ABSOlutely SummIng MUltiplier OPerators
in $L^p(G)$ for $p > 2$

Werner J. Ricker and Luis Rodríguez-Piazza

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Abstract. Let $G$ be an infinite compact abelian group. If its dual group $\Gamma$ contains an element of infinite order, then it is known that, for every $4 < p < \infty$, there exists a function $g \in L^p(G)$ whose associated convolution operator $C_g : f \mapsto f * g$ (on $L^p(G)$) is absolutely summing but the Fourier series of $g$ fails to be unconditionally convergent to $g$ in $L^p(G)$. It is shown that the restriction on $\Gamma$ containing an element of infinite order can be removed and also that the range of $p$ can be extended to arbitrary $p \in (2, \infty)$.

The idea of absolutely summing (also called 1-summing) operators is classical and has applications to many branches of mathematics; see e.g. [3], [4], [9], [13], [14]. Perhaps even more important are the $p$-multiplier operators arising in harmonic analysis; see [6], [7], [8], [10] and the references therein. Certain properties concerning operators which belong to both of these classes are exposed in [4], [2]. In the recent article [11] additional properties were identified which we briefly summarize. First some notation.

Let $G$ be an infinite compact abelian group with normalized Haar measure $\mu$ and dual group $\Gamma$. For $1 \leq p \leq \infty$ let $L^p(G)$ be the usual Banach space of functions on $G$ which are $p$-th power integrable. Since $G$ is always fixed (but arbitrary) we write $L^p(G)$ simply as $L^p$. Each $g \in L^1$ induces a bounded convolution operator $C^p_g : L^p \to L^p$ via $f \mapsto f * g$, for $f \in L^p$. Given $1 \leq p, q \leq \infty$, let $\Pi^{(1,p)}$ denote the Banach space of those bounded linear operators $T \in \mathcal{L}(L^q, L^p)$ which are absolutely summing from $L^q$ into $L^p$ and equipped with the usual 1-summing norm, which we denote by $\| \cdot \|_{\Pi^{(1,p)}}$, [1] p. 31. Those operators in $\mathcal{L}(L^q, L^p)$ which commute with translations form the space $\mathcal{M}^{(q,p)}$ of all $(q,p)$-multiplier operators. For $p = q$ we simply write $\Pi^{(p)}_1$ and $\mathcal{M}^{(p)}$. Set $\mathcal{M}^{(p)}_{\Pi_1} := \Pi^{(p)}_1 \cap \mathcal{M}^{(p)}$ and $\mathcal{M}^{(1,p)}_{\Pi_1} := \Pi^{(1,p)}_1 \cap \mathcal{M}^{(1,p)}$. Every operator in $\mathcal{M}^{(p)}_{\Pi_1}$ is of the form $C_g^{(p)}$ and every operator in $\mathcal{M}^{(1,p)}_{\Pi_1}$ is a convolution operator of the form $C_h^{(1,p)} : L^1 \to L^p$, for appropriate functions $g, h \in L^p$, respectively, [2] Ch.II, Propositions 7 & 8. For $1 \leq p \leq \infty$, define the Banach spaces

$\Sigma^{(p)}_1 := \{ g \in L^p : C_g^{(p)} \in \mathcal{M}^{(p)}_{\Pi_1} \}$ and $\Sigma^{(1,p)}_1 := \{ h \in L^p : C_h^{(1,p)} \in \mathcal{M}^{(1,p)}_{\Pi_1} \}$

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equipped with the norm
\[ \|g\|_{\Pi_1(p)} := \|C_g\|_{\Pi_1(p)} \quad \text{and} \quad \|h\|_{\Pi_1(1,p)} := \|C_h\|_{\Pi_1(1,p)}, \]
respectively. Finally, let \( S_p \) denote the linear space of all \( f \in L^p \) whose Fourier series \( \sum_{\gamma \in \Gamma} \hat{f}(\gamma)(\cdot, \gamma) \) is unconditionally convergent to \( f \) in \( L^p \), where \((\cdot, \gamma)\) denotes the character \( x \mapsto \gamma(x) \in \mathbb{C}, \) for \( x \in G \). Then \( S_p \) is a Banach space for the norm
\[ \|f\|_{S_p} := \sup_{F \in \mathcal{F}} \left\| \sum_{\gamma \in F} \hat{f}(\gamma)(\cdot, \gamma) \right\|_p, \]
where \( \mathcal{F} \) is the collection of all finite subsets of \( \Gamma \).

It is known that
\[ (1) \quad \Sigma_1(1,p) = \Sigma_1(p) = S_p = L^2, \quad 1 \leq p \leq 2; \]
see the Introduction in [11] where all the relevant references are provided. The point of departure in [11] is an investigation of the case when \( 2 < p < \infty \). It is proved there, with continuous inclusions, that
\[ (2) \quad \Sigma_1(1,p) \subseteq S_p \subseteq L^p, \quad 2 < p < \infty, \]
[11] Proposition 3.3(ii) and Corollary 4.6, and also with continuous inclusions that
\[ (3) \quad \Sigma_1(1,p) \subseteq \Sigma_1(p) \subseteq L^p, \quad 2 < p < \infty, \]
[11] Proposition 3.3(i)]. Moreover, one always has the containment
\[ \Sigma_1(p) \subseteq L^p, \quad 2 < p < \infty, \]
[11] Proposition 5.8]. However, obtaining results for the inclusion \( \Sigma_1(1,p) \subseteq \Sigma_1(p) \) to be proper or when \( \Sigma_1(p) \not\subseteq S_p \), for \( p > 2 \), turns out to be more elusive. The role of the Banach space \( C(G) \), consisting of all the continuous functions on \( G \), in relation to the spaces \( S_p \) and \( \Sigma_1(1,p) \) is also unclear. Under a certain restriction on the dual group \( \Gamma \) and on the index \( p > 2 \) we have the following known

**Facts.** Let \( G \) be an infinite compact abelian group such that \( \Gamma \) has an element of infinite order.

(i) For every \( 2 < p < \infty \) we have \( C(G) \not\subseteq S_p \). In particular, also \( C(G) \not\subseteq \Sigma_1(1,p) \).

(ii) For every \( 4 < p < \infty \) we have \( \Sigma_1(p) \not\subseteq S_p \).

(iii) For every \( 4 < p < \infty \) we have \( \Sigma_1(1,p) \not\subseteq \Sigma_1(p) \).

(iv) For \( G \) metrizable and every \( 4 < p < \infty \) the set of characters \( \{ (\cdot, \gamma) : \gamma \in \Gamma \} \)
does not form an unconditional basis for the Banach space \( \Sigma_1(p) \).

Fact (i) occurs in [11] Corollary 4.10] and Fact (ii) is precisely Proposition 5.5 of [11]. Facts (iii) and (iv) are then consequences of Fact (ii); see Corollaries 5.6 and 5.7 in [11], respectively.

In the recent article [12] it was observed (cf. Proposition 3.15 there) that the existence of an element of infinite order in \( \Gamma \) is not necessary for the validity of Fact (i) above. Indeed, for an arbitrary infinite compact abelian group \( G \), we even have \( C(G) \not\subseteq \bigcup_{2<p<\infty} S_p \).
The aim of this paper is to remove completely the restriction imposed on \( \Gamma \) in Facts (i) – (iv) above and to extend the range of \( p \) to include all \( 2 < p < \infty \), that is, to establish the following result.

**Theorem 1.** For every infinite compact abelian group \( G \) and every \( 2 < p < \infty \) we have

(a) \( \Sigma_1^{(p)} \not\subseteq S_p \),
(b) \( \Sigma_1^{(1,p)} \subsetneq \Sigma_1^{(p)} \), and
(c) \( \{\langle \cdot, \gamma \rangle : \gamma \in \Gamma \} \) is not an unconditional basis for \( \Sigma_1^{(p)} \) whenever \( G \) is metrizable.

In order to establish Theorem 1, we will require some preliminary results. Recall that the sequence of Rademacher functions \( r_j : [0, 1] \to \{-1, 1\} \), for \( j \in \mathbb{N} \), is given by \( r_j(t) := \text{sgn}(\sin 2^j \pi t) \), for \( t \in [0, 1] \). The cardinality of a finite set \( E \) is denoted by \( |E| \). Let \( p^* \) denote the conjugate index to \( p \), i.e., \( \frac{1}{p} + \frac{1}{p^*} = 1 \). The norm of a bounded linear operator \( T \) between Banach spaces is denoted by \( \|T\|_{\text{op}} \).

**Lemma 2.** Let \( G \) be an infinite compact abelian group. For every \( p \in (2, \infty) \) there exists a constant \( \alpha_p > 0 \) with the property that, for every finite subset \( E \subseteq \Gamma \), there is a choice of signs \( \{\sigma_{\gamma}\}_{\gamma \in E} \), with each \( \sigma_{\gamma} \in \{-1, 1\} \), such that

\[
(4) \quad \left\| \sum_{\gamma \in E} \sigma_{\gamma} \cdot \langle \cdot, \gamma \rangle \right\|_{L^p} \leq \alpha_p |E|^{1/2}.
\]

**Proof.** Let \( E = \{\gamma_1, \ldots, \gamma_n\} \), where \( n = |E| \). In order to estimate

\[
\int_0^1 \left| \sum_{j=1}^n r_j(t) \cdot \langle \cdot, \gamma_j \rangle \right|^p_{L^p(G)} dt = \int_0^1 \int_G \left| \sum_{j=1}^n r_j(t) \cdot \langle x, \gamma_j \rangle \right|^p d\mu(x) dt
\]

we interchange the order of integration and use Khinchin's inequality, [4] p. 10, to ensure the existence of \( \alpha_p > 0 \) such that

\[
\int_0^1 \left| \sum_{j=1}^n r_j(t) \cdot \langle x, \gamma_j \rangle \right|^p_{L^p(G)} dt = \int_G \int_0^1 \left| \sum_{j=1}^n r_j(t) \cdot \langle x, \gamma_j \rangle \right|^p dt d\mu(x)
\]

\[
\leq \int_G \alpha_p^n \left( \sum_{j=1}^n |\langle x, \gamma_j \rangle|^2 \right)^{p/2} d\mu(x) = \int_G \alpha_p^n |x|^{p/2} d\mu(x) = \alpha_p^n n^{p/2}.
\]

Thus, there exists \( t_0 \in [0, 1] \) such that \( \left\| \sum_{j=1}^n r_j(t_0) \cdot \langle \cdot, \gamma_j \rangle \right\|_{L^p(G)} \leq \alpha_p n^{1/2} \). Setting \( \sigma_{\gamma_j} := r_j(t_0) \), for \( 1 \leq j \leq n \), completes the proof. \( \square \)

**Lemma 3.** Let \( 2 < p < \infty \) and \( 1 \leq r < p^* \). Then, for every function \( f \in L^p \subseteq L^r \) and every multiplier operator \( T \in \mathcal{M}^{(r,p)} \), the function \( Tf \) belongs to \( \Sigma_1^{(p)} \subseteq L^p \). Moreover, there exists a constant \( \gamma_{p,r} > 0 \), depending only on \( p \) and \( r \), such that

\[
(5) \quad \|Tf\|_{\Sigma_1^{(p)}} \leq \gamma_{p,r} \|f\|_{L^{p^*}} \|T\|_{\text{op}}, \quad f \in L^{p^*}, \; T \in \mathcal{M}^{(r,p)}.
\]

**Proof.** Let \( f \in L^{p^*} \) and set \( h := Tf \in L^p \). An application of Hölder’s inequality shows that \( L^p \ast L^{p^*} \subseteq L^\infty \) continuously, i.e., \( C_{\infty}^{(p, \infty)} \in \mathcal{L}(L^p, L^\infty) \). So, if \( J^{(\infty,r)} : L^\infty \to L^r \) is the natural inclusion, then we have the factorization of \( C_{h}^{(p)} \in \mathcal{L}(L^p) \) given by

\[
(6) \quad C_{h}^{(p)} = T \circ J^{(\infty,r)} \circ C_{f}^{(p, \infty)},
\]
after recalling that \( T \) commutes with all convolution operators. Accordingly, \( C^h(p) \) is \( r \)-summing since \( J^{(\infty,r)} \) is \( r \)-summing (with \( \|J^{(\infty,r)}\|_{\Pi_1} = 1 \), [4, p. 40]. But, \( p > 2 \) implies that \( L^p(G) \) has cotype \( p \), [4, Theorem 11.6]. So, \( r < p^* \), we can apply [4, Corollary 11.16] to conclude that \( \Pi_1(p) = \Pi_1(p) \). Hence, \( C^h(p) \) is \( 1 \)-summing, i.e., \( Tf = h \in \Sigma_1(p) \).

Since \( \Pi_1(p) = \Pi_1(p) \), there is a constant \( \gamma_{p,r} \) (depending only on the cotype \( p \) constant of \( L^p(G) \) on \( r \)) such that

\[
\|h\|_{\Pi_1(p)} := \|C^h(p)\|_{\Pi_1(p)} \leq \gamma_{p,r} \|C^h(p)\|_{\Pi_1(p)} := \gamma_{p,r} \|h\|_{\Pi_1(p)}.
\]

Thanks to this inequality and (6) we obtain (5) because the \( r \)-summing norm \( \|J^{(\infty,r)}\|_{\Pi_1} = 1 \) and \( \|C^f(p,\infty)\|_{\text{op}} = \|f\|_{\text{op}} \).

**Remark 4.** (i) For \( r = 1 \) we observe that Lemma 3 reduces to Proposition 5.4 of [11] whenever \( p > 2 \). Indeed, for every \( p > 1 \) we have \( T \in M^{(1,p)} \) if and only if there exists \( g \in L^p \) such that \( T = C^g_{1,p} \), [10, Theorem 3.1.1], and so Lemma 3 yields \( L^p * L^p \subseteq \Sigma_1(p) \).

(ii) The statement of Lemma 3 is also valid for every \( 1 \leq p \leq 2 \), without any restriction on \( r \). That is, if \( 1 \leq r \leq \infty \), \( p \in [1,2] \), \( T \in M^{(r,p)} \) and \( f \in L^p \), then \( Tf \in \Sigma_1(p) \). Indeed, \( p^* \geq 2 \) implies that \( f \in L^2 \). Moreover, \( Tf \in L^2 \) because \( M^{(1,p)} \subseteq M^{(2,2)} \). The conclusion then follows from (1).

**Lemma 5.** Let \( 2 < p < \infty \), let \( E \) be a finite subset of \( \Gamma \) and let \( \varphi = \sum_{\gamma \in E} \sigma_{\gamma} \cdot (\cdot, \gamma) \), with each \( \sigma_{\gamma} \in \{-1,1\} \), be any trigonometric polynomial on \( G \) satisfying (1). Then, for every \( r \in (1,2) \), we have

\[
\|C_{\varphi}(r,p)\|_{\text{op}} \leq \alpha_p |E|^s, \quad \text{with } s := \frac{1}{2} - \frac{2}{p} + \frac{2}{rp}.
\]

**Proof.** For \( r = 1 \) it follows from (1) that

\[
\|C^{(1,p)}_{\varphi}\|_{\text{op}} \leq \|\varphi\|_p \leq \alpha_p |E|^{1/2}.
\]

On the other hand, \( (\frac{1}{p} - \frac{1}{2}) > 0 \). Define \( t \) via \( \frac{1}{t} = \frac{1}{2} + \frac{1}{p} \), i.e., \( \frac{1}{t} = \frac{1}{p} - \frac{1}{2} \). For every \( h \in L^2 \), it follows from the generalized Hölder inequality, [5, p. 527], that

\[
\|\hat{\varphi} \cdot \hat{h}\|_{p^*} \leq \|\hat{\varphi}\|_2 \|\hat{h}\|_r \leq \|\hat{\varphi}\|_2 \|\hat{h}\|_r = \|\hat{h}\|_r |E|^{1/t} = \|\hat{h}\|_r |E|^{1/p^* - 1/2}.
\]

Accordingly, the Hausdorff-Young inequality, [8, Theorem (31.10) and p. 227], applied in \( p^\ast \) (\( \Gamma \)) with \( G \) being the dual group of \( \Gamma \) yields

\[
\|h * \varphi\|_p \leq \|(h * \varphi)^\ast\|_{p^\ast} \leq \|\hat{h} \cdot \hat{\varphi}\|_{p^\ast} \leq \|\hat{h}\|_r |E|^{1/t} = \|h\|_{L^2(G)} |E|^{1/p^* - 1/2},
\]

and so we have

\[
\|C_{\varphi}^{(2,p)}\|_{\text{op}} \leq |E|^{1/p^* - 1/2}.
\]

By the Riesz-Thorin interpolation theorem, [7, p. 149], with \( 1 < r < 2 \) fixed, (7) and (8) yield that

\[
\|C_{\varphi}^{(r,p)}\|_{\text{op}} \leq \left( \alpha_p |E|^{1/2} \right)^{1 - \theta} \left( |E|^{1/p^* - 1/2} \right)^{\theta} = \alpha_p^{1 - \theta} |E|^s,
\]

where \( \theta \in [0,1] \) is determined by \( \frac{1}{r} = \theta \cdot \frac{1}{p} + (1-\theta) \cdot \frac{1}{2} \) (i.e., \( \theta = 2 - \frac{3}{2} \)) and \( s := \theta \left( \frac{1}{p} - \frac{1}{2} \right) + (1 - \theta) \cdot \frac{1}{2} \) (i.e., \( s = \frac{1}{2} - \frac{2}{p} = \frac{1}{2} - \frac{2}{p} + \frac{2}{rp} \)). This completes the proof since \( \alpha_p^{1 - \theta} \leq \alpha_p \).
The trigonometric polynomials

For this inequality we refer to (5.16) on p. 878 of [11], again noting that the argument of the following two cases.

We recall, for every $p > 2$, that $(S_p, \| \cdot \|_{S_p})$ is a (complex) Banach lattice for an equivalent lattice norm $\| \cdot \|_{S_p}$, where the positive cone is determined by $g \geq 0$ in $S_p$ if and only if $g \geq 0$ pointwise on $\Gamma$, [11], p. 864]. Moreover, via Propositions 3.3(ii) and 4.2 of [11] we have, for some constant $K_p > 0$, that

$$
\|f\|_p \leq \|f\|_{S_p} \leq K_p \|f\|_{S_p}, \quad f \in S_p.
$$

The proof of Theorem 1 is based on the above lemmata and on the existence of an increasing sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of finite subsets of $\Gamma$ and a sequence $\{V_n\}_{n=1}^{\infty}$ of trigonometric polynomials on $G$ satisfying the following four conditions:

1. $\sup_{n \in \mathbb{N}} |\Gamma_n| = \infty$ and
2. $\hat{\chi}_{\Gamma_n} = \chi_{\Gamma_n}$, for $n \in \mathbb{N}$.

For each $2 < p < \infty$, there exist positive constants $A_p, B_p$ satisfying

3. $\|\sum_{\gamma \in \Gamma_n} (\cdot, \gamma)\|_{S_p} \geq A_p |\Gamma_n|^{1/p^*}$, for $n \in \mathbb{N}$, and
4. $\|V_n\|_{p^*} \leq B_p |\Gamma_n|^{1/p}$, for $n \in \mathbb{N}$.

The existence of such sequences $\{\Gamma_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ is established via a consideration of the following two cases.

**Case (i).** $\Gamma$ contains an element of infinite order. Let $\gamma_0 \in \Gamma$ have infinite order and $\{v_m\}_{m=0}^{\infty}$ be the de la Vallée Poussin kernel on the circle group $T$. Define trigonometric polynomials $V_n : G \to \mathbb{C}$ by $V_n := v_{2^n} \circ \gamma_0$, for $n \in \mathbb{N}$. Then $\Gamma_n := \{k\gamma_0 : 0 \leq k < 2^n, k \in \mathbb{Z}\}$, for $n \in \mathbb{N}$, surely satisfies condition (I). It is established in the proof of Proposition 5.5 in [11] that condition (II) holds and that

$$
\|V_n\|_{p^*} \leq \epsilon_p (2^{n+1} + 1)^{1/p} + \beta_p (2^n)^{1/p} \leq (\epsilon_p 4^{1/p} + \beta_p) 2^{n/p}, \quad n \in \mathbb{N},
$$

for positive constants $\epsilon_p, \beta_p$; the argument given there holds (up to this stage) for all $p > 2$ (not just for $p > 4$). Since $|\Gamma_n| = 2^n$, for $n \in \mathbb{N}$, the previous inequality yields condition (IV). The proof of Proposition 5.5 in [11] also exhibits a sequence of trigonometric polynomials $\{P_n\}_{n=1}^{\infty}$ on $G$ such that $\sup(\hat{P}_n) = |\Gamma_n|$ and satisfying $P_n \ast V_n = P_n$, for $n \in \mathbb{N}$, with $\hat{P}_n(\gamma) \in \{-1, 1\}$ for each $\gamma \in \Gamma_n$. In particular, $|\hat{P}_n| = |\hat{V}_n|$, $\hat{P}_n = \chi_{\Gamma_n}$. Since $\| \cdot \|_{S_p}$ is a lattice norm we have

$$
\left\| \sum_{\gamma \in \Gamma_n} (\cdot, \gamma) \right\|_{S_p} = \|P_n\|_{S_p} \geq A_p 2^{n/p^*} = A_p |\Gamma_n|^{1/p^*}, \quad n \in \mathbb{N}.
$$

For this inequality we refer to (5.16) on p. 878 of [11], again noting that the argument applies for all $p > 2$. This is precisely condition (III).

**Case (ii).** Every element of $\Gamma$ has finite order. In this setting there exists a sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of finite subsets of $\Gamma$ satisfying $\{0\} \subsetneq \Gamma_1 \subsetneq \Gamma_2 \subsetneq \ldots$. Accordingly, $|\Gamma_{n+1}|/|\Gamma_n| \geq 2$, for $n \in \mathbb{N}$, and so $\sup_{n \in \mathbb{N}} |\Gamma_n| = \infty$, i.e., condition (I) holds. The trigonometric polynomials $V_n := \sum_{\gamma \in \Gamma_n} (\cdot, \gamma)$, for $n \in \mathbb{N}$, on $G$ clearly satisfy condition (II). Moreover, $V_n = |\chi_{\Gamma_n}| (with \ G_n := \{x \in G : (x, \gamma) = 1, \forall \gamma \in \Gamma_n\}) and \mu(G_n) = 1/|\Gamma_n|, for all n \in \mathbb{N}$; see Lemma 3.16 and its proof in [11]. Consequently, for every $2 < p < \infty$, we have

$$
\|V_n\|_{p^*} = |\Gamma_n|^{1/p}, \quad n \in \mathbb{N};
$$

i.e., condition (IV) holds. Moreover, since $\| \cdot \|_{S_p}$ is a lattice norm, it follows from [12], for every $2 < p < \infty$, that

$$
\left\| \sum_{\gamma \in \Gamma_n} (\cdot, \gamma) \right\|_{S_p} \geq \|V_n\|_{S_p} \geq \|V_n\|_{p^*} = |\Gamma_n|^{1/p^*}, \quad n \in \mathbb{N},
$$

which yields condition (III).
Proof of Theorem 1. (a) Fix \( p > 2 \). Let \( \{\Gamma_n\}_{n=1}^{\infty} \) and \( \{V_n\}_{n=1}^{\infty} \) be sequences satisfying conditions (I) – (IV) above. For each \( n \in \mathbb{N} \), apply Lemma 2 with \( E := \Gamma_n \) to obtain a trigonometric polynomial \( P_n \) on \( G \) satisfying (13), with \( \text{supp}(\hat{P}_n) = \Gamma_n \) and such that \( \hat{P}_n(\gamma) \in \{-1, 1\} \) for each \( \gamma \in \Gamma_n \). According to Lemma 5 we have, for each \( 1 < r < 2 \), that
\[
\|C(r, p)\|_{op} \leq \alpha_p|\Gamma_n|^s, \quad \text{with } s = s(r) = \frac{1}{2} - \frac{2}{p} + \frac{2}{rp}.
\]
as \( \|\cdot\|_p \) is a lattice norm, condition (III) yields
\[
\|P_n\|_{S_p} = \| \sum_{\gamma \in \Gamma_n} (\cdot, \gamma) \|_{S_p} \geq A_p|\Gamma_n|^{1/p^*}, \quad n \in \mathbb{N}.
\]
Condition (II) implies that
\[
\|P_n\|_{S_p} = \|V_n\|_{S_p} \quad \text{for each } n \in \mathbb{N}. \tag{10}
\]
Via (10), Lemma 3 (with \( T := C(r, p) \) and \( f := V_n \)) and condition (IV) we have, for all \( n \in \mathbb{N} \) and \( 1 \leq r < p^* \), that
\[
\|P_n\|_{S_p} \leq \gamma_{p, r} \|C(r, p)\|_{op}\|V_n\|_{p^*} \leq \gamma_{p, r} B_p\alpha_p|\Gamma_n|^{s(r)+1/p^*}.
\]
Then (11) implies, for a constant \( c_{p, r} > 0 \), that
\[
\frac{\|P_n\|_{S_p}}{c_{p, r}|\Gamma_n|^{s(r)+1/p^*}} \leq 1 \leq r < p^*, \quad n \in \mathbb{N}. \tag{12}
\]
But, \( \lim_{r \to p^*} (s(r) + \frac{1}{p} - \frac{1}{p^*}) = \left(\frac{2}{rp^*} - \frac{1}{2}\right) < 0 \). Accordingly, there is \( r_0 \in [1, p^*) \) such that \( (s(r_0) + \frac{1}{p} - \frac{1}{p^*}) < 0 \). Substituting \( r := r_0 \) into (12) we see for the sequence \( \{P_n\}_{n=1}^{\infty} \) that \( \lim_{n \to \infty} \frac{\|P_n\|_{S_p}}{\|P_n\|_{S_p}} = 0 \). It follows that there is no constant \( M > 0 \) satisfying
\[
M\|h\|_{S_p} \geq \|h\|_{S_p}, \quad h \in \Sigma_1(p), \tag{13}
\]
that is, \( \Sigma_1(p) \not\subseteq S_p \). The proof of Theorem 1(a) is thereby complete.

(b) and (c). These follow as in the proof of Corollary 5.6, respectively Corollary 5.7, in [11], where the use of [11] Proposition 5.5] there is now replaced with part (a) of Theorem 1.

A simpler proof, independent of Lemma 3 and Lemma 5, is available whenever \( 4 < p < \infty \). Indeed, if \( \Gamma \) has an element of infinite order, then there exist sequences of trigonometric polynomials \( \{P_n\}_{n=1}^{\infty} \) and \( \{V_n\}_{n=1}^{\infty} \) on \( G \) such that \( P_n \ast V_n = P_n \), with \( \|P_n \ast V_n\|_{S_p} \leq \|P_n\|_{p}\|V_n\|_{p^*} \) for every \( n \in \mathbb{N} \) and \( 1 \leq p < \infty \), [11] Proposition 5.4], and satisfying
\[
\frac{\|P_n \ast V_n\|_{S_p}}{\|P_n \ast V_n\|_{S_p}} \leq \frac{\|P_n\|_{p}\|V_n\|_{p^*}}{\|P_n \ast V_n\|_{S_p}} \leq F_p \cdot (2^n)^{\kappa(p)}, \quad n \in \mathbb{N}, \tag{14}
\]
for every \( 2 < p < \infty \) and some constant \( F_p > 0 \), where \( \kappa(p) := \frac{1}{p} + \frac{1}{2} - \frac{1}{p^*} \), [11] pp. 877-878].

On the other hand, if every element of \( \Gamma \) has finite order, then with \( \{V_n\}_{n=1}^{\infty} \) and \( \{\Gamma_n\}_{n=1}^{\infty} \) as in Case (ii) above we apply Lemma 2 to \( E := \Gamma_n \) to obtain a trigonometric polynomial \( P_n = \sum_{\gamma \in \Gamma_n} \sigma_{\gamma, n} (\cdot, \gamma) \) on \( G \), with each \( \sigma_{\gamma, n} \in \{-1, 1\} \), such that (4) holds, for every \( 1 < p < \infty \) and \( n \in \mathbb{N} \). Because of the identities
\[ P_n = V_n * P_n \quad \text{and} \quad \hat{P}_n = \hat{V}_n = \chi_{G_n}, \quad \text{for } n \in \mathbb{N}, \text{ the inequalities (9) and that } \| \cdot \|_{S_p} \text{ is a lattice norm, we have} \]

\[
\begin{align*}
(15) \quad \|V_n * P_n\|_{S_p} & \geq H_p \|V_n * P_n\|_{S_p} = H_p \|P_n\|_{S_p} = H_p \|V_n\|_{S_p} \\
& \geq H_p \|V_n\|_{S_p} \geq H_p \|V_n\|_p = H_p |\Gamma_n|^{1/p'}, \quad n \in \mathbb{N}.
\end{align*}
\]

On the other hand, via (4), the identities \( \|V_n\|_{p'} = |\Gamma_n|^{1/p} \) (see Case (ii) above) and Proposition 5.4, we have, for every \( p > 2 \), that

\[
\begin{align*}
(16) \quad \|V_n * P_n\|_{\Pi_1^{(p)}} & \leq \|P_n\|_p \|V_n\|_{p'} \leq \alpha_p |\Gamma_n|^{1/2 + 1/p}, \quad n \in \mathbb{N}.
\end{align*}
\]

It follows from (15) and (16) that

\[
\begin{align*}
(17) \quad \frac{\|P_n * V_n\|_{\Pi_1^{(p)}}}{\|P_n * V_n\|_{S_p}} & \leq \frac{\|P_n\|_p \|V_n\|_{p'}}{\|P_n \|_{S_p}} \leq \frac{\alpha_p}{H_p} |\Gamma_n|^{\kappa(p)}, \quad n \in \mathbb{N},
\end{align*}
\]

for every \( 2 < p < \infty \), with \( \sup_{n \in \mathbb{N}} |\Gamma_n| = \infty \). So, the strategy is to establish estimates (14) and (17), which then show that (13) cannot hold for any \( M > 0 \) (i.e., \( \Sigma_1^{(p)} \not\subseteq S_p \)) whenever \( \kappa(p) < 0 \), that is, precisely when \( 4 < p < \infty \). Unfortunately, this more direct approach cannot work for the missing indices \( 2 < p \leq 4 \), for which it suffices to establish, with continuous inclusions, that

\[
L^p * L^{p'} \subseteq \Sigma_1^{(1,p)} \subseteq S_p, \quad 2 < p \leq 4.
\]

The latter inclusion \( \Sigma_1^{(1,p)} \subseteq S_p \) was already observed to hold, even for \( 2 < p < \infty \); see (2). So, we only need to establish the following result, which is of independent interest.

**Proposition 6.** Let \( p > 2 \). Then \( L^p * L^{p'} \subseteq \Sigma_1^{(1,p)} \) if and only if \( p \leq 4 \), in which case, with \( \kappa_G \) denoting Grothendieck’s constant, we have

\[
\|g * h\|_{\Pi_1^{(1,p)}} \leq \kappa_G \|g\|_p \|h\|_{p'}, \quad g \in L^p, \quad h \in L^{p'}.
\]

**Proof.** Suppose that \( p \in (2,4] \). Fix functions \( g \in L^p \subseteq L^2 \) and \( h \in L^{p'} \). Then the convolution operator \( C_g^{(1,2)} : f \mapsto f * g \) is bounded from \( L^1 \) to \( L^2 \) with norm \( \|C_g^{(1,2)}\|_{op} \leq \|g\|_2 \). Moreover, \( C_g^{(1,2)} \) is then also 1-summing with \( \|C_g^{(1,2)}\|_{\Pi_1^{(1,2)}} \leq \kappa_G |\Gamma_n|^{\kappa(p)} \), by Proposition 3.4. So,

\[
(18) \quad \|C_g^{(1,2)}\|_{\Pi_1^{(1,2)}} \leq \kappa_G \|g\|_2 \leq \kappa_G \|g\|_p,
\]

where the last inequality holds because \( p > 2 \). Next we show that

\[
(19) \quad C_h^{(2,p)} \in \mathcal{L}(L^2, L^p) \text{ with } \|C_h^{(2,p)}\|_{op} \leq \|h\|_{p'}.
\]

To this effect, it follows from the Hausdorff-Young inequality applied in \( L^{p'} \) that \( \hat{h} \in \ell^p(\Gamma) \) with \( \|\hat{h}\|_{\ell^p(\Gamma)} \leq \|h\|_{p'} \). Consequently, for every \( f \in L^2 \) (in which case \( \hat{f} \in \ell^2(\Gamma) \)) it follows from the generalized Hölder inequality that \( (f * h) = \hat{f} \cdot \hat{h} \in \ell^r(\Gamma) \) with \( \|\hat{f} \cdot \hat{h}\|_{\ell^r(\Gamma)} \leq \|\hat{h}\|_{\ell^p(\Gamma)} \|f\|_{\ell^r(\Gamma)} \), where \( \frac{1}{r} = \frac{1}{2} + \frac{1}{p'} \). The previous two inequalities and Plancherel’s Theorem then yield

\[
(20) \quad \|(f * h)\|_{\ell^r(\Gamma)} \leq \|h\|_{p'} \|f\|_2.
\]

Moreover, \( p \leq 4 \) implies that \( \frac{1}{r} \geq (1 - \frac{1}{p'}) = \frac{1}{p} \) (i.e., \( r \leq p^* < 2 \)), and so \( \ell^r(\Gamma) \subseteq \ell^{p^*}(\Gamma) \) with

\[
(21) \quad \|\xi\|_{\ell^{p^*}(\Gamma)} \leq \|\xi\|_{\ell^{r}(\Gamma)} \quad \xi \in \ell^{r}(\Gamma).
\]
Accordingly, \((f \ast h)^\circ \in \mathcal{B}^* (\Gamma)\), and so we can again apply the Hausdorff-Young inequality (in \(\mathcal{B}^* (\Gamma)\) with \(G\) being the dual group of \(\Gamma\)) to conclude that \(\|f \ast h\|_p = \|((f \ast h)^\circ)^{\circ}\|_p \leq \|f \ast h\|_{p^*} (\Gamma)\). This inequality, combined with (20) and (21), yields
\[
\|C_h^{(2,p)}(f)\|_p = \|f \ast h\|_p \leq \|h\|_{p^*} \|f\|_2, \quad f \in L^2,
\]
from which (19) is immediate. Since \(C_{g,h} = C_h^{(2,p)} \circ C_g^{(1,2)}\), the ideal property of 1-summing operators, [4, p. 37], together with (18) and (19) yields the desired inequality
\[
\|g \ast h\|_{\Pi_1^{(1,p)}} := \|C_{g,h}^{(1,p)}\|_{\Pi_1^{(1,p)}} \leq \|C_g^{(1,2)}\|_{\Pi_1^{(1,2)}} \|C_h^{(2,p)}\|_{\op} \leq \kappa_G \|g\|_p \|h\|_{p^*}.
\]

Conversely, for some \(p > 2\), suppose that \(L^p \ast L^{p^*} \subseteq \Sigma_1^{(1,p)}\). Hence, also \(L^p \ast L^{p^*} \subseteq S_p^2\); see [2]. Fix \(g \in L^{p^*}\). Then the linear operator \(C_g : f \mapsto g \ast f\) maps \(L^p\) into \(S_p\). Let \(f_n \to 0\) in \(L^p\) and \(C_g f_n \to h\) in \(S_p\). Then also \(C_g f_n \to h\) in \(L^p\) as \(S_p \subseteq L^{p^*}\) continuously. Moreover, \(f_n \to 0\) in \(L^p\) implies that \(C_g f_n \to 0\) in \(L^p\) (as \(C_g^{(p)} \in \mathcal{L}(L^p)\)), and so we can conclude that \(h = 0\) in \(L^p\) and hence also in \(S_p\). Then the Closed Graph Theorem ensures that \(C_g \in \mathcal{L}(L^p, S_p)\), that is,
\[
\|f \ast g\|_{S_p} \leq \|C_g\|_{\op} \|f\|_p, \quad f \in L^p.
\]

Consider now the linear map \(g \mapsto C_g\) from \(L^{p^*}\) into the Banach space \(\mathcal{L}(L^p, S_p)\). Let \(g_n \to 0\) in \(L^{p^*}\) and \(C_g f_n \to T f\) in \(\mathcal{L}(L^p, S_p)\). Fix \(f \in L^p\). Then \(f \ast g_n \to 0\) in \(L^p\) and hence also in \(L^1\). Moreover, as \(f \ast g_n \to T f\) in \(S_p\) and \(S_p \subseteq L^p \subseteq L^1\), with continuous inclusions, it follows that \(f \ast g_n \to T f\) in \(L^1\). Accordingly, \(T f = 0\) in \(L^1\) and hence also in \(S_p\). Since \(f \in L^p\) is arbitrary, it follows that \(T = 0\). Again by the Closed Graph Theorem there is \(K > 0\) such that
\[
\|C_g\|_{\op} \leq K \|g\|_{p^*}, \quad g \in L^{p^*}.
\]
It is then clear from (22) and the previous inequality that
\[
\|f \ast g\|_{S_p} \leq K \|f\|_p \|g\|_{p^*}, \quad f \in L^p, \; g \in L^{p^*}.
\]

According to (14) and (17) there exists an increasing sequence \(\{\Gamma_n\}_{n=1}^\infty\) of finite subsets of \(\Gamma\) and sequences of trigonometric polynomials \(\{P_n\}_{n=1}^\infty\) and \(\{V_n\}_{n=1}^\infty\) on \(G\) such that
\[
\frac{\|V_n \ast P_n\|_{S_p}}{\|V_n\|_p \|P_n\|_{p^*}} \geq M_p |\Gamma_n|^{-\kappa(p)}, \quad n \in \mathbb{N},
\]
for some constant \(M_p > 0\) and with \(\sup_{n \in \mathbb{N}} |\Gamma_n| = \infty\), where we recall that \(\kappa(p) = \frac{1}{p} + \frac{1}{2} - \frac{1}{p^*}\). In view of (23) this is only possible if \(\kappa(p) \geq 0\), that is, if \(p \leq 4\). \(\square\)

**Remark 7.**

(i) Proposition 5.4 of [11] states that \(L^p \ast L^{p^*} \subseteq \Sigma_1^{(p)}\), for every \(1 \leq p < \infty\). Also, \(\Sigma_1^{(1,p)} \subseteq \Sigma_1^{(p)}\) (continuously) for every \(1 \leq p < \infty\); see [1] and [3]. Proposition 6 above asserts, for \(p > 2\), that \(L^p \ast L^{p^*}\) is contained in the smaller space \(\Sigma_1^{(1,p)}\) if and only if \(p \leq 4\). Of course, if \(1 \leq p \leq 2\), then \(L^p \ast L^{p^*} \subseteq L^2\), and so, via (1), we also have that \(L^p \ast L^{p^*} \subseteq \Sigma_1^{(1,p)}\).

(ii) Observe that the proof of Proposition 6 actually yields \(L^2 \ast L^{p^*} \subseteq \Sigma_1^{(1,p)}\) for all \(2 < p \leq 4\).
REFERENCES


MATHEMATICI-GENETRISCHEN FAKULTÄT, KATHOLISCHE UNIVERSITÄT, EICHSTÄTT-INGOLSTADT, D-85072 EICHSTÄTT, GERMANY

E-mail address: werner.ricker@ku-eichstaett.de

DEPARTMENT DE ANALISIS MATEMATICO AND IMUS, FACULTAD DE MATEMATICAS, UNIVERSIDAD DE SEVILLA, APTDO 1160, E-41080 SEVILLA, SPAIN

E-mail address: piazza@us.es