad \text{or} \quad a = b^{1-m} = w_1 & \text{if } \beta > 0, \\ a = b = 0 & \text{if } \beta \le 0, \end{cases}
$$

By (2.7) and (2.8) the lemma follows. □

Remark 2.2. When $\beta > 0$, $w_1 > w_\infty$ if and only if $\alpha > n\beta$.

Corollary 2.3. *Suppose there exist constants* $C_1 > C_2 > 0$ *such that*

$$
C_2 \le w(r) \le C_1 \quad \forall r \ge 1.
$$

Then (1.10) *holds.*

Lemma 2.4. *Let* $\eta > 0$, $\rho > 0$, $\beta > 0$, m, n, $\alpha \le n\beta$ *satisfy* (1.2) *and* (1.3). *Then*

$$
v(r) \ge \left(\eta^{m-1} + \frac{(1-m)\beta}{2(n-1)} r^2 \right)^{-\frac{1}{1-m}} \quad \forall r \ge 0. \qquad (2.9)
$$

Hence, there exists a constant $C_2 > 0$ *such that*

$$
w(r) \ge C_2 \quad \forall r \ge 1. \qquad (2.10)
$$

Proof. (2.9) is proved on page 22 of [DS2]. For the sake of completeness, we will give a simple different proof here. By (2.4),

$$\begin{aligned}
& -\frac{n-1}{m}(v^m)'(r) \le \beta r v(r) \quad \forall r>0\\
\Rightarrow \quad & -(n-1)v^{m-2}v'(r) \le \beta r \quad \forall r>0\\
\Rightarrow \quad & \frac{n-1}{1-m}(v(r)^{m-1}-\eta^{m-1}) \le \frac{\beta}{2}r^2 \quad \forall r>0
\end{aligned}$$

and (2.9) follows. By (2.9), we get (2.10) and the lemma follows. □

We now recall a result of [H2].

Lemma 2.5 (cf. Lemma 2.3 of [H2])**.** *Let $\eta>0$, $\rho>0$, m, n, $\alpha \ge n\beta > 0$ satisfy* (1.2) *and* (1.3)*. Then there exists a constant $C_1>0$ such that* (2.1) *holds.*

Proof. This result is proved in [H2]. For the sake of completeness, we will repeat the proof here. By (2.4), $v'(r)<0$ for all $r>0$. Then by (2.4),

$$\begin{aligned}
\frac{n-1}{m}r^{n-1}(v^m)'(r) &\le -\beta r^n v(r) - (\alpha - n\beta)\int_0^r z^{n-1}v(r)\,dz\\
&= -\frac{\alpha}{n}r^n v(r) \quad \forall r>0\\
\Rightarrow \quad v^{m-2}(r)v'(r) &\le -\frac{\alpha}{n(n-1)}r \qquad \forall r>0\\
\Rightarrow \quad v(r) &\le \left(\eta^{m-1}+\frac{\alpha(1-m)}{2n(n-1)}r^2\right)^{-\frac{1}{1-m}} \le \left(\frac{2n(n-1)}{\alpha(1-m)}r^{-2}\right)^{\frac{1}{1-m}} \quad \forall r>0.
\end{aligned}$$

Hence, (2.1) holds with $C_1 = \frac{2n(n-1)}{\alpha(1-m)}$ and the lemma follows. □

Lemma 2.6. *Let $\eta>0$, $\rho>0$, m, n, $0<\alpha\le n\beta$ satisfy* (1.2) *and* (1.3)*. Then there exists a constant $C_1>0$ such that* (2.1) *holds.*

Proof. Let $A=\{r\in[1,\infty): w'(r)\ge 0\}$. We now divide the proof into two cases.

Case 1: $A\cap[R_0,\infty)\ne\phi \quad \forall R_0>1$. We will use a modification of the proof of Lemma 3.2 of [H2] to prove this case. By Lemma 2.4 there exists a constant $C_2>0$ such that (2.10) holds. Hence, by (2.10),

$$r^n v(r) = r^{n-\frac{2}{1-m}}w(r)^{\frac{1}{1-m}} \ge C_2 r^{n-\frac{2}{1-m}} \quad \forall r\ge 1$$

$$\Rightarrow \quad r^n v(r)\to\infty \quad \text{as } r\to\infty. \tag{2.11}$$

We now claim that

$$\limsup_{\substack{r\in A\\ r\to\infty}} \frac{\int_0^r z^{n-1}v(z)\,dz}{r^n v(r)} \le \frac{1-m}{n(1-m)-2}. \tag{2.12}$$

We divide the proof of the above claim into two cases.

Case (1a): $\int_0^\infty z^{n-1}v(z)\,dz<\infty$. By (2.11) we get (2.12).

Case (1b): $\int_0^\infty z^{n-1}v(z)\,dz=\infty$. Since

$$\begin{aligned}
\frac{d}{dr}(r^n v(r)) &= \left(n-\frac{2}{1-m}\right)r^{n-1}v(r) + \frac{1}{1-m}r^{n-\frac{2}{1-m}}w^{\frac{m}{1-m}}(r)w'(r)\\
&\ge \left(n-\frac{2}{1-m}\right)r^{n-1}v(r) \quad \forall r\in A,
\end{aligned}$$

by (2.11) and the l'Hospital rule,

$$\begin{aligned}
&\limsup_{\substack{r\in A\\ r\to\infty}} \frac{\int_0^r z^{n-1}v(z)\,dz}{r^n v(r)}\\
&\quad = \limsup_{\substack{r\in A\\ r\to\infty}} \frac{r^{n-1}v(r)}{\left(n-\frac{2}{1-m}\right)r^{n-1}v(r)+\frac{1}{1-m}r^{n-\frac{2}{1-m}}w^{\frac{m}{1-m}}(r)w'(r)}\\
&\quad \le \left(n-\frac{2}{1-m}\right)^{-1}
\end{aligned}$$

and (2.12) follows. Let $0<\delta<\frac{\rho}{n(1-m)-2}$. By (2.12) there exists a constant $R_1>1$ such that

$$\begin{aligned}
&\frac{\int_0^r z^{n-1}v(z)\,dz}{r^n v(r)} < \frac{(1-m)}{n(1-m)-2}+\frac{\delta}{1+n\beta-\alpha} \qquad \forall r\ge R_1, r\in A,\\
(2.13)\quad &\Rightarrow \quad \int_0^r z^{n-1}v(z)\,dz \le \left(\frac{(1-m)}{n(1-m)-2}+\frac{\delta}{1+n\beta-\alpha}\right)r^n v(r) \quad \forall r\ge R_1, r\in A.
\end{aligned}$$

By (2.4) and (2.13),

$$\begin{aligned}
\frac{n-1}{m}r^{n-1}(v^m)'(r) &\le -\beta r^n v(r)+\left(\frac{(n\beta-\alpha)(1-m)}{n(1-m)-2}+\delta\right)r^n v(r)\\
&\le -\left(\frac{\rho}{n(1-m)-2}-\delta\right)r^n v(r) \quad \forall r\ge R_1, r\in A,\\
\Rightarrow \quad (n-1)v^{m-2}v'(r) &\le -\left(\frac{\rho}{n(1-m)-2}-\delta\right)r \qquad \forall r\ge R_1, r\in A.
\end{aligned}$$

Hence, there exists a constant $C_3>0$ such that

$$\begin{aligned}
&\frac{rv'(r)}{v(r)} \le -C_3 r^2 v(r)^{1-m} = -C_3 w(r) \quad \forall r\ge R_1, r\in A,\\
(2.14)\quad &\Rightarrow \quad 0\le w'(r) = \frac{2w(r)}{r}\left(1+\frac{1-m}{2}\cdot\frac{rv'(r)}{v(r)}\right)\\
&\qquad \le \frac{2w(r)}{r}\left(1-\frac{(1-m)C_3}{2}w(r)\right) \quad \forall r\ge R_1, r\in A,\\
&\Rightarrow \quad w(r)\le \frac{2}{(1-m)C_3} \quad \forall r\ge R_1, r\in A.
\end{aligned}$$

Let $r_1\in A\cap[R_1,\infty)$. Then for any $r'\in(r_1,\infty)\setminus A$, there exists $r_2\in A\cap[r_1,\infty)$ such that

$$\begin{aligned}
&w'(r)<0 \quad \forall r_2<r\le r' \quad \text{and} \quad w'(r_2)=0\\
(2.15)\quad &\Rightarrow \quad w(r')\le w(r_2)\le \frac{2}{(1-m)C_3} \quad \forall r'>r_1, r'\notin A \quad (\text{by } (2.14)).
\end{aligned}$$

By (2.14) and (2.15),

$$w(r)\le \frac{2}{(1-m)C_3} \quad \forall r\ge r_1$$

and (2.1) holds with $C_1=\max\left(\frac{2}{(1-m)C_3},\max_{1\le r\le r_1}w(r)\right)$.

Case 2: *There exists a constant* $R_0 > 1$ *such that* $A \cap [R_0, \infty) = \phi$. Then $w'(r) < 0$ for all $r \geq R_0$. Hence, (2.1) holds with $C_1 = \max_{1 \leq r \leq R_0} w(r)$ and the lemma follows. □

3. Proof of Theorem 1.1

We first recall a result of [H1]:

Lemma 3.1 (cf. Lemma 2.1 of [H1]). *Let* $\eta > 0$, m, n, $\alpha > 0$, $\beta \neq 0$ *satisfy* (1.2) *and*

$$\frac{m\alpha}{\beta} \leq n - 2.$$

Let v *be the solution of* (1.6), (1.7). *Then*

$$v(r) + \frac{\beta}{\alpha} r v'(r) > 0 \quad \forall r \geq 0 \tag{3.1}$$

and

$$v'(r) < 0 \qquad \forall r > 0. \tag{3.2}$$

Lemma 3.2. *Let* $\rho > 0$, m, n, $\alpha > n\beta$ *satisfy* (1.2), (1.3) *and* (1.9). *Then*

$$\lim_{r \to \infty} r^{n-2} v^m(r) = \infty. \tag{3.3}$$

Proof. Suppose (3.3) does not hold. Then there exists a sequence $\{r_i\}_{i=1}^{\infty}$, $r_i \to \infty$ as $i \to \infty$, such that $r_i^{n-2} v^m(r_i) \to a_1$ as $i \to \infty$ for some constant $a_1 \geq 0$. By Lemma 2.1, the sequence $\{r_i\}_{i=1}^{\infty}$ has a subsequence which we may assume without loss of generality to be the sequence itself such that $w(r_i) \to a_2$ as $i \to \infty$ where $a_2 = 0$, w_∞, or w_1 with w_1 being given by (2.3). By (2.5), Lemma 2.5, Lemma 2.6, and the l'Hospital rule,

$$\begin{aligned}
\frac{(n-1)}{m} a_1 =& \frac{(n-1)}{m} \lim_{i \to \infty} r_i^{n-2} v(r_i)^m \\
=& \beta \lim_{i \to \infty} \frac{\int_{r_i}^{\infty} s v(s)\, ds}{r_i^{2-n}} + \lim_{i \to \infty} \frac{\int_{r_i}^{\infty} \frac{\alpha - n\beta}{s^{n-1}} \left(\int_0^s z^{n-1} v(z)\, dz \right) ds}{r_i^{2-n}} \\
=& \frac{\beta}{n-2} \lim_{i \to \infty} r_i^n v(r_i) + \frac{\alpha - n\beta}{n-2} \lim_{i \to \infty} \int_0^{r_i} z^{n-1} v(z)\, dz \\
=& \frac{\beta}{n-2} \lim_{i \to \infty} r_i^{n-2} v(r_i)^m \cdot \lim_{i \to \infty} r_i^2 v(r_i)^{1-m} + \frac{\alpha - n\beta}{n-2} \int_0^{\infty} z^{n-1} v(z)\, dz \\
=& \frac{\beta}{n-2} a_1 a_2 + \frac{\alpha - n\beta}{n-2} \int_0^{\infty} z^{n-1} v(z)\, dz.
\end{aligned}$$

Hence,

$$\frac{\alpha - n\beta}{a_1} \int_0^{\infty} z^{n-1} v(z)\, dz = \frac{(n-1)(n-2)}{m} - \beta a_2. \tag{3.4}$$

By (2.4) and (3.4),

$$\begin{aligned}
-(n-1) \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} =& \beta \lim_{i \to \infty} r_i^2 v(r_i)^{1-m} + \lim_{i \to \infty} \frac{(\alpha - n\beta)}{r_i^{n-2} v(r_i)^m} \int_0^{r_i} z^{n-1} v(z)\, dz \\
=& \frac{(n-1)(n-2)}{m}.
\end{aligned} \tag{3.5}$$

Hence,

$$\lim_{i\to\infty}\frac{r_iv'(r_i)}{v(r_i)}=-\frac{(n-2)}{m}. \tag{3.6}$$

By (1.2), (1.3) and (1.9),

$$\frac{m\alpha}{\beta}<n-2$$

holds. Hence, there exists a constant $\varepsilon>0$ such that

$$\frac{m\alpha}{\beta}<n-2-\varepsilon. \tag{3.7}$$

By (3.7) and Lemma 3.1, (3.1) and (3.2) hold. Then by (3.1), (3.2) and (3.7),

$$\begin{aligned}&0>\frac{rv'(r)}{v(r)}>-\frac{\alpha}{\beta}>-\frac{n-2}{m}+\frac{\varepsilon}{m}\quad\forall r>0 \\ \Rightarrow\quad &\lim_{i\to\infty}\frac{r_iv'(r_i)}{v(r_i)}\ge-\frac{n-2}{m}+\frac{\varepsilon}{m},\end{aligned} \tag{3.8}$$

which contradicts (3.6). Hence, no such sequence $\{r_i\}_{i=1}^{\infty}$ exists, and the lemma follows. □

Lemma 3.3. *Let $\rho>0$, m, n, $\alpha>n\beta$ satisfy* (1.2), (1.3) *and* (1.9). *Then there exists a constant $\varepsilon\in(0,\min(1,w_\infty/2))$ such that for any $R_0>1$ there exists $r'>R_0$ such that*

$$w(r')\ge\varepsilon.$$

Proof. Suppose the lemma is false. Then

$$\lim_{r\to\infty}w(r)=0. \tag{3.9}$$

We claim that

$$\lim_{r\to\infty}\frac{rv'(r)}{v(r)}=0. \tag{3.10}$$

By the proof of Lemma 3.2 there exists a constant $\varepsilon>0$ such that (3.8) holds. Suppose (3.10) does not hold. Then by (3.8) and (3.9) there exists a sequence $\{r_i\}_{i=1}^{\infty}$, $r_i\to\infty$ as $i\to\infty$, such that $r_iv'(r_i)/v(r_i)\to a_3$ as $i\to\infty$ for some constant a_3 satisfying

$$-\frac{n-2}{m}+\frac{\varepsilon}{m}\le a_3<0 \tag{3.11}$$

and (3.5) holds. By Lemma 3.2, (3.5), (3.9) and (3.11), we get

$$-(n-1)\lim_{i\to\infty}\frac{r_iv'(r_i)}{v(r_i)}=0 \qquad\text{if } v\in L^1(\mathbb{R}^n),$$

and if $v\notin L^1(\mathbb{R}^n)$, then by the l'Hospital rule,

$$\begin{aligned}-(n-1)\lim_{i\to\infty}\frac{r_iv'(r_i)}{v(r_i)}&=(\alpha-n\beta)\lim_{i\to\infty}\frac{r_i^{n-1}v(r_i)}{(n-2)r_i^{n-3}v(r_i)^m+mr_i^{n-2}v(r_i)^{m-1}v'(r_i)}\\&=(\alpha-n\beta)\lim_{i\to\infty}\frac{r_i^2v(r_i)^{1-m}}{n-2+m(r_iv'(r_i)/v(r_i))}\\&=\frac{\alpha-n\beta}{n-2+ma_3}\cdot\lim_{i\to\infty}r_i^2v(r_i)^{1-m}\\&=0.\end{aligned}$$

Hence,

$$a_3 = \lim_{i\to\infty} \frac{r_i v'(r_i)}{v(r_i)} = 0,$$

which contradicts (3.11). Thus, no such sequence $\{r_i\}_{i=1}^\infty$ exists and (3.10) follows. Since

$$w'(r) = \frac{2w(r)}{r}\left(1 + \frac{1-m}{2} \cdot \frac{rv'(r)}{v(r)}\right),$$

by (3.10) there exists a constant $R_0 > 0$ such that

$$w'(r) > 0 \quad \forall r \ge R_0,$$

which contradicts (3.9) and the lemma follows. □

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into two cases.

Case 1: $\alpha \le n\beta$. By Corollary 2.3 , Lemma 2.4 and Lemma 2.6, we get (1.10).

Case 2: $\alpha > n\beta$. By Lemma 2.5 there exists a constant $C_1 > 0$ such that (2.1) holds. Let $0 < \varepsilon < \min(1, w_\infty/2)$ be as in Lemma 3.3. Suppose there exists a sequence $\{r_i\}_{i=1}^\infty$, $r_i \to \infty$ as $i \to \infty$, such that $w(r_i) < \varepsilon$ for all $i \in \mathbb{Z}^+$. Then by Lemma 3.3 there exists a subsequence of $\{r_i\}_{i=1}^\infty$ which we may assume without loss of generality to be the sequence itself and a sequence $\{r_i'\}_{i=1}^\infty$ such that $r_i < r_i' < r_{i+1}$ for all $i = 1, 2, \dots$ and

$$w(r_i) < \varepsilon < w(r_i') \quad \forall i = 1, 2, \dots. \tag{3.12}$$

By (3.12) and the intermediate value theorem, for any $i = 1, 2, \dots$, there exists $a_i \in (r_i, r_i')$ such that

$$w(a_i) = \varepsilon \quad \forall i = 1, 2, \dots.$$

Hence, $a_i \to \infty$ as $i \to \infty$ and

$$\lim_{i\to\infty} w(a_i) = \varepsilon.$$

This contradicts Lemma 2.1 and Remark 2.2. Hence no such sequence $\{r_i\}_{i=1}^\infty$ exists. Thus there exists a constant $R_1 > 1$ such that $w(r) \ge \varepsilon$ for all $r \ge R_1$. Hence (2.10) holds with $C_2 = \min(\varepsilon, \min_{1\le r\le R_1} w(r)) > 0$. By Corollary 2.3 we get (1.10) and the theorem follows. □

Proof of Corollary 1.3. By Theorem 1.1,

$$\begin{aligned} |x|^2 v_\lambda(x)^{1-m} &= (\lambda|x|)^2 v(\lambda x)^{1-m} \\ &\to \frac{2(n-1)(n(1-m)-2)}{(1-m)(\alpha(1-m)-2\beta)} \quad \text{uniformly on } \mathbb{R}^n \setminus B_R(0) \end{aligned}$$

as $\lambda \to \infty$ for any $R > 0$ and the corollary follows. □

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