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ABSOLUTELY SUMMING MULTIPLIER OPERATORS IN $L^p(G)$ FOR $p > 2$

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ABSTRACT. Let G be an infinite compact abelian group. If its dual group Γ contains an element of infinite order, then it is known that, for every $4 < p < \infty$, there exists a function $g \in L^p(G)$ whose associated convolution operator $C_g : f \mapsto f * g$ (on $L^p(G)$) is absolutely summing but the Fourier series of g fails to be unconditionally convergent to g in $L^p(G)$. It is shown that the restriction on Γ containing an element of infinite order can be removed and also that the range of p can be extended to arbitrary $p \in (2, \infty)$.

The idea of absolutely summing (also called 1-summing) operators is classical and has applications to many branches of mathematics; see e.g. [3], [4], [9], [13], [14]. Perhaps even more important are the p-multiplier operators arising in harmonic analysis; see [6], [7], [8], [10] and the references therein. Certain properties concerning operators which belong to both of these classes are exposed in [1], [2]. In the recent article [11] additional properties were identified which we briefly summarize. First some notation.

Let G be an infinite compact abelian group with normalized Haar measure μ and dual group Γ. For $1 \leq p \leq \infty$ let $L^p(G)$ be the usual Banach space of functions on G which are p-th power integrable. Since G is always fixed (but arbitrary) we write $L^p(G)$ simply as L^p. Each $g \in L^1$ induces a bounded convolution operator $C_g^{(p)} : L^p \to L^p$ via $f \mapsto f * g$, for $f \in L^p$. Given $1 \leq p, q \leq \infty$, let $\Pi_1^{(q,p)}$ denote the Banach space of those bounded linear operators $T \in \mathcal{L}(L^q, L^p)$ which are absolutely summing from L^q into L^p and equipped with the usual 1-summing norm, which we denote by $\|\cdot\|_{\Pi_1^{(q,p)}}$, [4, p. 31]. Those operators in $\mathcal{L}(L^q, L^p)$ which commute with translations form the space $\mathcal{M}^{(q,p)}$ of all (q,p)-multiplier operators. For $p = q$ we simply write $\Pi_1^{(p)}$ and $\mathcal{M}^{(p)}$. Set $\mathcal{M}_{\Pi_1}^{(p)} := \Pi_1^{(p)} \cap \mathcal{M}^{(p)}$ and $\mathcal{M}_{\Pi_1}^{(1,p)} := \Pi_1^{(1,p)} \cap \mathcal{M}^{(1,p)}$. Every operator in $\mathcal{M}_{\Pi_1}^{(p)}$ is of the form $C_g^{(p)}$ and every operator in $\mathcal{M}_{\Pi_1}^{(1,p)}$ is a convolution operator of the form $C_h^{(1,p)} : L^1 \to L^p$, for appropriate functions $g, h \in L^p$, respectively, [2, Ch.II, Propositions 7 & 8]. For $1 \leq p \leq \infty$, define the Banach spaces

$$\Sigma_1^{(p)} := \left\{ g \in L^p : C_g^{(p)} \in \mathcal{M}_{\Pi_1}^{(p)} \right\} \text{ and } \Sigma_1^{(1,p)} := \left\{ h \in L^p : C_h^{(1,p)} \in \mathcal{M}_{\Pi_1}^{(1,p)} \right\}$$

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equipped with the norm

$$\|g\|_{\Pi_1^{(p)}} := \|C_g^{(p)}\|_{\Pi_1^{(p)}} \ \text{ and } \ \|h\|_{\Pi_1^{(1,p)}} := \|C_h^{(1,p)}\|_{\Pi_1^{(1,p)}},$$

respectively. Finally, let S_p denote the linear space of all $f \in L^p$ whose Fourier series $\sum_{\gamma\in\Gamma} \widehat{f}(\gamma)(\cdot,\gamma)$ is unconditionally convergent to f in L^p, where $(\cdot,\gamma)$ denotes the character $x \mapsto (x,\gamma) := \gamma(x) \in \mathbb{C}$, for $x \in G$. Then S_p is a Banach space for the norm

$$\|f\|_{S_p} := \sup_{F\in\mathcal{F}} \Big\| \sum_{\gamma\in F} \widehat{f}(\gamma)(\cdot,\gamma)\Big\|_p, \qquad f \in S_p,$$

where $\mathcal{F}$ is the collection of all finite subsets of Γ.

It is known that

$$\Sigma_1^{(1,p)} = \Sigma_1^{(p)} = S_p = L^2, \qquad 1 \le p \le 2; \tag{1}$$

see the Introduction in [11] where all the relevant references are provided. The point of departure in [11] is an investigation of the case when $2 < p < \infty$. It is proved there, with continuous inclusions, that

$$\Sigma_1^{(1,p)} \subsetneqq S_p \subsetneqq L^p, \qquad 2 < p < \infty, \tag{2}$$

[11, Proposition 3.3(ii) and Corollary 4.6], and also with continuous inclusions that

$$\Sigma_1^{(1,p)} \subseteq \Sigma_1^{(p)} \subseteq L^p, \qquad 2 < p < \infty, \tag{3}$$

[11, Proposition 3.3(i)]. Moreover, one always has the containment

$$\Sigma_1^{(p)} \subsetneqq L^p, \qquad 2 < p < \infty,$$

[11, Proposition 5.8]. However, obtaining results for the inclusion $\Sigma_1^{(1,p)} \subseteq \Sigma_1^{(p)}$ to be *proper* or when $\Sigma_1^{(p)} \nsubseteq S_p$, for $p > 2$, turns out to be more elusive. The role of the Banach space $C(G)$, consisting of all the continuous functions on G, in relation to the spaces S_p and $\Sigma_1^{(1,p)}$ is also unclear. Under a certain restriction on the dual group Γ and on the index $p > 2$ we have the following known

Facts. Let G be an infinite compact abelian group such that Γ has an element of infinite order.

(i) For every $2 < p < \infty$ we have $C(G) \nsubseteq S_p$. In particular, also $C(G) \nsubseteq \Sigma_1^{(1,p)}$.
(ii) For every $4 < p < \infty$ we have $\Sigma_1^{(p)} \nsubseteq S_p$.
(iii) For every $4 < p < \infty$ we have $\Sigma_1^{(1,p)} \subsetneqq \Sigma_1^{(p)}$.
(iv) For G metrizable and every $4 < p < \infty$ the set of characters $\{(\cdot,\gamma) : \gamma \in \Gamma\}$ does not form an unconditional basis for the Banach space $\Sigma_1^{(p)}$.

Fact (i) occurs in [11, Corollary 4.10] and Fact (ii) is precisely Proposition 5.5 of [11]. Facts (iii) and (iv) are then consequences of Fact (ii); see Corollaries 5.6 and 5.7 in [11], respectively.

In the recent article [12] it was observed (cf. Proposition 3.15 there) that the existence of an element of infinite order in Γ is not necessary for the validity of Fact (i) above. Indeed, for an arbitrary infinite compact abelian group G, we even have $C(G) \nsubseteq \bigcup_{2<p<\infty} S_p$.

The aim of this paper is to remove completely the restriction imposed on Γ in Facts (i) – (iv) above and to extend the range of p to include all $2 < p < \infty$, that is, to establish the following result.

Theorem 1. *For every infinite compact abelian group G and every $2 < p < \infty$ we have*

(a) $\Sigma_1^{(p)} \not\subseteq S_p$,
(b) $\Sigma_1^{(1,p)} \subsetneqq \Sigma_1^{(p)}$, *and*
(c) $\{(\cdot,\gamma) : \gamma \in \Gamma\}$ *is not an unconditional basis for* $\Sigma_1^{(p)}$ *whenever G is metrizable.*

In order to establish Theorem 1 we will require some preliminary results. Recall that the sequence of Rademacher functions $r_j : [0,1] \to \{-1,1\}$, for $j \in \mathbb{N}$, is given by $r_j(t) := \operatorname{sign}(\sin 2^j \pi t)$, for $t \in [0,1]$. The cardinality of a finite set E is denoted by $|E|$. Let p^* denote the conjugate index to p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. The norm of a bounded linear operator T between Banach spaces is denoted by $\|T\|_{\mathrm{op}}$.

Lemma 2. *Let G be an infinite compact abelian group. For every $p \in (2,\infty)$ there exists a constant $\alpha_p > 0$ with the property that, for every finite subset $E \subseteq \Gamma$, there is a choice of signs $\{\sigma_\gamma\}_{\gamma\in E}$, with each $\sigma_\gamma \in \{-1,1\}$, such that*

$$\Big\| \sum_{\gamma\in E} \sigma_\gamma \cdot (\cdot,\gamma) \Big\|_{L^p} \le \alpha_p |E|^{1/2}. \tag{4}$$

Proof. Let $E = \{\gamma_1,\dots,\gamma_n\}$, where $n = |E|$. In order to estimate

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t)(\cdot,\gamma_j) \Big\|_{L^p(G)}^p \,dt = \int_0^1 \int_G \Big| \sum_{j=1}^n r_j(t)(x,\gamma_j) \Big|^p d\mu(x)\,dt$$

we interchange the order of integration and use Khinchin's inequality, [4, p. 10], to ensure the existence of $\alpha_p > 0$ such that

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t)(\cdot,\gamma_j) \Big\|_{L^p(G)}^p \,dt = \int_G \int_0^1 \Big| \sum_{j=1}^n r_j(t)(x,\gamma_j) \Big|^p \,dt\,d\mu(x)$$
$$\le \int_G \alpha_p^p \Big(\sum_{j=1}^n |(x,\gamma_j)|^2 \Big)^{p/2} d\mu(x) = \int_G \alpha_p^p\, n^{p/2}\, d\mu(x) = \alpha_p^p\, n^{p/2}.$$

Thus, there exists $t_0 \in [0,1]$ such that $\big\| \sum_{j=1}^n r_j(t_0)(\cdot,\gamma_j) \big\|_{L^p(G)} \le \alpha_p n^{1/2}$. Setting $\sigma_{\gamma_j} := r_j(t_0)$, for $1 \le j \le n$, completes the proof. □

Lemma 3. *Let $2 < p < \infty$ and $1 \le r < p^*$. Then, for every function $f \in L^{p^*} \subseteq L^r$ and every multiplier operator $T \in \mathcal{M}^{(r,p)}$, the function Tf belongs to $\Sigma_1^{(p)} \subseteq L^p$. Moreover, there exists a constant $\gamma_{p,r} > 0$, depending only on p and r, such that*

$$\|Tf\|_{\Pi_1^{(p)}} \le \gamma_{p,r} \|f\|_{p^*} \|T\|_{\mathrm{op}}, \qquad f \in L^{p^*},\ T \in \mathcal{M}^{(r,p)}. \tag{5}$$

Proof. Let $f \in L^{p^*}$ and set $h := Tf \in L^p$. An application of Hölder's inequality shows that $L^p * L^{p^*} \subseteq L^\infty$ continuously, i.e., $C_f^{(p,\infty)} \in \mathcal{L}(L^p, L^\infty)$. So, if $J^{(\infty,r)} : L^\infty \to L^r$ is the natural inclusion, then we have the factorization of $C_h^{(p)} \in \mathcal{L}(L^p)$ given by

$$C_h^{(p)} = T \circ J^{(\infty,r)} \circ C_f^{(p,\infty)}, \tag{6}$$

after recalling that T commutes with all convolution operators. Accordingly, $C_h^{(p)}$ is r-summing since $J^{(\infty,r)}$ is r-summing (with $\|J^{(\infty,r)}\|_{\Pi_r} = 1$), [4, p. 40]. But, $p > 2$ implies that $L^p(G)$ has cotype p, [4, Theorem 11.6]. Since $r < p^*$, we can apply [4, Corollary 11.16] to conclude that $\Pi_r^{(p)} = \Pi_1^{(p)}$. Hence, $C_h^{(p)}$ is 1-summing, i.e., $Tf = h \in \Sigma_1^{(p)}$.

Since $\Pi_r^{(p)} = \Pi_1^{(p)}$, there is a constant $\gamma_{p,r}$ (depending only on the cotype p constant of $L^p(G)$ and on r) such that

$$\|h\|_{\Pi_1^{(p)}} := \|C_h^{(p)}\|_{\Pi_1^{(p)}} \leq \gamma_{p,r}\|C_h^{(p)}\|_{\Pi_r^{(p)}} := \gamma_{p,r}\|h\|_{\Pi_r^{(p)}}.$$

Thanks to this inequality and (6) we obtain (5) because the r-summing norm $\|J^{(\infty,r)}\|_{\Pi_r} = 1$ and $\|C_f^{(p,\infty)}\|_{\mathrm{op}} = \|f\|_{p^*}$. □

Remark 4. (i) For $r = 1$ we observe that Lemma 3 reduces to Proposition 5.4 of [11] whenever $p > 2$. Indeed, for every $p > 1$ we have $T \in \mathcal{M}^{(1,p)}$ if and only if there exists $g \in L^p$ such that $T = C_g^{(1,p)}$, [10, Theorem 3.1.1], and so Lemma 3 yields $L^p * L^{p^*} \subseteq \Sigma_1^{(p)}$.

(ii) The statement of Lemma 3 is also valid for every $1 \leq p \leq 2$, without any restriction on r. That is, if $1 \leq r \leq \infty$, $p \in [1,2]$, $T \in \mathcal{M}^{(r,p)}$ and $f \in L^{p^*}$, then $Tf \in \Sigma_1^{(p)}$. Indeed, $p^* \geq 2$ implies that $f \in L^2$. Moreover, $Tf \in L^2$ because $\mathcal{M}^{(1,p)} \subseteq \mathcal{M}^{(2,2)}$. The conclusion then follows from (1).

Lemma 5. *Let $2 < p < \infty$, let E be a finite subset of Γ and let $\varphi = \Sigma_{\gamma\in E}\, \sigma_\gamma \cdot (\cdot, \gamma)$, with each $\sigma_\gamma \in \{-1, 1\}$, be any trigonometric polynomial on G satisfying* (4). *Then, for every $r \in (1,2)$, we have*

$$\|C_\varphi^{(r,p)}\|_{\mathrm{op}} \leq \alpha_p |E|^s, \qquad \text{with } s := \frac{1}{2} - \frac{2}{p} + \frac{2}{rp}.$$

Proof. For $r = 1$ it follows from (4) that

$$\|C_\varphi^{(1,p)}\|_{\mathrm{op}} \leq \|\varphi\|_p \leq \alpha_p |E|^{1/2}. \tag{7}$$

On the other hand, $(\frac{1}{p^*} - \frac{1}{2}) > 0$. Define t via $\frac{1}{p^*} = \frac{1}{2} + \frac{1}{t}$, i.e., $\frac{1}{t} = \frac{1}{p^*} - \frac{1}{2}$. For every $h \in L^2$, it follows from the generalized Hölder inequality, [5, p. 527], that $\|\widehat{\varphi}\cdot\widehat{h}\|_{p^*} \leq \|\widehat{h}\|_2\, \|\widehat{\varphi}\|_t = \|\widehat{h}\|_2\, \|\chi_E\|_t = \|\widehat{h}\|_2\, |E|^{1/t} = \|\widehat{h}\|_2 \cdot |E|^{1/p^* - 1/2}$. Accordingly, the Hausdorff-Young inequality, [8, Theorem (31.10) and p. 227], applied in $\ell^{p^*}(\Gamma)$ with G being the dual group of Γ yields

$$\|h * \varphi\|_p \leq \|(h*\varphi)\widehat{\ }\|_{\ell^{p^*}(\Gamma)} = \|\widehat{h}\cdot\widehat{\varphi}\|_{\ell^{p^*}(\Gamma)} \leq \|\widehat{h}\|_2\, |E|^{1/t} = \|h\|_{L^2(G)} |E|^{1/p^* - 1/2},$$

and so we have

$$\left\|C_\varphi^{(2,p)}\right\|_{\mathrm{op}} \leq |E|^{1/p^* - 1/2}. \tag{8}$$

By the Riesz-Thorin interpolation theorem, [7, p. 149], with $1 < r < 2$ fixed, (7) and (8) yield that

$$\|C_\varphi^{(r,p)}\|_{\mathrm{op}} \leq \left(\alpha_p |E|^{1/2}\right)^{1-\theta} \left(|E|^{1/p^* - 1/2}\right)^{\theta} = \alpha_p^{1-\theta} |E|^s,$$

where $\theta \in [0,1]$ is determined by $\frac{1}{r} = \frac{\theta}{2} + \frac{(1-\theta)}{1}$ (i.e., $\theta = 2 - \frac{2}{r}$) and $s := \theta(\frac{1}{p^*} - \frac{1}{2}) + (1-\theta)\cdot\frac{1}{2}$ (i.e., $s = \frac{1}{2} - \frac{\theta}{p} = \frac{1}{2} - \frac{2}{p} + \frac{2}{rp}$). This completes the proof since $\alpha_p^{1-\theta} \leq \alpha_p$. □

We recall, for every $p > 2$, that $(S_p, \|\cdot\|_{S_p})$ is a (complex) Banach lattice for an equivalent *lattice norm* $|\!|\!|\cdot|\!|\!|_{S_p}$, where the positive cone is determined by $g \geq 0$ in S_p if and only if $\widehat{g} \geq 0$ pointwise on Γ, [11, p. 864]. Moreover, via Propositions 3.3(ii) and 4.2 of [11] we have, for some constant $K_p > 0$, that

$$\|f\|_p \leq \|f\|_{S_p} \leq |\!|\!|f|\!|\!|_{S_p} \leq K_p \|f\|_{S_p}, \qquad f \in S_p. \tag{9}$$

The proof of Theorem 1 is based on the above lemmata and on the existence of an increasing sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of finite subsets of Γ and a sequence $\{V_n\}_{n=1}^{\infty}$ of trigonometric polynomials on G satisfying the following four conditions:

(I) $\sup_{n\in\mathbb{N}} |\Gamma_n| = \infty$ and

(II) $\widehat{V}_n \cdot \chi_{\Gamma_n} = \chi_{\Gamma_n}$, for $n \in \mathbb{N}$.

For each $2 < p < \infty$, there exist positive constants A_p, B_p satisfying

(III) $|\!|\!|\Sigma_{\gamma\in\Gamma_n}(\cdot,\gamma)|\!|\!|_{S_p} \geq A_p |\Gamma_n|^{1/p^*}$, for $n \in \mathbb{N}$, and

(IV) $\|V_n\|_{p^*} \leq B_p |\Gamma_n|^{1/p}$, for $n \in \mathbb{N}$.

The existence of such sequences $\{\Gamma_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ is established via a consideration of the following two cases.

Case (i). *Γ contains an element of infinite order.* Let $\gamma_0 \in \Gamma$ have infinite order and $\{v_m\}_{m=0}^{\infty}$ be the de la Vallée Poussin kernel on the circle group $\mathbb{T}$. Define trigonometric polynomials $V_n : G \to \mathbb{C}$ by $V_n := v_{2^n} \circ \gamma_0$, for $n \in \mathbb{N}$. Then $\Gamma_n := \{k\gamma_0 : 0 \leq k < 2^n, k \in \mathbb{Z}\}$, for $n \in \mathbb{N}$, surely satisfies condition (I). It is established in the proof of Proposition 5.5 in [11] that condition (II) holds and that

$$\|V_n\|_{p^*} \leq \epsilon_p (2^{n+1}+1)^{1/p} + \beta_p (2^n)^{1/p} \leq (\epsilon_p 4^{1/p} + \beta_p) 2^{n/p}, \qquad n \in \mathbb{N},$$

for positive constants ϵ_p, β_p; the argument given there holds (up to this stage) for all $p > 2$ (not just for $p > 4$). Since $|\Gamma_n| = 2^n$, for $n \in \mathbb{N}$, the previous inequality yields condition (IV). The proof of Proposition 5.5 in [11] also exhibits a sequence of trigonometric polynomials $\{P_n\}_{n=1}^{\infty}$ on G such that $\mathrm{supp}(\widehat{P}_n) = \Gamma_n$ and satisfying $P_n * V_n = P_n$, for $n \in \mathbb{N}$, with $\widehat{P}_n(\gamma) \in \{-1, 1\}$ for each $\gamma \in \Gamma_n$. In particular, $|\widehat{P}_n| = |\widehat{V}_n| \cdot |\widehat{P}_n| = \chi_{\Gamma_n}$. Since $|\!|\!|\cdot|\!|\!|_{S_p}$ is a lattice norm we have

$$\Big|\!\Big|\!\Big| \sum_{\gamma\in\Gamma_n} (\cdot,\gamma) \Big|\!\Big|\!\Big|_{S_p} = |\!|\!|P_n|\!|\!|_{S_p} \geq A_p 2^{n/p^*} = A_p |\Gamma_n|^{1/p^*}, \qquad n \in \mathbb{N}.$$

For this inequality we refer to (5.16) on p. 878 of [11], again noting that the argument applies for all $p > 2$. This is precisely condition (III).

Case (ii). *Every element of Γ has finite order.* In this setting there exists a sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of finite subgroups of Γ satisfying $\{0\} \subsetneqq \Gamma_1 \subsetneqq \Gamma_2 \subsetneqq \ldots$. Accordingly, $|\Gamma_{n+1}|/|\Gamma_n| \geq 2$, for $n \in \mathbb{N}$, and so $\sup_{n\in\mathbb{N}} |\Gamma_n| = \infty$, i.e., condition (I) holds. The trigonometric polynomials $V_n := \sum_{\gamma\in\Gamma_n} (\cdot,\gamma)$, for $n \in \mathbb{N}$, on G clearly satisfy condition (II). Moreover, $V_n = |\Gamma_n| \chi_{G_n}$ (with $G_n := \{x \in G : (x,\gamma) = 1, \forall\, \gamma \in \Gamma_n\}$) and $\mu(G_n) = 1/|\Gamma_n|$, for all $n \in \mathbb{N}$; see Lemma 3.16 and its proof in [11]. Consequently, for every $2 < p < \infty$, we have

$$\|V_n\|_{p^*} = |\Gamma_n|^{1/p}, \qquad n \in \mathbb{N};$$

i.e., condition (IV) holds. Moreover, since $|\!|\!|\cdot|\!|\!|_{S_p}$ is a lattice norm, it follows from (9), for every $2 < p < \infty$, that

$$\Big|\!\Big|\!\Big| \sum_{\gamma\in\Gamma_n} (\cdot,\gamma) \Big|\!\Big|\!\Big|_{S_p} = |\!|\!|V_n|\!|\!|_{S_p} \geq \|V_n\|_{S_p} \geq \|V_n\|_p = |\Gamma_n|^{1/p^*}, \qquad n \in \mathbb{N},$$

which yields condition (III).

Proof of Theorem 1. (a) Fix $p>2$. Let $\{\Gamma_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ be sequences satisfying conditions (I) – (IV) above. For each $n \in \mathbb{N}$, apply Lemma 2 with $E := \Gamma_n$ to obtain a trigonometric polynomial P_n on G satisfying (4), with $\operatorname{supp}(\widehat{P}_n) = \Gamma_n$ and such that $\widehat{P}_n(\gamma) \in \{-1,1\}$ for each $\gamma \in \Gamma_n$. According to Lemma 5 we have, for each $1<r<2$, that

$$\big\|C_{P_n}^{(r,p)}\big\|_{\mathrm{op}} \le \alpha_p |\Gamma_n|^s, \qquad \text{with } s = s(r) = \frac{1}{2} - \frac{2}{p} + \frac{2}{rp}. \tag{10}$$

As $\|\!|\cdot|\!\|_{S_p}$ is a lattice norm, condition (III) yields

$$\|\!|P_n|\!\|_{S_p} = \Big\|\!\Big|\sum_{\gamma\in\Gamma_n} (\cdot,\gamma)\Big|\!\Big\|_{S_p} \ge A_p |\Gamma_n|^{1/p^*}, \qquad n \in \mathbb{N}. \tag{11}$$

Condition (II) implies that $P_n = V_n * P_n$, that is, $C_{P_n}^{(r,p)}(V_n) := P_n * V_n = P_n$, for $n \in \mathbb{N}$. Via (10), Lemma 3 (with $T := C_{P_n}^{(r,p)}$ and $f := V_n$) and condition (IV) we have, for all $n \in \mathbb{N}$ and $1 \le r < p^*$, that

$$\|P_n\|_{\Pi_1^{(p)}} \le \gamma_{p,r} \|C_{P_n}^{(r,p)}\|_{\mathrm{op}} \|V_n\|_{p^*} \le \gamma_{p,r} B_p \alpha_p |\Gamma_n|^{s(r)+1/p}.$$

Then (11) implies, for a constant $c_{p,r}>0$, that

$$\frac{\|P_n\|_{\Pi_1^{(p)}}}{\|\!|P_n|\!\|_{S_p}} \le c_{p,r} |\Gamma_n|^{s(r)+1/p-1/p^*}, \qquad 1 \le r < p^*, \quad n \in \mathbb{N}. \tag{12}$$

But, $\lim_{r\uparrow p^*}(s(r) + \frac{1}{p} - \frac{1}{p^*}) = (\frac{2}{pp^*} - \frac{1}{2}) < 0$. Accordingly, there is $r_0 \in [1,p^*)$ such that $(s(r_0) + \frac{1}{p} - \frac{1}{p^*}) < 0$. Substituting $r := r_0$ into (12) we see for the sequence $\{P_n\}_{n=1}^{\infty}$ that $\lim_{n\to\infty} \|P_n\|_{\Pi_1^{(p)}} / \|\!|P_n|\!\|_{S_p} = 0$. It follows that there is no constant $M>0$ satisfying

$$M\|h\|_{\Pi_1^{(p)}} \ge \|\!|h|\!\|_{S_p}, \qquad h \in \Sigma_1^{(p)}, \tag{13}$$

that is, $\Sigma_1^{(p)} \nsubseteq S_p$. The proof of Theorem 1(a) is thereby complete.

(b) and (c). These follow as in the proof of Corollary 5.6, respectively Corollary 5.7, in [11], where the use of [11, Proposition 5.5] there is now replaced with part (a) of Theorem 1. □

A simpler proof, independent of Lemma 3 and Lemma 5, is available whenever $4<p<\infty$. Indeed, if Γ has an element of infinite order, then there exist sequences of trigonometric polynomials $\{P_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ on G such that $P_n * V_n = P_n$, with $\|P_n * V_n\|_{\Pi_1^{(p)}} \le \|P_n\|_p \|V_n\|_{p^*}$ for every $n \in \mathbb{N}$ and $1 \le p < \infty$, [11, Proposition 5.4], and satisfying

$$\frac{\|P_n * V_n\|_{\Pi_1^{(p)}}}{\|\!|P_n * V_n|\!\|_{S_p}} \le \frac{\|P_n\|_p \|V_n\|_{p^*}}{\|\!|P_n * V_n|\!\|_{S_p}} \le F_p \cdot (2^n)^{\kappa(p)}, \qquad n \in \mathbb{N}, \tag{14}$$

for every $2<p<\infty$ and some constant $F_p>0$, where $\kappa(p) := \frac{1}{p} + \frac{1}{2} - \frac{1}{p^*}$, [11, pp. 877-878].

On the other hand, if every element of Γ has finite order, then with $\{V_n\}_{n=1}^{\infty}$ and $\{\Gamma_n\}_{n=1}^{\infty}$ as in Case (ii) above we apply Lemma 2 to $E := \Gamma_n$ to obtain a trigonometric polynomial $P_n = \sum_{\gamma\in\Gamma_n} \sigma_{\gamma,n} \cdot (\cdot,\gamma)$ on G, with each $\sigma_{\gamma,n} \in \{-1,1\}$, such that (4) holds, for every $1<p<\infty$ and $n \in \mathbb{N}$. Because of the identities

$P_n = V_n * P_n$ and $|\widehat{P}_n| = |\widehat{V}_n| = \chi_{\Gamma_n}$, for $n \in \mathbb{N}$, the inequalities (9) and that $|\!|\!| \cdot |\!|\!|_{S_p}$ is a lattice norm, we have

$$\begin{aligned} \|V_n * P_n\|_{S_p} &\geq H_p |\!|\!| V_n * P_n |\!|\!|_{S_p} = H_p |\!|\!| P_n |\!|\!|_{S_p} = H_p |\!|\!| V_n |\!|\!|_{S_p} \\ &\geq H_p \|V_n\|_{S_p} \geq H_p \|V_n\|_p = H_p |\Gamma_n|^{1/p^*}, \qquad n \in \mathbb{N}. \end{aligned} \tag{15}$$

On the other hand, via (4), the identities $\|V_n\|_{p^*} = |\Gamma_n|^{1/p}$ (see Case (ii) above) and [11, Proposition 5.4], we have, for every $p > 2$, that

$$\|V_n * P_n\|_{\Pi_1^{(p)}} \leq \|P_n\|_p \|V_n\|_{p^*} \leq \alpha_p |\Gamma_n|^{1/2+1/p}, \qquad n \in \mathbb{N}. \tag{16}$$

It follows from (15) and (16) that

$$\frac{\|P_n * V_n\|_{\Pi_1^{(p)}}}{\|P_n * V_n\|_{S_p}} \leq \frac{\|P_n\|_p \|V_n\|_{p^*}}{\|P_n * V_n\|_{S_p}} \leq \frac{\alpha_p}{H_p} \, |\Gamma_n|^{\kappa(p)}, \qquad n \in \mathbb{N}, \tag{17}$$

for every $2 < p < \infty$, with $\sup_{n\in\mathbb{N}} |\Gamma_n| = \infty$. So, the strategy is to establish estimates (14) and (17), which then show that (13) cannot hold for any $M > 0$ (i.e., $\Sigma_1^{(p)} \nsubseteq S_p$) whenever $\kappa(p) < 0$, that is, precisely when $4 < p < \infty$. Unfortunately, this more direct approach *cannot* work for the missing indices $2 < p \leq 4$, for which it suffices to establish, with continuous inclusions, that

$$L^p * L^{p^*} \subseteq \Sigma_1^{(1,p)} \subseteq S_p, \qquad 2 < p \leq 4.$$

The latter inclusion $\Sigma_1^{(1,p)} \subseteq S_p$ was already observed to hold, even for $2 < p < \infty$; see (2). So, we only need to establish the following result, which is of independent interest.

Proposition 6. *Let $p > 2$. Then $L^p * L^{p^*} \subseteq \Sigma_1^{(1,p)}$ if and only if $p \leq 4$, in which case, with κ_G denoting Grothendieck's constant, we have*

$$\|g * h\|_{\Pi_1^{(1,p)}} \leq \kappa_G \, \|g\|_p \, \|h\|_{p^*}, \qquad g \in L^p, \ h \in L^{p^*}.$$

Proof. Suppose that $p \in (2, 4]$. Fix functions $g \in L^p \subseteq L^2$ and $h \in L^{p^*}$. Then the convolution operator $C_g^{(1,2)} : f \mapsto f * g$ is bounded from L^1 to L^2 with norm $\|C_g^{(1,2)}\|_{\mathrm{op}} \leq \|g\|_2$. Moreover, $C_g^{(1,2)}$ is then also 1-summing with $\|C_g^{(1,2)}\|_{\Pi_1^{(1,2)}} \leq \kappa_G \|C_g^{(1,2)}\|_{\mathrm{op}}$, [4, Theorem 3.4]. So,

$$\|C_g^{(1,2)}\|_{\Pi_1^{(1,2)}} \leq \kappa_G \, \|g\|_2 \leq \kappa_G \, \|g\|_p, \tag{18}$$

where the last inequality holds because $p > 2$. Next we show that

$$C_h^{(2,p)} \in \mathcal{L}(L^2, L^p) \text{ with } \|C_h^{(2,p)}\|_{\mathrm{op}} \leq \|h\|_{p^*}. \tag{19}$$

To this effect, it follows from the Hausdorff-Young inequality applied in L^{p^*} that $\widehat{h} \in \ell^p(\Gamma)$ with $\|\widehat{h}\|_{\ell^p(\Gamma)} \leq \|h\|_{p^*}$. Consequently, for every $f \in L^2$ (in which case $\widehat{f} \in \ell^2(\Gamma)$) it follows from the generalized Hölder inequality that $(f * h)\widehat{\ } = \widehat{f} \cdot \widehat{h} \in \ell^r(\Gamma)$ with $\|\widehat{h} \cdot \widehat{f}\|_{\ell^r(\Gamma)} \leq \|\widehat{h}\|_{\ell^p(\Gamma)} \|\widehat{f}\|_{\ell^2(\Gamma)}$, where $\frac{1}{r} = \frac{1}{2} + \frac{1}{p}$. The previous two inequalities and Plancherel's Theorem then yield

$$\|(f * h)\widehat{\ }\|_{\ell^r(\Gamma)} \leq \|h\|_{p^*} \|f\|_2. \tag{20}$$

Moreover, $p \leq 4$ implies that $\frac{1}{r} \geq (1 - \frac{1}{p}) = \frac{1}{p^*}$ (i.e., $r \leq p^* < 2$), and so $\ell^r(\Gamma) \subseteq \ell^{p^*}(\Gamma)$ with

$$\|\xi\|_{\ell^{p^*}(\Gamma)} \leq \|\xi\|_{\ell^r(\Gamma)}, \qquad \xi \in \ell^r(\Gamma). \tag{21}$$

Accordingly, $(f*h)^\frown \in \ell^{p^*}(\Gamma)$, and so we can again apply the Hausdorff-Young inequality (in $\ell^{p^*}(\Gamma)$ with G being the dual group of Γ) to conclude that $\|f*h\|_p = \|((f*h)^\frown)^\smile\|_p \le \|(f*h)^\frown\|_{\ell^{p^*}(\Gamma)}$. This inequality, combined with (20) and (21), yields

$$\|C_h^{(2,p)}(f)\|_p = \|f*h\|_p \le \|h\|_{p^*}\|f\|_2, \qquad f \in L^2,$$

from which (19) is immediate. Since $C_{g*h}^{(1,p)} = C_h^{(2,p)} \circ C_g^{(1,2)}$, the ideal property of 1-summing operators, [4, p. 37], together with (18) and (19) yields the desired inequality

$$\|g*h\|_{\Pi_1^{(1,p)}} := \|C_{g*h}^{(1,p)}\|_{\Pi_1^{(1,p)}} \le \|C_g^{(1,2)}\|_{\Pi_1^{(1,2)}}\|C_h^{(2,p)}\|_{\mathrm{op}} \le \kappa_G\,\|g\|_p\|h\|_{p^*}.$$

Conversely, for some $p>2$, suppose that $L^p*L^{p^*} \subseteq \Sigma_1^{(1,p)}$. Hence, also $L^p*L^{p^*} \subseteq S_p$; see (2). Fix $g \in L^{p^*}$. Then the linear operator $C_g : f \mapsto g*f$ maps L^p into S_p. Let $f_n \to 0$ in L^p and $C_g f_n \to h$ in S_p. Then also $C_g f_n \to h$ in L^p as $S_p \subseteq L^p$ continuously. Moreover, $f_n \to 0$ in L^p implies that $C_g f_n \to 0$ in L^p (as $C_g^{(p)} \in \mathcal{L}(L^p)$), and so we can conclude that $h=0$ in L^p and hence also in S_p. Then the Closed Graph Theorem ensures that $C_g \in \mathcal{L}(L^p, S_p)$, that is,

$$\|f*g\|_{S_p} \le \|C_g\|_{\mathrm{op}}\|f\|_p, \qquad f \in L^p. \tag{22}$$

Consider now the linear map $g \mapsto C_g$ from L^{p^*} into the Banach space $\mathcal{L}(L^p, S_p)$. Let $g_n \to 0$ in L^{p^*} and $C_{g_n} \to T$ in $\mathcal{L}(L^p, S_p)$. Fix $f \in L^p$. Then $f*g_n \to 0$ in L^{p^*} and hence also in L^1. Moreover, as $f*g_n \to Tf$ in S_p and $S_p \subseteq L^p \subseteq L^1$, with continuous inclusions, it follows that $f*g_n \to Tf$ in L^1. Accordingly, $Tf=0$ in L^1 and hence also in S_p. Since $f \in L^p$ is arbitrary, it follows that $T=0$. Again by the Closed Graph Theorem there is $K>0$ such that

$$\|C_g\|_{\mathrm{op}} \le K\|g\|_{p^*}, \qquad g \in L^{p^*}.$$

It is then clear from (22) and the previous inequality that

$$\|f*g\|_{S_p} \le K\|f\|_p\|g\|_{p^*}, \qquad f \in L^p,\ g \in L^{p^*}. \tag{23}$$

According to (14) and (17) there exists an increasing sequence $\{\Gamma_n\}_{n=1}^\infty$ of finite subsets of Γ and sequences of trigonometric polynomials $\{P_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ on G such that

$$\frac{\|V_n*P_n\|_{S_p}}{\|V_n\|_p\|P_n\|_{p^*}} \ge M_p|\Gamma_n|^{-\kappa(p)}, \qquad n \in \mathbb{N},$$

for some constant $M_p>0$ and with $\sup_{n\in\mathbb{N}}|\Gamma_n| = \infty$, where we recall that $\kappa(p) = \frac{1}{p}+\frac{1}{2}-\frac{1}{p^*}$. In view of (23) this is only possible if $\kappa(p) \ge 0$, that is, if $p \le 4$. □

Remark 7. (i) Proposition 5.4 of [11] states that $L^p*L^{p^*} \subseteq \Sigma_1^{(p)}$, for every $1 \le p < \infty$. Also, $\Sigma_1^{(1,p)} \subseteq \Sigma_1^{(p)}$ (continuously) for every $1 \le p < \infty$; see (1) and (3). Proposition 6 above asserts, for $p>2$, that $L^p*L^{p^*}$ is contained in the *smaller* space $\Sigma^{(1,p)}$ if and only if $p \le 4$. Of course, if $1 \le p \le 2$, then $L^p*L^{p^*} \subseteq L^2$, and so, via (1), we also have that $L^p*L^{p^*} \subseteq \Sigma_1^{(1,p)}$.

(ii) Observe that the proof of Proposition 6 actually yields $L^2*L^{p^*} \subseteq \Sigma_1^{(1,p)}$ for all $2<p\le 4$.

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