COMMUTATORS OF SMALL RANK AND REDUCIBILITY OF OPERATOR SEMIGROUPS

ALI JAFARIAN, ALEXEY I. POPOV, MEHDI RADJABALIPOUR, AND HEYDAR RADJAVI

(Communicated by Pamela B. Gorkin)

Abstract. It is easy to see that if $G$ is a non-abelian group of unitary matrices, then for no members $A$ and $B$ of $G$ can the rank of $AB - BA$ be one. We examine the consequences of the assumption that this rank is at most two for a general semigroup $S$ of linear operators. Our conclusion is that under obviously necessary, but trivial, size conditions, $S$ is reducible. In the case of a unitary group satisfying the hypothesis, we show that it is contained in the direct sum $G_1 \oplus G_2$, where $G_1$ is at most $3 \times 3$ and $G_2$ is abelian.

1. Introduction

It is easy to see that if $G$ is a non-abelian group of unitary matrices, then for no members $A$ and $B$ of $G$ can the rank of $AB - BA$ be one. Indeed, suppose that $A, B \in G$ be such that $AB \neq BA$. Then $ABA^{-1}B^{-1} - I = (AB - BA)A^{-1}B^{-1}$. Since $ABA^{-1}B^{-1}$ is a member of $G$, it is a unitary matrix; hence it is diagonalizable via a unitary similarity. If the rank of $AB - BA$ were equal to one, exactly one diagonal entry of $ABA^{-1}B^{-1}$ would be different from one, so that $\det(ABA^{-1}B^{-1})$ would be different from one, which is, clearly, a contradiction. In particular, this shows that the condition $\text{rank}(AB - BA) \leq 1$ for all $A, B$ in a unitary group $G$ implies that $G$ is abelian.

For semigroups of matrices and, more generally, linear operators on Banach spaces, the corresponding problem is more difficult. The following result was obtained in [6, Corollary 2].

Theorem 1.1 ([6]). Let $S$ be a semigroup of Schatten $p$-class operators on a Hilbert space. If $\text{rank}(AB - BA) \leq 1$ for all $A, B \in S$, then $S$ is triangularizable.

This was generalized to compact operators on arbitrary Banach spaces in [7, Theorem 9.2.10]. For non-compact operators, this question was studied in a series of papers. In [2, Lemma 5], the authors showed that the same conclusion holds for semigroups of algebraic operators, and in [3], it was shown that every non-commutative doubly generated semigroup $S$ with the condition that

$$\text{rank}(AB - BA) \leq 1$$

Received by the editors January 16, 2013.

2010 Mathematics Subject Classification. Primary 47D03, 20M20; Secondary 47B47, 51F25.

Key words and phrases. Semigroup of operators, unitary group, commutator, rank, invariant subspace.

The second and fourth authors’ research was supported in part by NSERC (Canada).

The third author’s research was supported in part by the Iranian National Science Foundation.
for all \( A, B \in S \) has a hyperinvariant subspace. Finally, it was generalized to arbitrary operators on Banach spaces in [4] as follows:

**Theorem 1.2** ([4]). Let \( X \) be a Banach space of dimension at least two. Let \( S \) be a non-commutative semigroup of operators on \( X \). If \( \text{rank}(AB - BA) \leq 1 \) for all \( A, B \in S \), then \( S \) is reducible.

It is natural to try to replace the rank-one condition in the above statements with the condition \( \text{rank}(AB - BA) \leq r \), where \( r \in \mathbb{N} \) is fixed. The following quick example shows that one cannot expect the same answer as in Theorem 1.2, even for semigroups of finite-rank operators.

**Example 1.3.** Let \( \mathcal{H} \) be a finite- or infinite-dimensional Hilbert space. For all \( i, j = 1, 2, \ldots \), denote the \( i, j \)-matrix unit by \( E_{ij} \). That is, for a fixed orthonormal basis \( (e_i) \), we have \( E_{ij}(e_k) = \delta_{jk}e_i \). The semigroup
\[
S = \{ E_{ij} : i, j \in \mathbb{N} \} \cup \{0\}
\]
is an irreducible semigroup of operators of rank \( \leq 1 \) such that \( \text{rank}(AB - BA) \leq 2 \) for all \( A, B \in S \).

In the present paper, we obtain results regarding the following question: when does the assumption \( \text{rank}(AB - BA) \leq 2 \) for all operators \( A \) and \( B \) in a semigroup \( S \) imply reducibility of \( S \)? Our main argument uses special unitary groups whose structure is also of some independent interest and is a subject of study in the last section of this paper.

Throughout the paper, the linear space \( \mathbb{C}^n \) is considered as a Hilbert space with the standard inner product \( \langle \cdot, \cdot \rangle \). In the case of infinite-dimensional spaces, the term *operator* is reserved for the bounded linear operators. The set of operators on a Banach space \( X \) is denoted by \( B(X) \). The term *invariant subspace* means a non-trivial invariant subspace. A *semigroup* is a set \( S \) of operators on \( X \) such that \( AB \in S \) for all \( A, B \in S \). A semigroup \( S \subseteq B(X) \) is *reducible* if it admits an invariant subspace, and it is *triangularizable* if there exists a chain \( C \) that is maximal as a chain of subspaces of \( X \) and that has the property that every member of \( C \) is \( S \)-invariant (see [7, Definition 7.1.1]). A semigroup \( S \subseteq B(X) \) is *irreducible* if it is not reducible. The symbol \( \text{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) denotes the \( n \times n \) diagonal matrix with \( \alpha_1, \alpha_2, \ldots, \alpha_n \) on the diagonal. The symbol \( \text{nul}(A) \) denotes the dimension of \( \ker A \). Finally, we will write \( A \equiv B \) if the matrices \( A \) and \( B \) are unitarily similar.

## 2. Reducibility of semigroups

We will start by investigating the structure of certain very special groups of unitaries.

**Definition 2.1.** Let \( p \) and \( q \) be two prime numbers. The symbol \( \mathcal{G}(p, q, A) \) will denote the group of unitaries generated by the \( p \times p \) matrices
\[
S = \begin{bmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
A = \begin{bmatrix}
\omega_1 & 0 & \ldots & 0 & 0 \\
0 & \omega_2 & \ldots & 0 & 0 \\
\vdots \\
0 & 0 & \ldots & \omega_{p-1} & 0 \\
0 & 0 & \ldots & 0 & \omega_p
\end{bmatrix},
\]
where \( A \) is not a scalar multiple of the identity and \( \omega_i^q = 1 \) for all \( i = 1, 2, \ldots, p \).
Our interest in these groups stems from the fact that if \( G \) is a minimal non-abelian group of matrices, then \( G \) admits a subgroup \( G_0 \) whose restriction to a \( G_0 \)-invariant subspace is closely related to a group of the form \( G(p, q, A) \) (see [7, Lemma 4.2.9]).

**Proposition 2.2.** Let \( p, q \) be two prime numbers and \( A \) be a \( p \times p \) matrix as in Definition 2.1. If rank \((XY - YX) \leq 2\) for all \( X, Y \in G(p, q, A) \), with the equality achieved on some members of it, then either

(i) \( p = 2 \) or
(ii) \( p = 3 \) and \( q = 2 \).

**Proof.** Denote the group \( G(p, q, A) \) by \( G \), for simplicity of notation. It is not hard to see that every member of \( G \) can be written in the form \( DS^k \), where \( D \) is a diagonal matrix whose diagonal entries are \( q \)-roots of unity, \( S \) is the cyclic permutation as in Definition 2.1 and \( 0 \leq k < p \). Moreover, if \( X_1 = D_1 S^{k_1} \) and \( X_2 = D_2 S^{k_2} \), then \( X_1 X_2 = D_3 S^{k_1 + k_2} \), for some diagonal matrix \( D_3 \).

Let \( X \) and \( Y \) be arbitrary members of \( G \). It follows from the above observation that \( XYX^{-1}Y^{-1} \) is a diagonal matrix. It is clear that if

\[
\text{rank}(XY - YX) = 2,
\]

then exactly two eigenvalues of \( XYX^{-1}Y^{-1} \) are not equal to one. Since

\[
\text{det}(XYX^{-1}Y^{-1}) = 1,
\]

we conclude that \( XYX^{-1}Y^{-1} \) is of the form

\[
\text{diag}(1, \ldots, 1, \omega, 1, \ldots, 1, \bar{\omega}, 1, \ldots, 1),
\]

where \( \omega \neq 1 \) and \( \omega^q = 1 \), and each of the series of ones between \( \omega \) and \( \bar{\omega} \) could be absent.

Observe that if \( D = \text{diag}(d_1, \ldots, d_{p-1}, d_p) \), then \( SDS^{-1} = \text{diag}(d_p, d_1, \ldots, d_{p-1}) \). It follows that \( G \) has a member of the form

\[
A_0 = \text{diag}(\omega, 1, \ldots, 1, \bar{\omega}, 1, \ldots, 1),
\]

where \( \omega \neq 1 \), \( \omega^q = 1 \), and the series of ones between \( \omega \) and \( \bar{\omega} \) is shorter than the series of ones following \( \bar{\omega} \).

Suppose that \( p > 3 \), so that \( p \geq 5 \). If the series of ones between \( \omega \) and \( \bar{\omega} \) is not absent, then consider \( B = SA_0^{-1}S^{-1} \). It follows that

\[
A_0B = A_0SA_0^{-1}S^{-1} = \text{diag}(\omega, \bar{\omega}, 1, \ldots, 1, \omega, \bar{\omega}, 1, \ldots, 1),
\]

so that rank \((A_0S - SA_0) = \text{rank}(A_0SA_0^{-1}S^{-1} - I) = 4\), contrary to the assumptions. So, the series of ones between \( \omega \) and \( \bar{\omega} \) must be absent, and

\[
A_0 = \text{diag}(\omega, \bar{\omega}, 1, \ldots, 1).
\]

However, in this case we may consider \( C = S^2A_0^{-1}S^{-2} \). We get

\[
A_0C = A_0S^2A_0^{-1}S^{-2} = \text{diag}(\omega, \bar{\omega}, \omega, \bar{\omega}, 1, \ldots, 1),
\]

so that rank \((A_0S^2 - S^2A_0) = 4\).

This shows that either \( p = 2 \) or \( p = 3 \). Suppose that \( p = 3 \). We claim that, necessarily, \( q = 2 \). Assume that \( q > 2 \). Then, by the same argument as above,

\[
A_0 = \text{diag}(\omega, \bar{\omega}, 1) \in G,
\]

where \( \omega \neq 1 \) and \( \omega^q = 1 \). Clearly, \( SA_0^{-1}S^{-1} = \text{diag}(1, \bar{\omega}, \omega) \), so that

\[
A_0SA_0^{-1}S^{-1} = \text{diag}(\omega, \bar{\omega}^2, \omega).
\]
If $q > 2$, then all the diagonal entries of this matrix are different from 1, so that $	ext{rank}(A_0 S - S A_0) = 3$, a contradiction. \hfill \Box

The next proposition records certain observations about the groups $G$ satisfying \text{rank}(XY - YX) \leq 2$ for all $X, Y \in G$. We will need the following notation.

**Definition 2.3.** Let $S$ be a set of $n \times n$ matrices and $M$ be a linear subspace of $\mathbb{C}^n$. Then we put

$$S(M) = \{ T \in S : TM \subseteq M \}.$$ 

**Proposition 2.4.** Let $G$ be a non-abelian group of unitary $n \times n$ matrices, and assume \text{rank}(AB - BA) \leq 2 for all $A, B \in G$. If $M$ is a linear subspace of $\mathbb{C}^n$, then $G(M) = G(M^⊥)$ is a subgroup of $G$ and at least one of the unitary groups $G(M)|_M$ or $G(M)|_{M^⊥}$ is abelian.

**Proof.** If $n \leq 2$, then the conclusions of the proposition are evident. Therefore, we will assume in the proof that $n \geq 3$.

Let $A, B \in G$. Since $ABA^{-1}B^{-1}$ is a unitary and \text{rank}(ABA^{-1}B^{-1} - I) = \text{rank}(AB - BA) \leq 2$, it follows from the first paragraph of the introduction that \text{rank}(AB - BA) is 0 or 2, and hence \text{rank}(ABA^{-1}B^{-1} - I) = \text{diag}(\omega, \omega', 1, 1, \ldots, 1) \neq I$ for some $\omega \neq 1 \neq \omega'$. Also, since $1 = \text{det}(ABA^{-1}B^{-1}) = \omega \omega'$, it follows that $\omega' = \bar{\omega}$.

Next, assume $M$ is a linear subspace of $\mathbb{C}^n$. Clearly, $G(M) = G(M^⊥)$. Assume, if possible, that both $G(M)|_M$ and $G(M)|_{M^⊥}$ are non-abelian. For $i = 1, 2$, choose $A_i = C_i + D_i \in G(M)$ decomposed according to $\mathbb{C}^n = M \oplus M^⊥$, such that $C_1C_2 \neq C_2C_1$. Notice that the condition $D_1D_2 \neq D_2D_1$ would imply

$$\text{rank}(A_1A_2 - A_2A_1) = 4;$$

hence $D_1D_2 = D_2D_1$. Assume, if possible, that $D_1$ is not in the centre of $G(M)|_{M^⊥}$. In this case, choose $A_3 = C_3 + D_3 \in G(M)$ such that $D_1D_3 \neq D_3D_1$ and, consequently, $C_3C_1 = C_1C_3$. Then $C_1(C_2C_3) \neq C_2C_1C_3 = (C_2C_3)C_1$ and $D_1(D_2D_3) = D_2D_1D_3 \neq (D_2D_3)D_1$, which is a contradiction. Thus $D_1$ and, by symmetry, $D_2$ belong to the centre of $G(M)|_{M^⊥}$. Now, since $G(M)|_{M^⊥}$ is not abelian, there exist $A_3 = C_3 + D_3$ and $A_4 = C_4 + D_4$ in $G(M)$ such that $C_3C_4 = C_4C_3$ and $D_3D_4 \neq D_4D_3$. Another symmetrical argument reveals that $C_3, C_4$ belong to the centre of $G(M)|_M$. Then $(C_1C_3)(C_2C_4) = C_1C_2C_3C_4 \neq C_2C_1C_3C_4 = (C_2C_4)(C_1C_3)$ and $(D_1D_3)(D_2D_4) = D_2D_3D_4D_1 \neq D_2D_3D_4D_1 = (D_2D_4)(D_1D_3)$, and, hence,

$$\text{rank}[(A_1A_3)(A_2A_4) - (A_2A_4)(A_1A_3)] = 4,$$ a contradiction. \hfill \Box

Before we state our main theorem, we need two lemmas.

**Lemma 2.5.** Let $G$ be a non-abelian unitary group on $\mathbb{C}^n$ and $N \subseteq \mathbb{C}^n$ be a 3-dimensional subspace. Assume that $G$ has a subgroup $G_0$ such that $N$ is $G_0$-invariant and, in some basis $\{ e_1, e_2, e_3 \}$ of $N$, $G_0|_N = G(3, q, A)$, where $q$ is a prime number and $A$ is a diagonal matrix as in Definition 2.1. If \text{rank}(XY - YX) \leq 2 for all $X, Y \in G$, then $N$ is $G$-invariant. Moreover, $G|_N$ is irreducible and $G|_{N^⊥}$ is abelian.

**Proof.** Since $G$ is non-abelian, in view of the observation made at the beginning of the introduction, \text{rank}(XY - YX) = 2, for some $X, Y \in G$. By Proposition 2.2, $q$ must be equal to 2. Considering matrices of the form $XYX^{-1}Y^{-1}$, as in the proof of Proposition 2.2, we conclude that $G(3, 2, A)$ admits a diagonal matrix $B$ with
eigenvalues \{1, -1, -1\}. Considering $SB^{-1}$ and $S^2B^{-2}$, where $S$ is the cyclic permutation as in Definition 2.1, we conclude that the matrices
\[
\text{diag}(1, -1, -1), \quad \text{diag}(-1, 1, -1), \quad \text{and} \quad \text{diag}(-1, -1, 1)
\]
all belong to $G(3, 2, A)$.

Pick an arbitrary $Z \in G$ and assume that $N$ is not $Z$-invariant. Fix a matrix $\tilde{S} \in G$ such that $\tilde{S}|_N = S$. Choose a basis $\{e_1, e_2, \ldots, e_n\}$ for $N^\perp$ consisting of eigenvectors of $\tilde{S}$. Since $N$ (and, hence, $N^\perp$) is not $Z$-invariant, there exist $i \leq 3$ and $j \geq 4$ such that $\langle Ze_j, e_i \rangle \neq 0$. Due to the cyclic nature of the conditions of the theorem with respect to the ordered triple $(e_1, e_2, e_3)$, we may and shall assume without loss of generality that $i = 1$. Let $M$ be the 2-dimensional subspace of $C^n$ spanned by $\{e_1, e_2\}$ and write
\[
Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}
\]
with respect to $C^n = M \oplus M^\perp$.

The matrix $	ext{diag}(-1, -1, 1) \in G(3, 2, A)$ can be obtained as $CSC^{-1}S^{-1}$, where $C = \text{diag}(1, -1, -1) \in G(3, 2, A)$. This shows that if $C \in G$ is such that $C|_N = C$, then the matrix
\[
T = \tilde{C}S\tilde{C}^{-1}\tilde{S}^{-1} = \text{diag}(-1, -1, 1, \ldots, 1) \in G.
\]

With respect to $C^n = M \oplus M^\perp$, this matrix has the form
\[
T = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.
\]

Then
\[
TZ - ZT = \begin{bmatrix} 0 & 2Z_{12} \\ -2Z_{21} & 0 \end{bmatrix}.
\]

Since rank $(TZ - ZT) \leq 2$, we conclude that rank $(Z_{12}) = \text{rank} (Z_{21}) = 1$, for none of $Z_{12}$ and $Z_{21}$ are zero. It follows that $Z_{12}e_k (k = 3, 4, \ldots, n)$ are multiples of $Z_{12}e_j$. Replacing $Z$ by $Z\tilde{S}$ changes the first column $Z_{12}e_3$ of $Z_{12}Z_{11}e_1$ and its $(j - 2)$nd column $Z_{12}e_j$ to $\lambda_j Z_{12}e_j$, where $\lambda_j$ is the eigenvalue of $\tilde{S}$ corresponding to $e_j$. Thus, again, $Z_{11}e_2$ is a multiple of $Z_{12}e_j$. Another replacement of $Z$ by $Z\tilde{S}^2$ reveals that the first two rows of $Z$ are linearly dependent, a contradiction. This shows that $N$ is $G$-invariant.

Finally, the irreducibility of $G|_N$ follows from the fact that $G(p, q, A)$ is irreducible (see, e.g., [7 Lemma 4.2.8]), and the commutativity of $G|_{N^\perp}$ was established in Proposition 2.4.\]

\textbf{Lemma 2.6.} Let $S$ be a semigroup of $n \times n$ matrices and $N$ be a subspace of $C^n$ such that, with respect to the decomposition $C^n = N \oplus N^\perp$, the representation of every member $Z \in S$,
\[
Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},
\]
has the property that rank $(Z_{21}) \leq 1$. Then each $Z \in S$ admits an invariant subspace $N_Z$ such that either $N_Z \subseteq N$, in which case $\dim(N/N_Z) \leq 1$, or $N \subseteq N_Z$, in which case $\dim(N_Z/N) \leq 1$ and $N_Z = \text{span}\{N, ZN\}$.
contains an eigenvector in space span \( n \) for all \( A, B \).

Clearly, there is no loss of generality in assuming that 2 \( \leq k \leq n - 2 \).

Proof. Denote the dimension of \( N \) by \( k \). Clearly, there is no loss of generality in assuming that 2 \( \leq k \leq n - 2 \).

Let \( Z \in S \) be such that \( N \) is not \( Z \)-invariant. Since \( \text{rank}(Z_{21}) = 1 \), by choosing appropriate bases \( \{e_1, \ldots, e_k\} \) for \( N \) and \( \{e_{k+1}, \ldots, e_n\} \) for \( N^\perp \), we may assume that only the \((1,1)\)-entry of \( Z_{21} \) is non-zero.

Consider the matrix \( Z^2 \in S \). Its \((2,1)\)-block is equal to \( Z_{21}Z_{11} + Z_{22}Z_{21} \). Notice that only the first row of the matrix \( Z_{21}Z_{11} \) may contain non-zero entries and only the first column of the matrix \( Z_{22}Z_{21} \) may contain non-zero entries. Since the rank of \( Z_{21}Z_{11} + Z_{22}Z_{21} \) is assumed to be at most one, we conclude that one of the matrices \( Z_{21}Z_{11} \) or \( Z_{22}Z_{21} \) must satisfy the property that all its entries, except perhaps the \((1,1)\)-entry, are equal to zero. If all but the \((1,1)\)-entry of \( Z_{21}Z_{11} \) are zero, then \( Z_{11} \) (and, hence, \( Z \)) leaves invariant the space \( \text{span}\{e_2, \ldots, e_k\} \). If all but the \((1,1)\)-entry of \( Z_{22}Z_{21} \) are zero, then in the first column of \( Z_{22} \) only the first entry may be non-zero, so that \( Z \) leaves invariant the space \( \text{span}\{e_1, \ldots, e_{k-1}, e_k\} \). \( \square \)

Now we are ready to prove the main theorem of the paper.

Theorem 2.7. Let \( \mathcal{G} \) be a group of unitary \( n \times n \) matrices. If \( \text{rank}(AB - BA) \leq 2 \) for all \( A, B \in \mathcal{G} \), then there is a subspace \( M \) of \( \mathbb{C}^n \) such that 1 \( \leq \dim M \leq 3 \) and \( \mathcal{G} \subset \mathcal{G}_1 \oplus \mathcal{G}_2 \) with \( \mathcal{G}_2 \) abelian, where the direct sum is with respect to the decomposition \( \mathbb{C}^n = M \oplus M^\perp \).

Proof. Clearly, there is no loss of generality in assuming that \( \mathcal{G} \) is not abelian and \( n \geq 4 \). Moreover, we may also assume that \( \mathcal{G} = \mathbb{T}\mathcal{G}_0 \), where \( \mathbb{T} \) is the unit circle on the complex plane.

Since \( \mathcal{G} = \mathbb{T}\mathcal{G}_0 \), it is a compact Lie group, so \([1, \text{Theorem 5}] \) implies that \( \mathcal{G} \) contains a finite non-abelian subgroup. It follows that \( \mathcal{G} \) contains a minimal non-abelian subgroup. By \([7, \text{Lemma 4.2.9}] \), every minimal non-abelian finite group admits an invariant subspace \( N \) such that the restriction of the group to \( N \) is, after a similarity, generated by two matrices \( \alpha A \) and \( \beta S \), where \( A \) is a non-scalar diagonal matrix, \( S \) is the cyclic permutation, and \( \alpha, \beta \in \mathbb{T} \). Since \( \mathcal{G} = \mathbb{T}\mathcal{G}_0 \), we conclude that \( \mathcal{G} \) contains a subgroup \( \mathcal{G}_0 \) whose restriction to \( N \) is equal (in an appropriate basis) to the group \( \mathcal{G}(p, q, A) \).

It follows from Proposition 2.2 and Lemma 2.5 that, without loss of generality, \( p = 2 \). Since \( \mathcal{G}(2, q, A) \) is not abelian, it contains a matrix \( C \) of the form \( XY^{-1}X^{-1}Y \) different from the identity. By the properties of \( \mathcal{G}(p, q, A) \), this matrix is necessarily diagonal, and its diagonal entries are \( q \)-roots of the unity. Since \( \det(C) = 1 \), we have \( C = \text{diag}(\omega, \tilde{\omega}) \), for some \( \omega \neq 1 \), \( \omega^q = 1 \).

If \( Z \in \mathcal{G} \) is an arbitrary matrix, then, considering the rank of \( ZC - CZ \), we conclude that, with respect to the decomposition \( \mathbb{C}^n = N \oplus N^\perp \), \( Z \) is represented as

\[
Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},
\]

where \( \text{rank}(Z_{21}) \leq 1 \). By Lemma 2.4, either \( Z \) admits an eigenvector in \( N \) or the space \( \text{span}\{N, ZN\} \) has dimension 3 and is \( Z \)-invariant. Notice that this space contains \( N \) as a subspace of codimension one.

First, we claim that, assuming \( N \) is not \( \mathcal{G} \)-invariant, \( \mathcal{G} \) admits a matrix without an eigenvector in \( N \).
Indeed, let $V \in \mathcal{G}$ be such that $\mathcal{N}$ is not $V$-invariant. If $V$ does not have eigenvectors in $\mathcal{N}$, we are done. Suppose that $V$ has an eigenvector in $\mathcal{N}$. Write $V$ as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$ 

Let $f \in \mathcal{N}$ be an eigenvector of $V$. Clearly, $\text{span}\{f\} = \ker(V_{21})$ and $f$ is an eigenvector for $V_{11}$. Since $G(2, q, A)$ is irreducible (see, e.g., [7, Lemma 4.2.8]), there exists $U \in G(2, q, A)$ such that $f$ is not an eigenvector of $UV_{11}$. There exists a matrix $Z \in \mathcal{G}$ of the form $U \oplus D$, where $D$ is a unitary $(n - 2) \times (n - 2)$ matrix. Since $\ker(DV_{21}) = \ker(V_{21}) = \text{span}\{f\}$, the matrix $ZV$ does not admit eigenvectors in $\mathcal{N}$.

Let $T \in \mathcal{G}$ be a matrix without eigenvectors in $\mathcal{N}$. Since $T$ is a unitary matrix, every invariant subspace of it is reducing. By Lemma 2.6, there exist an orthonormal basis $\{e_1, e_2\}$ of $\mathcal{N}$ and a unit vector $e_3$ in $\mathcal{N}^\perp$ such that, relative to the decomposition $\mathbb{C}^n = \text{span}\{e_1\} \oplus \text{span}\{e_2\} \oplus \text{span}\{e_3\} \oplus (\mathcal{N}^\perp \oplus \text{span}\{e_3\})$, $T$ is written in the form

$$T = \begin{bmatrix} p & q & w & 0 \\ r & s & u & 0 \\ 0 & t & v & 0 \\ 0 & 0 & 0 & U \end{bmatrix},$$

where $r \neq 0$, $t \neq 0$, and $U$ is an $(n - 3) \times (n - 3)$ unitary matrix.

Let $S \in \mathcal{G}$ be arbitrary. Write, relative to the same decomposition,

$$S = \begin{bmatrix} a & b & * & * \\ c & d & * & * \\ e & f & * & * \\ g & h & * & * \end{bmatrix},$$

where $a, b, c, d, e, f$ are complex numbers, $g$ and $h$ are $(n - 3)$-vectors, and the symbol $*$ stands for a number or a matrix whose value does not concern us. Multiplying $T$ by $S$, we get

$$TS = \begin{bmatrix} * & * & * \\ ar + cs + eu & br + ds + fu & * & * \\ ct + ev & dt + fv & * & * \\ Ug & Uh & * & * \end{bmatrix}.$$ 

Recall that

$$\text{rank} \left( \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \leq 1 \quad \text{and} \quad \text{rank} \left( \begin{bmatrix} ct + ev & dt + fv \\ Ug & Uh \end{bmatrix} \right) \leq 1.$$

Suppose that one of the vectors $g$ and $h$ is not zero, say, $g \neq 0$. Then there exists $\alpha \in \mathbb{C}$ such that $h = \alpha g$, $f = \alpha e$ and $dt + fv = \alpha(ct + ev)$. Since $t \neq 0$, we conclude that $d = \alpha c$. It follows that

$$\text{rank} \left( \begin{bmatrix} c & d \\ e & f \\ g & h \end{bmatrix} \right) = 1.$$

Repeating the same argument with the matrix $TS$ replacing the matrix $S$, we obtain

$$\text{rank} \left( \begin{bmatrix} ar + cs + eu & br + ds + fu \\ ct + ev & dt + fv \\ Ug & Uh \end{bmatrix} \right) = 1.$$
It follows that \( br + ds + fu = \alpha(ar + cs + eu) \). Since \( r \neq 0 \), the only possibility is that \( b = \alpha a \). However, this implies that
\[
\text{rank} \left( \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} \right) = 1.
\]
This is impossible since the matrix \( S \) is unitary, hence invertible.

The case \( h \neq 0 \) brings us to the same conclusion. Therefore \( g = h = 0 \). Since \( S \) was chosen arbitrarily, this implies that \( M = GN \) is \( G \)-invariant and \( M \perp (N^\perp \ominus \text{span}\{e_3\}) \). Under the assumption that \( N \) is not \( G \)-invariant, this means that \( M = \text{span}\{e_1, e_2, e_3\} \), a 3-dimensional \( G \)-invariant subspace. The rest of the conclusions of the theorem follow from Proposition 2.4.

**Corollary 2.8.** Let \( X \) be a Banach space and \( S = \mathbb{R}^+ S \) be a semigroup of operators on \( X \) containing a non-zero compact operator such that the minimal rank of non-zero operators in \( S \) is at least 4. If \( \text{rank}(AB - BA) \leq 2 \) for all \( A, B \in S \), then \( S \) is reducible.

**Proof.** Suppose that \( S \) is irreducible. It is well-known that a non-trivial ideal of an irreducible semigroup is irreducible. Thus, there is no loss of generality in assuming that \( S \) consists of compact operators.

Denote the minimal non-zero rank of operators in \( S \) by \( r \). By \cite{10} Lemma 8.1.15, \( r \) is finite and there exists an idempotent \( E \in S \) of rank \( r \). Let \( S_0 = ESE|\text{Range } E \). Then \( S_0 \) is represented as a semigroup of \( r \times r \) matrices. Moreover, every member of this semigroup is either invertible or zero, by the minimality of the rank \( r \) in \( S \). Also, as a compression of an irreducible semigroup, the semigroup \( S_0 \) must be irreducible. By \cite{10} Lemma 3.1.6, \( S_0 \setminus \{0\} \) is a group of matrices. Moreover, there exists a group \( G \) of unitary matrices such that, after a similarity, \( S_0 \setminus \{0\} \subseteq \mathbb{R}^+ G \). Clearly, \( G \) must be irreducible, too. Also, the proof of \cite{10} Lemma 3.1.6 shows that the group \( G \) is, in fact, similar to the group \( \{ \frac{1}{r(T)} T : T \in G_0 \} \). Hence, the condition \( \text{rank}(AB - BA) \leq 2 \) holds for all \( A, B \in G \). This, obviously, contradicts the conclusion of Theorem 2.7.

We remark that the condition about the rank in Corollary 2.8 cannot be improved. This is clear if the minimal rank is allowed to be equal to 2 (take, for example, the group of \( 2 \times 2 \) unitaries). The following proposition exhibits an example of an irreducible group of \( 3 \times 3 \) unitary matrices with the property \( \text{rank}(AB - BA) \leq 2 \) for all \( A, B \) in the group.

**Proposition 2.9.** Let
\[
T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]
Then the group \( G = \langle T, S \rangle \) is irreducible and satisfies the condition that
\[
\text{rank}(AB - BA) \leq 2
\]
for all \( A, B \in G \).
Proof. By [7, Lemma 4.2.8], the group \( G \) is irreducible. Let us show that
\[
\operatorname{rank}(AB - BA) \leq 2
\]
for all \( A, B \in G \).

Observe that every member of \( G \), being a finite product of matrices \( T, S, T^{-1} \) and \( S^{-1} \), can be written in one of the following three forms:
\[
\begin{bmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \gamma
\end{bmatrix},
\begin{bmatrix}
0 & \alpha & 0 \\
0 & 0 & \beta \\
\gamma & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & \alpha \\
\beta & 0 & 0 \\
0 & \gamma & 0
\end{bmatrix},
\]
with \( \alpha, \beta, \gamma \in \{1, -1\} \). Moreover, among the numbers \( \alpha, \beta, \gamma \) exactly two or none are equal to \(-1\), the rest being equal to \(1\). For \( A \in G \), let us refer to the particular form of \( A \) among the three forms above as the pattern of \( A \).

A routine check shows that for all matrices \( A \) and \( B \in G \), the patterns of \( AB \) and \( BA \) are the same. Hence, the difference \( AB - BA \) must have the same pattern, too. Now, since there are exactly zero or two elements equal to \(-1\) among the non-zero elements of \( AB \) and \( BA \), a quick check shows that there is at least one entry \((i, j)\) such that \((AB)_{ij}\) and \((BA)_{ij}\) are both equal to \(1\) or \(-1\) simultaneously. But this means that the difference \( AB - BA \) has at most two non-zero entries, so that \( \operatorname{rank}(AB - BA) \leq 2 \). \( \square \)

3. ON THE STRUCTURE OF THE GROUP \( G(p, q, A) \)

For prime numbers \( p \) and \( q \), let \( G = G(p, q, A) \) be the irreducible group with generators \( A \) and \( S \) as defined before. These groups played a central role in our arguments from Section 2. In the present section, we will further study the structure of these groups in terms of the following parameters:
\[
\begin{align*}
(3.1) \quad \rho &= \min \{ \operatorname{rank}(D - I) \neq 0 : D \in G ; D \text{ diagonal} \}, \\
(3.2) \quad r &= \max \{ \operatorname{rank}(XYX^{-1}Y^{-1} - I) : X, Y \in G \}.
\end{align*}
\]

Note. Clearly, \( 1 \leq \rho \leq r \leq p \).

Throughout the remainder of the paper, \( G = G(p, q, A) \) for some \( p, q, A \). If \( p, q \) are fixed, we may also write \( G_A, \rho_A \) and \( r_A \) to denote \( G(p, q, A) \), \( \rho \) and \( r \), respectively.

Theorem 3.1. Let \( D_A \) be the collection of all diagonal matrices in \( G_A = G(p, q, A) \) and let \( \mathcal{S} \) be the subgroup generated by \( S \). Also, let \( C_A \) be the commutator subgroup of \( G_A \). Then \( G_A = D_A \mathcal{S} = \mathcal{S} D_A \) and \( C_A \subset D_A \). Moreover, if \( C_A \neq D_A \), then one of the following cases holds:

(i) \( C_A \) contains no non-scalar matrix. Then \( p/2 \leq \rho_A \leq r_A = p = q \) and \( C_A = \{ \eta I : \eta^p = 1 \} \).

(ii) \( C_A \) contains no non-scalar matrices, and for any non-scalar \( B \in C_A \), \( \rho_B = D_B, 2 \leq \rho_B \leq r_B \leq r_A \) and \( \rho_A \leq \rho_B \leq 2\rho_A \).

Proof. For convenience, we drop the subscript \( A \) and will only maintain the subscript \( B \) to avoid confusion. Consider the general word
\[
(3.3) \quad G = A^{\alpha_1}S^{\beta_1}A^{\alpha_2}S^{\beta_2} \cdots A^{\alpha_m}S^{\beta_m} \in G
\]
for some integers \( m, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m \). Since
\[
(3.4) \quad S^\beta A^\alpha S^{-\beta} \in D, \forall \alpha, \beta \in \mathbb{Z},
\]

it follows that every word of the form \((3.3)\) can be rewritten as
\[
G = DS^\gamma, \text{ for some } D \in \mathcal{D},
\]
where \(\gamma = \beta_1 + \beta_2 + \cdots + \beta_m\). Now, \(G\) is diagonal if and only if \(\gamma = 0\) (mod \(p\)). Then \(\mathcal{G} = D\mathcal{S}\) and \(\mathcal{C} \subset \mathcal{D}\). Since \(\mathcal{G}^{-1} = \mathcal{G}\), it follows that \(\mathcal{G} = \mathcal{G}\mathcal{D}\).

To prove (i), assume \(\mathcal{C}\) contains no non-scalar matrix. Since \(\mathcal{C} \neq \{I\}\), there exists \(C = \eta I\) for some complex number \(\eta \neq 1\) and some \(C \in \mathcal{C}\). It is easy to see that \(\omega^\eta = 1\). Also, \(\omega^p = \det(C) = 1\). Hence, \(q|p\), and thus \(q = p = r\). Since \(C\) is a group, \(\mathcal{C} = \{\eta I : \ \eta^p = 1\}\). Now, if \(\text{rank}(D - I) = \rho < p/2\), then \(D\) and \(SD^{-1}S^{-1}\) each have at most \(\rho\) entries different from 1 and, hence, \(DSD^{-1}S^{-1} \neq \eta I\) for some \(\eta \in \mathbb{C}\), a contradiction.

For (ii), assume there exists a non-scalar \(B \in \mathcal{C}\). Then the subgroup \(\mathcal{G}_B\) of \(\mathcal{G}\) is non-abelian and the relations \((3.4)\) and \((3.5)\) can be sharpened as follows:
\[
S^\beta B^\alpha S^{-\beta} = B^\alpha B^{-\alpha} S^\beta B^{-\beta} \in \mathcal{C}_B, \ \forall \alpha, \beta \in \mathbb{Z},
\]
\[
G = DS^\gamma, \text{ for some } D \in \mathcal{C}_B.
\]
This shows that \(\mathcal{D}_B \subset \mathcal{C}_B\), which proves \(\mathcal{D}_B = \mathcal{C}_B\). Since \(\det(C) = 1\) for all \(C \in \mathcal{C}\), it follows that \(\text{rank}(D - I) \geq 2\) whenever \(I \neq D \in \mathcal{D}\). The inequality \(\rho_B \leq 2\rho_A\) follows from the fact that if \(\text{rank}(D - I) = \rho\), then \(\text{rank}(DSD^{-1}S^{-1} - I) \leq 2\rho\) and the rest of (ii) is clear.

The next corollary studies the case \(\rho = 1\). We continue to use the notation established in the previous paragraphs.

**Corollary 3.2.** It is always true that \(2 \leq r \leq p\) and, if \(I \neq C \in \mathcal{C}\), then
\[
\text{rank}(C - I) \geq 2.
\]
In particular, if \(\rho = 1\), then one of the following cases holds:

(i) In this case, \(\mathcal{C} = \{I, -I\} \subset \mathcal{D} = \{I, -I, \text{diag}(1, -1), \text{diag}(-1, 1)\}\).

(ii) \(\mathcal{C}\) contains non-scalar matrices, and for any non-scalar \(B \in \mathcal{C}\), \(\rho_B \geq 2\) and \(\mathcal{C}_B = \mathcal{D}_B\). The lower bound 2 is attained for some \(B\).

If \(p = q = 2\), then \(p = 1\), \(\mathcal{D} = \{I, -I, \text{diag}(1, -1), -\text{diag}(1, -1)\}\) and \(\mathcal{C} = \{I, -I\}\).

**Proof.** Observe that if \(\text{rank}(X^{-1}Y^{-1}XY - I) = 1\), then \(I \neq \text{det}(X^{-1}Y^{-1}XY) = 1\), a contradiction. Thus, \(2 \leq r \leq p\). Now, if \(D \in \mathcal{D}\) and \(\text{rank}(D - I) = 1\), then \(\text{det}(D) \neq 1\) and, hence, \(D \notin \mathcal{C}\). In particular, if \(\rho = 1\), then \(\mathcal{D} \neq \mathcal{C}\) and, in view of Theorem 3.1 one of the following cases holds.

**Case 1.** \(p/2 \leq 1 \leq r = p = q\), which implies that \(r = p = q = 2\) and \(\mathcal{D} \neq \mathcal{C} = \{I, -I\}\). Thus \(\mathcal{D} = \{I, -I, \text{diag}(1, -1), \text{diag}(-1, 1)\}\) is the only choice left.

**Case 2.** There exists a non-scalar \(B \in \mathcal{C}\), and for any such \(B\), \(\mathcal{C}_B = \mathcal{D}_B\) and \(\rho_B \geq 2\). Now, if \(D \in \mathcal{D}\) has exactly one diagonal entry different from 1, then \(DSD^{-1}S^{-1}\) is a commutator with exactly two diagonal entries different from 1.

Conversely, if \(p = q = 2\), then \(\text{rank}(C - I) = 2\) whenever \(I \neq C \in \mathcal{C}\), which implies that \(\mathcal{D} \neq \{I, -I\} = \mathcal{C}\). Thus, \(\mathcal{D} = \mathcal{C} \cup \{\text{diag}(1, -1), \text{diag}(-1, 1)\}\) and, hence, \(\rho = 1\).

The following theorem studies the case \(\rho = 2\).
Theorem 3.3. If \( r = 2 \), then either

(i) \( r = p \) and \( q > 2 \) or
(ii) \( r = p - 1 \), \( q = 2 \).

Proof. If \( p = 2 \), then \( r = 2 \). Also, \( q > 2 = p \) by Corollary 3.2.

So, we assume \( p \geq 3 \). Let \( D_2 \) be the (non-empty) collection of all matrices \( D \in D \) such that exactly \( p - 2 \) entries on the main diagonal of \( D \) are equal to 1. We claim there exists \( \Delta \in D_2 \) for which exactly the first two diagonal entries are different from 1. Let \( s \) be the minimal positive integer for which there exist a positive integer \( h \) and a matrix \( D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p) \in D_2 \) such that \( \lambda_h \neq 1 \) and \( \lambda_{h+s} \neq 1 \). Examining \( S^{-h+1} DS^{h-1} \) and \( S^{-h-s+1} DS^{h+s-1} \) reveals that \( 1 \leq s < p/2 \) and allows us to assume without loss of generality that \( h = 1 \). Let \( p - 1 = ms + t \) for some non-negative integers \( m, t \) with \( 0 \leq t \leq s - 1 \), and, in fact, since \( p \) is an odd prime, it follows that either \( s = 1 \) or \( 0 \leq t \leq s - 2 \). Let \( \lambda_1 = \omega \) and \( \lambda_{s+1} = \omega^n \neq 1 \) for some primitive \( q \)th root \( \omega \) of 1 and some positive integer \( a < q \). For \( 1 \leq k \leq m - 1 \), assume \( \Delta_1, \Delta_2, \ldots, \Delta_k \in D_2 \) are constructed such that \( \Delta_1 = D \) and the first and the \((ks+1)\)th diagonal entries of \( \Delta_k \) are \( \omega^k \) and \( \omega^{ak} \), respectively, where \( \varepsilon_k := (-1)^{k+1} \). Define \( \Delta_{k+1} = S^{ks} D^k S^{-ks} \Delta_k^{-1} \). This finite induction yields \( \Delta_m \in D_2 \), whose first diagonal entry is \( \omega^m \) and whose \((ms+1)\)th diagonal entry is \( \omega^{am} \) (necessarily, \( \neq 1 \)). Now, observe that the first and the \((t+2)\)nd diagonal entries of \( S^{t+1} \Delta_m S^{-t-1} \in D_2 \) are \( \omega^m \) and \( \omega^{am} \), respectively. Since all other entries are equal to 1 and \( \omega^{am} \neq 1 \), it follows that \( \omega^m \neq 1 \). By minimality, \( t + 2 \geq s + 1 \); hence, \( s = 1 \) and \( t = 0 \).

Thus, there exists \( k \in \{1, 2, \ldots, q - 1\} \) such that

(3.8) \[ \Delta = \text{diag}(\omega, \omega^k, 1, 1, \ldots, 1) \in D_2. \]

Let \( \Omega := \Gamma S T^{-1} S^{-1} \in C \), where

(3.9) \[ \Gamma = \text{diag}(\omega, \omega^k, \omega, \omega^k, \ldots, \omega, \omega^k, 1) = \Pi_{j=0}^{(p-3)/2} S^{2j} \Delta S^{-2j} \in D. \]

Hence

(3.10) \[ \Omega = \text{diag}(\omega, \omega^{k-1}, \omega^{-1}, \omega^{k-1}, \ldots, \omega^{-1}, \omega^{k-1}, \omega^{-k}). \]

Let us assume \( q \geq 3 \) and settle the problem in this case. We claim \( k \geq 2 \); otherwise,

(3.11) \[ S^{-1} \Delta S \Pi_{i=1}^{p-2} S^i \Delta^{-1} S^{-i} = \text{diag}(\omega^2, 1, 1, \ldots, 1) \in D \]

and \( \text{rank}(D - I) = 1 \), a contradiction. Therefore, \( k \geq 2 \) and the proof of part (i) follows from the fact that \( r = \text{rank}(\Omega - I) = p \).

All we have to do now is settle the case \( p > q = 2 \). In (3.9), \( \omega = \omega^k = -1 \), and one can deduce that

(3.12) \[ \Delta' := \Delta S \Delta S^{-1} = \text{diag}(-1, 1, -1, 1, \ldots, 1) \in D_2. \]

Choose a positive integer \( u \) such that \( p = 4u \pm 1 \). Define \( \Omega' := \Gamma' S (\Gamma')^{-1} S^{-1} \in C \), where

(3.13) \[ \Gamma' = \text{diag}(-1, 1, -1, 1, \ldots, -1, 1, -1) = \Pi_{j=0}^{u-1} S^{4j} \Delta' S^{-4j} \in D. \]

Hence,

(3.14) \[ \Omega' = \text{diag}(1, -1, -1, -1, \ldots, -1, 1, 1). \]

Since \( r \geq \text{rank}(\Omega' - I) = p - 1 \), it follows that \( p - 1 \leq r \leq p \). Also, since \( \det(C) = 1 \) for all \( C \in C \), it follows that \( \text{rank}(C) \neq p \), and we are done. \( \square \)
Based on Theorem 3.3, we can sharpen Corollary 3.2 as follows.

**Corollary 3.4.** If \( \rho = 1 \), then one of the following cases holds:

(i) \( r = p = q = 2 \). In this case,
\[
\mathcal{C} = \{I, -I\} \subset \mathcal{D} = \{I, -I, \text{diag}(1, -1), \text{diag}(-1, 1)\}.
\]

(ii) \( p = r \) and \( q > 2 \).

(iii) \( r = p - 1 \) and \( q = 2 \).

**Proof.** Part (i) is the same as part (i) of Corollary 3.2. Let \( B \in \mathcal{C} \) be as in part (ii) of Corollary 3.2 such that \( \rho_B = 2 \). By Theorem 3.3, we have one of the following cases.

Case 1. \( r_B = p \) and \( q > 2 \). Then \( p \leq r \leq p \), which proves (ii).

Case 2. \( r_B = p - 1 \) and \( q = 2 \). Then \( r_B \) is even and, hence, \( p \) is odd. If \( r \) were equal to \( p \), we would have \( -I \in \mathcal{C} \), which is impossible since the determinant of every member of \( \mathcal{C} \) is equal to one. This proves (iii). \( \square \)

The following corollary studies the case \( r = 2 \); its easy proof is left to the interested reader.

**Corollary 3.5.** If \( r = 2 \), then one of the following cases holds:

(i) \( \rho = 1 \) and \( p = q = 2 \). In this case,
\[
\mathcal{C} = \{I, -I\} \subset \mathcal{D} = \{I, -I, \text{diag}(1, -1), \text{diag}(-1, 1)\}.
\]

(ii) \( \rho = 1 \), \( p = 2 \) and \( q > 2 \). In this case,
\[
\mathcal{C} = \{\text{diag}(\omega, \bar{\omega}) : \omega^q = 1\} \subset \mathcal{D} = \{\text{diag}(\omega, \eta) : \omega^q = \eta^q = 1\}.
\]

(iii) \( \rho = 1 \), \( p = 3 \) and \( q = 2 \). In this case,
\[
\mathcal{C} = \text{diag}(\omega, \bar{\omega}) : \omega^q = 1\}.
\]

(iv) \( \rho = 2 \), \( p = 2 \) and \( q > 2 \). In this case, \( \mathcal{C} = \mathcal{D} = \{\text{diag}(\omega, \eta) : \omega^q = \eta^q = 1\} \).

(v) \( \rho = 2 \), \( p = 3 \) and \( q = 2 \). In this case,
\[
\mathcal{C} = \mathcal{D} = \{\text{diag}(\omega, \bar{\omega}) : \omega^q = 1\}.
\]

**References**


University of New Haven, 300 Boston Post Road, West Haven, Connecticut 06516

*E-mail address*: ajafarian@newhaven.edu

Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1, Canada

*E-mail address*: a4popov@uwaterloo.ca

Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1, Canada (on sabbatical from the Iranian Academy of Sciences, Tehran, Iran)

*E-mail address*: radjabalipour@ias.ac.ir

Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1, Canada

*E-mail address*: hradjavi@uwaterloo.ca