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POINTS NEAR REAL ALGEBRAIC SETS

W. M. SCHMIDT AND U. ZANNIER

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ABSTRACT. Given a real algebraic set X and a box $\mathcal{B}$ in $\mathbb{R}^n$, which is a union of cubes of equal size and with disjoint interiors, we bound the number of cubes that intersect X. As a consequence, we bound the volume of the set of points having distance at most δ from $X \cap \mathcal{B}$, and we estimate the number of integer points in a domain $\mathcal{D} \subset \mathbb{R}^n$ bounded by algebraic hypersurfaces.

1. Introduction

When I is an ideal in the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$, we will write $X(I)$ for the real algebraic set consisting of the common zeros in $\mathbb{R}^n$ of the polynomials in I. The *dimension* of a real algebraic set X is the maximum of the dimensions of the real analytic manifolds it contains, and is also the maximal transcendence degree of the function fields of its irreducible components. See, e.g., [1], sections 3.4.1, 3.4.2.

Given positive integers $c_1, \ldots, c_n$, *a box of type* $(c_1, \ldots, c_n)$ will be a box $\mathcal{B} \subset \mathbb{R}^n$ defined by

$$a_i \leq x_i \leq b_i \qquad (i = 1, \ldots, n) \tag{1.1}$$

where $b_i - a_i = c_i$. Such a box has volume $C = c_1 \cdots c_n$ and is the union of C closed cubes $\mathcal{C}$ of side 1 parallel to the coordinate axes, whose interiors are disjoint. The vertices of these cubes form a grid Γ consisting of the points

$$(a_1 + j_1, \ldots, a_n + j_n) \quad \text{with} \quad 0 \leq j_i \leq c_i \quad (i = 1, \ldots, n). \tag{1.2}$$

We will employ quantities $D_0 = 1$ and

$$D_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} (c_{i_1} + 1) \cdots (c_{i_k} + 1) \quad (k = 1, \ldots, n). \tag{1.3}$$

The cardinality of Γ equals D_n.

We have $D_k \ll F_k$ for $1 \leq k \leq n$, where F_k is the maximal volume of the k-dimensional faces of $\mathcal{B}$, and where the constant implied by $\ll$ depends only on n. When $c_1 \geq \cdots \geq c_n$, then

$$D_k \ll F_k = c_1 \ldots c_k \leq c_1^k.$$

Write $F_0 = D_0$.

The quantity

$$N_{\mathcal{B}}(d, \ell) = 2^{n-1}(D_\ell d^{n-\ell} + D_{\ell-1} d^{n-\ell+1} + \cdots + D_0 d^n) \tag{1.4}$$

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has

$$N_{\mathcal{B}}(d,\ell) \ll F_\ell d^{n-\ell} + F_{\ell-1} d^{n-\ell+1} + \cdots + F_0 d^n.$$

Theorem. *Suppose $X = X(I)$ where I is generated by polynomials of total degree at most d, and suppose $\ell = \dim X < n$. Let S_X be the union of all $(n-l)$-dimensional planes parallel to a coordinate subspace which intersect X.*

S_X is an algebraic set properly contained in $\mathbb{R}^n$. For any box $\mathcal{B}$ of type $(c_1,\dots,c_n)$ whose grid Γ does not intersect S_X, fewer than $N_{\mathcal{B}}(d,\ell)$ of the cubes $\mathcal{C} \in \mathcal{B}$ intersect X.

The condition that Γ not intersect S_X will cause little trouble in applications. The Theorem will be proved in Section 2. We now are going to list some consequences which will be derived in Section 3. In Section 4 we will remark on points near complex algebraic sets in C^n.

When $\mathcal{B}$ is a box given by (1.1), let $\mathcal{B}^\circ$ be the half-open box given by $a_i \le x_i < b_i$ $(i = 1,\dots,n)$.

Corollary 1. *Let $f_1,\dots,f_r$ be polynomials in $\mathbb{R}[x_1,\dots,x_n]$ of total degree at most d, and $\mathcal{D}$ the set of points $\mathbf{x}$ in $\mathbb{R}^n$ with*

$$f_q(\mathbf{x}) \ge 0 \quad (q = 1,\dots,r).$$

Let $\mathcal{B}$ be a box of type $(c_1,\dots,c_n)$ and write $V(\mathcal{D},\mathcal{B})$ for the volume of $\mathcal{D}\cap\mathcal{B}$, and $Z(\mathcal{D},\mathcal{B}^\circ)$ for the number of integer points in $\mathcal{D}\cap\mathcal{B}^\circ$. Then

$$|Z(\mathcal{D},\mathcal{B}^\circ) - V(\mathcal{D},\mathcal{B})| < r N_{\mathcal{B}}(d, n-1). \tag{1.5}$$

Remarks. The corollary is similar to a theorem of Davenport [2], where instead of the D_k, certain quantities V_k occur, which are the sums of the volumes of the projections of $\mathcal{D}$ on the k-dimensional coordinate subspaces.

The number $Z(\mathcal{D},\mathcal{B})$ of integer points in $\mathcal{D}\cap\mathcal{B}$ is less than

$$V(\mathcal{D},\mathcal{B}) + r N_{\mathcal{B}'}(d, n-1)$$

where $\mathcal{B}'$ is the box $a_i \le x_i \le b_i + 1$ $(i = 1,\dots,n)$, and hence involves $c_i' = c_i + 1$ in place of c_i $(i = 1,\dots,n)$.

By *distance* in $\mathbb{R}^n$ we will understand the distance in terms of the maximum norm.

Corollary 2. *Let $\mathcal{B}$ be a box (1.1) where $c_i = b_i - a_i > 0$ $(i = 1,\dots,n)$, but with $c_1,\dots,c_n$ not necessarily integers. Define D_k $(k = 0,\dots,n)$ and $N_{\mathcal{B}}(d,\ell)$ as before. Let $X = X(I)$ be as in the Theorem, and $0 < \delta \le \min(c_1,\dots,c_n)$.*

Then the set of points in $\mathbb{R}^n$ having distance at most δ from $X\cap\mathcal{B}$ has volume

$$V(X,\mathcal{B},\delta) < 9^n N_{\mathcal{B}}(\delta d,\ell).$$

When $1 \le \delta \le \min(c_1,\dots,c_n)$, then the number $Z(X,\mathcal{B},\delta)$ of integer points having distance at most δ from $X\cap\mathcal{B}$ is less than $12^n N_{\mathcal{B}}(\delta d,\ell)$.

Our estimates, in particular $9^n, 12^n$, could be somewhat improved. Corollary 2 implies that $V(X,\mathcal{B},\delta)$ and $Z(X,\mathcal{B},\delta)$ are

$$\ll F_\ell (d\delta)^{n-\ell} + F_{\ell-1}(d\delta)^{n-\ell+1} + \cdots + F_0 (d\delta)^n,$$

and when $0 < \delta < 1/d$ they are $\ll F_\ell (d\delta)^{n-\ell}$.

2. Proof of the Theorem

We will use two facts from real algebraic geometry. First, a set $X = X(I) \subset \mathbb{R}^n$ where I is generated by polynomials of total degree at most d, has fewer than $2^{n-1}d^n$ connected components. See [1], Proposition 3.9.4.

Next, let $\sigma = \{i_1, \dots, i_k\}$ where $1 \le k \le n$ be a k-element subset of $\{1, \dots, n\}$. Given $(\xi_{i_1}, \dots, \xi_{i_k}) \in \mathbb{R}^k$, let $M(\xi_{i_1}, \dots, \xi_{i_k})$ be the $(n-k)$-dimensional space consisting of points $\mathbf{x} = (x_1, \dots, x_n)$ with $x_{i_1} = \xi_{i_1}, \dots, x_{i_k} = \xi_{i_k}$, i.e., with $x_i = \xi_i$ for $i \in \sigma$. Observe that $M(\xi_{i_1}, \dots, \xi_{i_k})$ depends not only on the numbers $\xi_{i_1}, \dots, \xi_{i_k}$ but also on their subscripts. For $k = n$, $M(\xi_{i_1}, \dots, \xi_{i_n})$ consists of the single point $(\xi_{i_1}, \dots, \xi_{i_n})$. When $X \subset \mathbb{R}^n$ is a real algebraic set, then so is each set $X\langle \xi_{i_1}, \dots, \xi_{i_k} \rangle = X \cap M(\xi_{i_1}, \dots, \xi_{i_k})$.

Suppose now that $\ell = \dim X < n$, and set $h = \ell + 1$. For $\sigma = \{i_1, \dots, i_h\}$ let $T(i_1, \dots, i_h)$ consist of the points $(\xi_{i_1}, \dots, \xi_{i_h}) \in \mathbb{R}^h$ where $X\langle \xi_{i_1}, \dots, \xi_{i_h} \rangle$ is non-empty. Then $T(i_1, \dots, i_h)$ is an algebraic set properly contained in $\mathbb{R}^h$. The corresponding fact for complex algebraic sets is well known (see, e.g., [3], Corollary 11.13), and for real algebraic sets it may be derived by a complexification argument (see [1], section 3.3). It follows that the set $T^+(i_1, \dots, i_h)$ of points $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ with $(x_{i_1}, \dots, x_{i_h}) \in T(i_1, \dots, i_h)$ is algebraic and properly contained in $\mathbb{R}^n$.

We now define S_X to be the union of the sets $T^+(i_1, \dots, i_h)$ where $h = \ell + 1$ and $\sigma = \{i_1, \dots, i_h\} \subset \{1, \dots, n\}$. Thus S_X is the union of the $(n-h)$-dimensional planes parallel to a coordinate subspace which intersect X. For instance, when X is a curve, so that $\ell = 1$, then when $n = 2$, the set S_X is just X, and when $n = 3$, then S_X is the union of the lines parallel to a coordinate axis and meeting X. The set S_X has the property that when $\mathbf{x} = (x_1, \dots, x_n) \notin S_X$ and $\{i_1, \dots, i_h\}$ is as above, then $X\langle x_{i_1}, \dots, x_{i_h} \rangle$ is empty.

A cube in $\mathbb{R}^n$ has "faces" of every dimension s where $0 \le s \le n$. For instance, its vertices are faces of dimension 0, and the cube itself is a face of dimension n. Faces of dimension s of the cubes $\mathcal{C} \subset \mathcal{B}$ will be called s-cubes. When $0 < s \le n$, the boundary of such an s-cube is the union of certain $(s-1)$-cubes.

Lemma 2.1. *Let $\mathcal{B}$ be a box as in the Theorem, so that in particular all the points* (1.2)*, i.e. the vertices of the cubes $\mathcal{C} \subset \mathcal{B}$, lie outside S_X. Then:*

(i) No s-cube with $s < n - \ell$ intersects X.

(ii) Given s with $n - \ell \le s \le n$, there are at most

$$2^{s-1} D_{n-s} d^s \tag{2.1}$$

s-cubes which intersect X, but whose boundaries do not.

Proof. Set $k = n - s$. For $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ and $j_{i_1}, \dots, j_{i_k}$ with

$$0 \le j_{i_1} \le c_{i_1}, \dots, 0 \le j_{i_k} \le c_{i_k}, \tag{2.2}$$

set

$$M_{\mathbf{i}}(\mathbf{j}) = M(a_{i_1} + j_{i_1}, \dots, a_{i_k} + j_{i_k}).$$

Each of these is an $s = (n-k)$-dimensional space. Each s-cube is contained in some $M_{\mathbf{i}}(\mathbf{j})$.

(i) It suffices here to deal with $s = n - \ell - 1$, so that $k = \ell + 1 = h$. Now $M_{\mathbf{i}}(\mathbf{j})$ does not intersect X by the property of S_X.

(ii) When $0 < s < n$, then $X_{\mathbf{i}}(\mathbf{j}) := X \cap M_{\mathbf{i}}(\mathbf{j})$ is an algebraic set defined in terms of polynomials of degree at most d, lying in s-dimensional space $X_{\mathbf{i}}(\mathbf{j})$. When $s = n$, recall that $X \subset R^n$. If some s-cube with $s < n$ intersects $X_{\mathbf{i}}(\mathbf{j})$ but its boundary does not, or when an n-cube (i.e. a cube $\mathcal{C} \subset \mathcal{B}$) intersects X but its boundary does not, then some connected component of $X_{\mathbf{i}}(j)$, or some connected component of X, lies in the interior of the cube. By what was said at the beginning, there are fewer than $2^{s-1}d^s$ such components. Therefore the number of s-cubes in question lying in $X_{\mathbf{i}}(\mathbf{j})$ or in X is bounded by this quantity. When $s = n$, this $2^{n-1}d^n = 2^{n-1}D_0 d^n$ is our final bound. When $s < n$, we need to sum over the spaces $M_{\mathbf{i}}(\mathbf{j})$. Summation over $j_{i_1}, \ldots, j_{i_k}$ with (2.2) gives

$$2^{s-1}(c_{i_1}+1)\cdots(c_{i_k}+1)d^s.$$

Since $k = n - s$, summation over the sets $\{i_1, \ldots, i_k\}$ gives the claimed bound

$$2^{s-1}D_{n-s}d^s. \qquad \square$$

The proof of the Theorem is now easily completed. A cube $\mathcal{C} \subset \mathcal{B}$ which intersects X will have a smallest s such that one of its s-dimensional faces intersects X. Here $s \geq n - \ell$ by Lemma 2.1. The boundary of such an s-cube will not intersect X. The number of s-cubes in question is bounded by (2.1). Given an s-cube, there are 2^{n-s} cubes in $\mathbb{R}^n$ of side 1, of which it is an s-dimensional face, hence at most 2^{n-s} cubes $\mathcal{C}$ of which it is a face. Therefore for given s we obtain the bound $2^{n-s}2^{s-1}D_{n-s}d^s = 2^{n-1}D_{n-s}d^s$. Taking the sum over s in $n-\ell \leq s \leq n$ we obtain the estimate

$$2^{n-1}\sum_{s=n-\ell}^{n} D_{n-s}d^s = N_{\mathcal{B}}(d,\ell). \qquad \square$$

3. Deduction of the corollaries

Proof of Corollary 1. Given an interval $\mathcal{I} : a \leq x \leq b$, the half-open interval $a \leq x < b$ will be denoted $\mathcal{I}^\circ$. When $b - a \in \mathbb{N}$, an interval *properly shifted from* $\mathcal{I}$ will be any interval $\hat{a} \leq x \leq \hat{b}$ where $\hat{b} - \hat{a} = b - a$ and $\lceil a \rceil - 1 < \hat{a} < a$. When $\widehat{\mathcal{I}}$ is properly shifted from $\mathcal{I}$, then $\widehat{\mathcal{I}}^\circ$ and $\mathcal{I}^\circ$ contain the same integers. More generally, when $\mathcal{B}$ is a box of type $(c_1, \ldots, c_n)$ given by (1.1), a box $\widehat{\mathcal{B}}$ will be said to be properly shifted from $\mathcal{B}$ if it is given by $\hat{a}_i \leq x_i \leq \hat{b}_i$ $(i = 1, \ldots, n)$ where $\hat{b}_i - \hat{a}_i = c_i$ and $\lceil a_i \rceil - 1 < \hat{a}_i < a_i$. Then $\mathcal{B}^\circ, \widehat{\mathcal{B}}^\circ$ contain the same integer points, so that in the context of Corollary 1, $Z(\mathcal{D}, \mathcal{B}^\circ) = Z(\mathcal{D}, \widehat{\mathcal{B}}^\circ)$. Since the set $S_X \subset \mathbb{R}^n$ of the Theorem is nowhere dense, there are properly shifted boxes $\widehat{\mathcal{B}}$ whose grid $\widehat{\Gamma}$ does not intersect S_X, and with arbitrarily small shift; hence $V(\mathcal{D}, \widehat{\mathcal{B}})$ is arbitrarily close to $V(\mathcal{D}, \mathcal{B})$. Therefore it will suffice to prove Corollary 1 in the case when the grid Γ of $\mathcal{B}$ does not intersect S_X, so that the Theorem applies.

Each cube $\mathcal{C} \subset \mathcal{B}$ gives rise in an obvious way to a half-open cube $\mathcal{C}^\circ$. The cubes $\mathcal{C}^\circ$ give a partition of $\mathcal{B}^\circ$. Each integer point in $\mathcal{D} \cap \mathcal{B}^\circ$ lies in a unique cube $\mathcal{C}^\circ$, and each cube $\mathcal{C}^\circ$ contains one integer point. Therefore $Z(\mathcal{D}, \mathcal{B}^\circ)$ is bounded from above by the number of cubes $\mathcal{C}^\circ$ which lie in $\mathcal{D}$ or intersect the boundary of $\mathcal{D}$, and is bounded from below by the number of cubes $\mathcal{C}^\circ$ which intersect $\mathcal{D}$ but not its boundary. As a consequence, $|Z(\mathcal{D}, \mathcal{B}^\circ) - V(\mathcal{D}, \mathcal{B})|$ is bounded by the number of cubes $\mathcal{C}^\circ$ which intersect the boundary of $\mathcal{D}$. This boundary is contained in the hypersurfaces $X(f_q)$ $(q = 1, \ldots, r)$ consisting of the zeros of f_q. By the case $\ell = n - 1$ of the Theorem, the estimate (1.5) follows. $\square$

Proof of Corollary 2. Set $c_i^* = \lceil c_i/\delta \rceil$, so that in view of $\delta \le c_i$ we have $c_i^* + 1 < 3c_i/\delta$. Let $\mathcal{B}^*$ be the box given by $a_i \le x_i \le a_i + \delta c_i^*$ $(i = 1, \ldots, n)$, This box contains $\mathcal{B}$ and is the union of $C^* = c_1^* \cdots c_n^*$ closed cubes $\mathcal{C}^*$ of side δ. Set $D_0^* = 1$, and for $1 \le k \le n$,

$$D_k^* =: \sum_{i_1 < \cdots < i_k} (c_{i_1}^* + 1) \cdots (c_{i_k}^* + 1) < (3/\delta)^k D_k \le 3^n \delta^{-k} D_k.$$

Since $\mathcal{B} \subset \mathcal{B}^*$, it will suffice to prove the claimed bounds for the volume $V(X, \mathcal{B}^*, \delta)$, and the number $Z(X, \mathcal{B}^*, \delta)$ of integer points, of the set of points having distance at most δ from $X \cap \mathcal{B}^*$.

The vertices of the C^* cubes $\mathcal{C}^* \subset \mathcal{B}^*$ form a grid Γ^* consisting of the points $(a_1 + \delta j_1, \ldots, a_n + \delta j_n)$ with $0 \le j_i \le c_i$ $(i = 1, \ldots, n)$. An easy variation on the Theorem shows that when Γ^* does not intersect S_X, then $X \cap \mathcal{B}^*$ is contained in cubes $\mathcal{C}_1^*, \ldots, \mathcal{C}_t^*$ of side δ where

$$\begin{aligned} t &< 2^{n-1}(D_\ell^* d^{n-\ell} + D_{\ell-1}^* d^{n-\ell+1} + \cdots + D_0^* d^n) \\ &< 2^{n-1} 3^n (D_\ell \delta^{-\ell} d^{n-\ell} + D_{\ell-1} \delta^{1-\ell} d^{n-\ell+1} + \cdots + D_0 d^n). \end{aligned}$$

An argument as for Corollary 1 shows that we may assume that in fact Γ^* and S_X are disjoint. Points having distance at most δ from $X \cap \mathcal{B}^*$ have distance less than δ from some $\mathcal{C}_q^*$, and hence lie in a cube $\mathcal{C}_q^{**} \supset \mathcal{C}_q^*$ of side 3δ. Such a cube has volume $(3\delta)^n$, so that the total volume is at most

$$\begin{aligned} (3\delta)^n t &< 9^n 2^{n-1} (D_\ell (\delta d)^{n-\ell} + D_{\ell-1} (\delta d)^{n-\ell+1} + \cdots + D_0 (\delta d)^n) \\ &= 9^n N_{\mathcal{B}}(\delta d, \ell). \end{aligned}$$

When $1 \le \delta \le \min(c_1, \ldots, c_n)$, each cube $\mathcal{C}_q^{**}$ contains not more than $(3\delta + 1)^n \le (4\delta)^n$ integer points, so that the number $Z(X, \mathcal{B}^*, \delta)$ of integer points is less than

$$(4\delta)^n t < 12^n N_{\mathcal{B}}(\delta d, \ell). \qquad \square$$

4. On algebraic sets in $\mathbb{C}^n$

Let I be an ideal in $\mathbb{C}[z_1, \ldots, z_n]$ generated by polynomials of total degree at most d. Suppose the set $Y(I)$ consisting of the common zeros in $\mathbb{C}^n$ of the polynomials in I has dimension $\ell < n$. Write $z_j = x_j + iy_j$ with real x_j, y_j $(j = 1, \ldots, n)$. For $f \in I$ set

$$\begin{aligned} f(z_1, \ldots, z_n) &= f(x_1 + iy_1, \ldots, x_n + iy_n) \\ &= g_f(x_1, y_1, \ldots, x_n, y_n) + i h_f(x_1, y_1, \ldots, x_n, y_n) \end{aligned}$$

with g_f, h_f in $\mathbb{R}[x_1, y_1, \ldots, x_n, y_n]$. Since g_f, h_f are in $\mathbb{R}[x_1, y_1, \ldots, x_n, y_n]$ and we may interpret $Y(I)$ to lie in $\mathbb{R}^{2n}$, we may interpret $Y(I)$ to be $X(I_{\mathbb{R}})$, where $I_{\mathbb{R}}$ is generated by the polynomials g_f, h_f with $f \in I$.

Suppose $\mathcal{B} \subset \mathbb{C}^n = \mathbb{R}^{2n}$ is a box of type $(c_1, \ldots, c_{2n})$ and define D_k $(k = 0, \ldots, 2n)$ in an obvious way. By the Theorem, fewer than $N_{\mathcal{B}}(d, 2\ell)$ of the cubes $\mathcal{C} \subset \mathcal{B}$ intersect $Y(I) = X(I_{\mathbb{R}})$. A suitable version of Corollary 2 also holds.

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Department of Mathematics, University of Colorado, Boulder, Colorado 80309-0395

Scuola Normale Superiore, Piazza de Cevalier, 56100 Pisa, Italy