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COMMUTATORS OF SMALL RANK AND REDUCIBILITY OF OPERATOR SEMIGROUPS

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Abstract. It is easy to see that if $\mathcal{G}$ is a non-abelian group of unitary matrices, then for no members A and B of $\mathcal{G}$ can the rank of $AB - BA$ be one. We examine the consequences of the assumption that this rank is at most two for a general semigroup $\mathcal{S}$ of linear operators. Our conclusion is that under obviously necessary, but trivial, size conditions, $\mathcal{S}$ is reducible. In the case of a unitary group satisfying the hypothesis, we show that it is contained in the direct sum $\mathcal{G}_1 \oplus \mathcal{G}_2$, where $\mathcal{G}_1$ is at most 3×3 and $\mathcal{G}_2$ is abelian.

1. Introduction

It is easy to see that if $\mathcal{G}$ is a non-abelian group of unitary matrices, then for no members A and B of $\mathcal{G}$ can the rank of $AB - BA$ be one. Indeed, suppose that $A, B \in \mathcal{G}$ be such that $AB \neq BA$. Then $ABA^{-1}B^{-1} - I = (AB - BA)A^{-1}B^{-1}$. Since $ABA^{-1}B^{-1}$ is a member of $\mathcal{G}$, it is a unitary matrix; hence it is diagonalizable via a unitary similarity. If the rank of $AB - BA$ were equal to one, exactly one diagonal entry of $ABA^{-1}B^{-1}$ would be different from one, so that $\det(ABA^{-1}B^{-1})$ would be different from one, which is, clearly, a contradiction. In particular, this shows that the condition $\operatorname{rank}(AB - BA) \leqslant 1$ for all A, B in a unitary group $\mathcal{G}$ implies that $\mathcal{G}$ is abelian.

For semigroups of matrices and, more generally, linear operators on Banach spaces, the corresponding problem is more difficult. The following result was obtained in [6, Corollary 2].

Theorem 1.1 ([6])**.** *Let $\mathcal{S}$ be a semigroup of Schatten p-class operators on a Hilbert space. If $\operatorname{rank}(AB - BA) \leqslant 1$ for all $A, B \in \mathcal{S}$, then $\mathcal{S}$ is triangularizable.*

This was generalized to compact operators on arbitrary Banach spaces in [7, Theorem 9.2.10]. For non-compact operators, this question was studied in a series of papers. In [2, Lemma 5], the authors showed that the same conclusion holds for semigroups of algebraic operators, and in [3], it was shown that every non-commutative doubly generated semigroup $\mathcal{S}$ with the condition that

$$\operatorname{rank}(AB - BA) \leqslant 1$$

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for all $A, B \in \mathcal{S}$ has a hyperinvariant subspace. Finally, it was generalized to arbitrary operators on Banach spaces in [4] as follows:

Theorem 1.2 ([4])**.** *Let X be a Banach space of dimension at least two. Let $\mathcal{S}$ be a non-commutative semigroup of operators on X. If* $\operatorname{rank}(AB - BA) \leqslant 1$ *for all $A, B \in \mathcal{S}$, then $\mathcal{S}$ is reducible.*

It is natural to try to replace the rank-one condition in the above statements with the condition $\operatorname{rank}(AB - BA) \leqslant r$, where $r \in \mathbb{N}$ is fixed. The following quick example shows that one cannot expect the same answer as in Theorem 1.2, even for semigroups of finite-rank operators.

Example 1.3. Let $\mathcal{H}$ be a finite- or infinite-dimensional Hilbert space. For all $i, j = 1, 2, \ldots,$ denote the i, j-matrix unit by E_{ij} . That is, for a fixed orthonormal basis (e_i), we have $E_{ij}(e_k) = \delta_{jk} e_i$. The semigroup

$$\mathcal{S} = \{E_{ij} \ : \ i, j \in \mathbb{N}\} \cup \{0\}$$

is an irreducible semigroup of operators of rank $\leqslant 1$ such that $\operatorname{rank}(AB - BA) \leqslant 2$ for all $A, B \in \mathcal{S}$.

In the present paper, we obtain results regarding the following question: when does the assumption $\operatorname{rank}(AB - BA) \leqslant 2$ for all operators A and B in a semigroup $\mathcal{S}$ imply reducibility of $\mathcal{S}$? Our main argument uses special unitary groups whose structure is also of some independent interest and is a subject of study in the last section of this paper.

Throughout the paper, the linear space $\mathbb{C}^n$ is considered as a Hilbert space with the standard inner product $\langle \cdot, \cdot \rangle$. In the case of infinite-dimensional spaces, the term *operator* is reserved for the bounded linear operators. The set of operators on a Banach space X is denoted by $\mathcal{B}(X)$. The term *invariant subspace* means a non-trivial invariant subspace. A *semigroup* is a set $\mathcal{S}$ of operators on X such that $AB \in \mathcal{S}$ for all $A, B \in \mathcal{S}$. A semigroup $\mathcal{S} \subseteq \mathcal{B}(X)$ is *reducible* if it admits an invariant subspace, and it is *triangularizable* if there exists a chain $\mathcal{C}$ that is maximal as a chain of subspaces of X and that has the property that every member of $\mathcal{C}$ is $\mathcal{S}$-invariant (see [7, Definition 7.1.1]). A semigroup $\mathcal{S} \subseteq \mathcal{B}(X)$ is *irreducible* if it is not reducible. The symbol $\operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ denotes the $n \times n$ diagonal matrix with $\alpha_1, \alpha_2, \ldots, \alpha_n$ on the diagonal. The symbol $\operatorname{nul}(A)$ denotes the dimension of $\ker A$. Finally, we will write $A \equiv B$ if the matrices A and B are unitarily similar.

2. Reducibility of semigroups

We will start by investigating the structure of certain very special groups of unitaries.

Definition 2.1. Let p and q be two prime numbers. The symbol $\mathcal{G}(p, q, A)$ will denote the group of unitaries generated by the $p \times p$ matrices

$$S = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \omega_1 & 0 & \ldots & 0 & 0 \\ 0 & \omega_2 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & \omega_{p-1} & 0 \\ 0 & 0 & \ldots & 0 & \omega_p \end{bmatrix},$$

where A is not a scalar multiple of the identity and $\omega_i^q = 1$ for all $i = 1, 2, \ldots, p$.

Our interest in these groups stems from the fact that if $\mathcal{G}$ is a minimal non-abelian group of matrices, then $\mathcal{G}$ admits a subgroup $\mathcal{G}_0$ whose restriction to a $\mathcal{G}_0$-invariant subspace is closely related to a group of the form $\mathcal{G}(p, q, A)$ (see [7, Lemma 4.2.9]).

Proposition 2.2. *Let p, q be two prime numbers and A be a $p \times p$ matrix as in Definition* 2.1. *If* $\operatorname{rank}(XY - YX) \leqslant 2$ *for all $X, Y \in \mathcal{G}(p, q, A)$, with the equality achieved on some members of it, then either*

(i) $p = 2$ *or*
(ii) $p = 3$ *and* $q = 2$.

Proof. Denote the group $\mathcal{G}(p, q, A)$ by $\mathcal{G}$, for simplicity of notation. It is not hard to see that every member of $\mathcal{G}$ can be written in the form DS^k, where D is a diagonal matrix whose diagonal entries are q-roots of unity, S is the cyclic permutation as in Definition 2.1, and $0 \leqslant k < p$. Moreover, if $X_1 = D_1 S^{k_1}$ and $X_2 = D_2 S^{k_2}$, then $X_1 X_2 = D_3 S^{k_1 + k_2}$, for some diagonal matrix D_3.

Let X and Y be arbitrary members of $\mathcal{G}$. It follows from the above observation that $XYX^{-1}Y^{-1}$ is a diagonal matrix. It is clear that if

$$\operatorname{rank}(XY - YX) = 2,$$

then exactly two eigenvalues of $XYX^{-1}Y^{-1}$ are not equal to one. Since

$$\det(XYX^{-1}Y^{-1}) = 1,$$

we conclude that $XYX^{-1}Y^{-1}$ is of the form

$$\operatorname{diag}(1, \ldots, 1, \omega, 1, \ldots, 1, \bar{\omega}, 1, \ldots, 1),$$

where $\omega \neq 1$ and $\omega^q = 1$, and each of the series of ones between ω and $\bar{\omega}$ could be absent.

Observe that if $D = \operatorname{diag}(d_1, \ldots, d_{p-1}, d_p)$, then $SDS^{-1} = \operatorname{diag}(d_p, d_1, \ldots, d_{p-1})$. It follows that $\mathcal{G}$ has a member of the form

$$A_0 = \operatorname{diag}(\omega, 1, \ldots, 1, \bar{\omega}, 1, \ldots, 1),$$

where $\omega \neq 1$, $\omega^q = 1$, and the series of ones between ω and $\bar{\omega}$ is shorter than the series of ones following $\bar{\omega}$.

Suppose that $p > 3$, so that $p \geqslant 5$. If the series of ones between ω and $\bar{\omega}$ is not absent, then consider $B = SA_0^{-1}S^{-1}$. It follows that

$$A_0 B = A_0 S A_0^{-1} S^{-1} = \operatorname{diag}(\omega, \bar{\omega}, 1, \ldots, 1, \bar{\omega}, \omega, 1, \ldots, 1),$$

so that $\operatorname{rank}(A_0 S - S A_0) = \operatorname{rank}(A_0 S A_0^{-1} S^{-1} - I) = 4$, contrary to the assumptions. So, the series of ones between ω and $\bar{\omega}$ must be absent, and

$$A_0 = \operatorname{diag}(\omega, \bar{\omega}, 1, \ldots, 1).$$

However, in this case we may consider $C = S^2 A_0^{-1} S^{-2}$. We get

$$A_0 C = A_0 S^2 A_0^{-1} S^{-2} = \operatorname{diag}(\omega, \bar{\omega}, \bar{\omega}, \omega, 1, \ldots, 1),$$

so that $\operatorname{rank}(A_0 S^2 - S^2 A_0) = 4$.

This shows that either $p = 2$ or $p = 3$. Suppose that $p = 3$. We claim that, necessarily, $q = 2$. Assume that $q > 2$. Then, by the same argument as above,

$$A_0 = \operatorname{diag}(\omega, \bar{\omega}, 1) \in \mathcal{G},$$

where $\omega \neq 1$ and $\omega^q = 1$. Clearly, $SA_0^{-1}S^{-1} = \operatorname{diag}(1, \bar{\omega}, \omega)$, so that

$$A_0 S A_0^{-1} S^{-1} = \operatorname{diag}(\omega, \bar{\omega}^2, \omega).$$

If $q > 2$, then all the diagonal entries of this matrix are different from 1, so that rank $(A_0S - SA_0) = 3$, a contradiction. □

The next proposition records certain observations about the groups $\mathcal{G}$ satisfying rank $(XY - YX) \leqslant 2$ for all $X, Y \in \mathcal{G}$. We will need the following notation.

Definition 2.3. Let $\mathcal{S}$ be a set of $n \times n$ matrices and $\mathcal{M}$ be a linear subspace of $\mathbb{C}^n$. Then we put

$$\mathcal{S}(\mathcal{M}) = \{T \in \mathcal{S} \,:\, T\mathcal{M} \subseteq \mathcal{M}\}.$$

Proposition 2.4. *Let $\mathcal{G}$ be a non-abelian group of unitary $n \times n$ matrices, and assume* $\operatorname{rank}(AB - BA) \leq 2$ *for all $A, B \in \mathcal{G}$. If $\mathcal{M}$ is a linear subspace of $\mathbb{C}^n$, then $\mathcal{G}(\mathcal{M}) = \mathcal{G}(\mathcal{M}^\perp)$ is a subgroup of $\mathcal{G}$ and at least one of the unitary groups $\mathcal{G}(\mathcal{M})|_{\mathcal{M}}$ or $\mathcal{G}(\mathcal{M})|_{\mathcal{M}^\perp}$ is abelian.*

Proof. If $n \leqslant 2$, then the conclusions of the proposition are evident. Therefore, we will assume in the proof that $n \geqslant 3$.

Let $A, B \in \mathcal{G}$. Since $ABA^{-1}B^{-1}$ is a unitary and rank $(ABA^{-1}B^{-1} - I) = \operatorname{rank}(AB - BA) \leq 2$, it follows from the first paragraph of the introduction that rank $(AB - BA)$ is 0 or 2, and hence $ABA^{-1}B^{-1} \equiv \operatorname{diag}(\omega, \omega', 1, 1, \cdots, 1) \neq I$ for some $\omega \neq 1 \neq \omega'$. Also, since $1 = \det(ABA^{-1}B^{-1}) = \omega\omega'$, it follows that $\omega' = \bar{\omega}$.

Next, assume $\mathcal{M}$ is a linear subspace of $\mathbb{C}^n$. Clearly, $\mathcal{G}(\mathcal{M}) = \mathcal{G}(\mathcal{M}^\perp)$. Assume, if possible, that both $\mathcal{G}(\mathcal{M})|_{\mathcal{M}}$ and $\mathcal{G}(\mathcal{M})|_{\mathcal{M}^\perp}$ are non-abelian. For $i = 1, 2$, choose $A_i = C_i \oplus D_i \in \mathcal{G}(\mathcal{M})$ decomposed according to $\mathbb{C}^n = \mathcal{M} \oplus \mathcal{M}^\perp$, such that $C_1C_2 \neq C_2C_1$. Notice that the condition $D_1D_2 \neq D_2D_1$ would imply

$$\operatorname{rank}(A_1A_2 - A_2A_1) = 4;$$

hence $D_1D_2 = D_2D_1$. Assume, if possible, that D_1 is not in the centre of $\mathcal{G}(\mathcal{M})|_{\mathcal{M}^\perp}$. In this case, choose $A_3 = C_3 \oplus D_3 \in \mathcal{G}(\mathcal{M})$ such that $D_1D_3 \neq D_3D_1$ and, consequently, $C_3C_1 = C_1C_3$. Then $C_1(C_2C_3) \neq C_2C_1C_3 = (C_2C_3)C_1$ and $D_1(D_2D_3) = D_2D_1D_3 \neq (D_2D_3)D_1$, which is a contradiction. Thus D_1 and, by symmetry, D_2 belong to the centre of $\mathcal{G}(\mathcal{M})|_{\mathcal{M}^\perp}$. Now, since $\mathcal{G}(\mathcal{M})|_{\mathcal{M}^\perp}$ is not abelian, there exist $A_3 = C_3 \oplus D_3$ and $A_4 = C_4 \oplus D_4$ in $\mathcal{G}(\mathcal{M})$ such that $C_3C_4 = C_4C_3$ and $D_3D_4 \neq D_4D_3$. Another symmetrical argument reveals that C_3, C_4 belong to the centre of $\mathcal{G}(\mathcal{M})|_{\mathcal{M}}$. Then $(C_1C_3)(C_2C_4) = C_1C_2C_3C_4 \neq C_2C_1C_3C_4 = (C_2C_4)(C_1C_3)$ and $(D_1D_3)(D_2D_4) = D_2D_3D_4D_1 \neq D_2D_4D_3D_1 = (D_2D_4)(D_1D_3)$, and, hence, $\operatorname{rank}[(A_1A_3)(A_2A_4) - (A_2A_4)(A_1A_3)] = 4$, a contradiction. □

Before we state our main theorem, we need two lemmas.

Lemma 2.5. *Let $\mathcal{G}$ be a non-abelian unitary group on $\mathbb{C}^n$ and $\mathcal{N} \subseteq \mathbb{C}^n$ be a 3-dimensional subspace. Assume that $\mathcal{G}$ has a subgroup $\mathcal{G}_0$ such that $\mathcal{N}$ is $\mathcal{G}_0$-invariant and, in some basis $\{e_1, e_2, e_3\}$ of $\mathcal{N}$, $\mathcal{G}_0|_{\mathcal{N}} = \mathcal{G}(3, q, A)$, where q is a prime number and A is a diagonal matrix as in Definition* 2.1. *If* rank $(XY - YX) \leqslant 2$ *for all $X, Y \in \mathcal{G}$, then $\mathcal{N}$ is $\mathcal{G}$-invariant. Moreover, $\mathcal{G}|_{\mathcal{N}}$ is irreducible and $\mathcal{G}|_{\mathcal{N}^\perp}$ is abelian.*

Proof. Since G is non-abelian, in view of the observation made at the beginning of the introduction, rank $(XY - YX) = 2$, for some X, $Y \in \mathcal{G}$. By Proposition 2.2, q must be equal to 2. Considering matrices of the form $XYX^{-1}Y^{-1}$, as in the proof of Proposition 2.2, we conclude that $\mathcal{G}(3, 2, A)$ admits a diagonal matrix B with eigenvalues $\{1,-1,-1\}$. Considering SBS^{-1} and S^2BS^{-2}, where S is the cyclic permutation as in Definition 2.1, we conclude that the matrices

$$\operatorname{diag}(1,-1,-1), \quad \operatorname{diag}(-1,1,-1), \quad \text{and} \quad \operatorname{diag}(-1,-1,1)$$

all belong to $\mathcal{G}(3,2,A)$.

Pick an arbitrary $Z \in \mathcal{G}$ and assume that $\mathcal{N}$ is not Z-invariant. Fix a matrix $\widetilde{S} \in \mathcal{G}$ such that $\widetilde{S}|_{\mathcal{N}} = S$. Choose a basis $\{e_4, e_5, \cdots, e_n\}$ for $\mathcal{N}^{\perp}$ consisting of eigenvectors of $\widetilde{S}$. Since $\mathcal{N}$ (and, hence, $\mathcal{N}^{\perp}$) is not Z-invariant, there exist $i \leq 3$ and $j \geq 4$ such that $\langle Ze_j, e_i \rangle \neq 0$. Due to the cyclic nature of the conditions of the theorem with respect to the ordered triple (e_1, e_2, e_3), we may and shall assume without loss of generality that $i = 1$. Let $\mathcal{M}$ be the 2-dimensional subspace of $\mathbb{C}^n$ spanned by $\{e_1, e_2\}$ and write

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \text{ with respect to } \mathbb{C}^n = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

The matrix $\operatorname{diag}(-1,-1,1) \in \mathcal{G}(3,2,A)$ can be obtained as $CSC^{-1}S^{-1}$, where $C = \operatorname{diag}(1,-1,-1) \in \mathcal{G}(3,2,A)$. This shows that if $\widetilde{C} \in \mathcal{G}$ is such that $\widetilde{C}|_{\mathcal{N}} = C$, then the matrix

$$T = \widetilde{C}\widetilde{S}\widetilde{C}^{-1}\widetilde{S}^{-1} = \operatorname{diag}(-1,-1,1,1,\ldots,1) \in \mathcal{G}.$$

With respect to $\mathbb{C}^n = \mathcal{M} \oplus \mathcal{M}^{\perp}$, this matrix has the form

$$T = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

Then

$$TZ - ZT = \begin{bmatrix} 0 & 2Z_{12} \\ -2Z_{21} & 0 \end{bmatrix}.$$

Since $\operatorname{rank}(TZ - ZT) \leqslant 2$, we conclude that $\operatorname{rank}(Z_{12}) = \operatorname{rank}(Z_{21}) = 1$, for none of Z_{12} and Z_{21} are zero. It follows that $Z_{12}e_k$ $(k = 3, 4, \cdots, n)$ are multiples of $Z_{12}e_j$. Replacing Z by $Z\widetilde{S}$ changes the first column $Z_{12}e_3$ of Z_{12} to $Z_{11}e_1$ and its $(j-2)^{\text{nd}}$ column $Z_{12}e_j$ to $\lambda_j Z_{12}e_j$, where λ_j is the eigenvalue of $\widetilde{S}$ corresponding to e_j. Thus, again, $Z_{11}e_2$ is a multiple of $Z_{12}e_j$. Another replacement of Z by $Z\widetilde{S}^2$ reveals that the first two rows of Z are linearly dependent, a contradiction. This shows that $\mathcal{N}$ is $\mathcal{G}$-invariant.

Finally, the irreducibility of $\mathcal{G}|_{\mathcal{N}}$ follows from the fact that $\mathcal{G}(p,q,A)$ is irreducible (see, e.g., [7, Lemma 4.2.8]), and the commutativity of $\mathcal{G}|_{\mathcal{N}^{\perp}}$ was established in Proposition 2.4. ☐

Lemma 2.6. *Let $\mathcal{S}$ be a semigroup of $n \times n$ matrices and $\mathcal{N}$ be a subspace of $\mathbb{C}^n$ such that, with respect to the decomposition $\mathbb{C}^n = \mathcal{N} \oplus \mathcal{N}^{\perp}$, the representation of every member $Z \in \mathcal{S}$,*

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

has the property that $\operatorname{rank}(Z_{21}) \leqslant 1$. *Then each $Z \in \mathcal{S}$ admits an invariant subspace $\mathcal{N}_Z$ such that either $\mathcal{N}_Z \subseteq \mathcal{N}$, in which case* $\dim(\mathcal{N}/\mathcal{N}_Z) \leqslant 1$, *or $\mathcal{N} \subseteq \mathcal{N}_Z$, in which case* $\dim(\mathcal{N}_Z/\mathcal{N}) \leqslant 1$ *and* $\mathcal{N}_Z = \operatorname{span}\{\mathcal{N}, Z\mathcal{N}\}$.

Proof. Denote the dimension of $\mathcal{N}$ by k. Clearly, there is no loss of generality in assuming that $2 \leqslant k \leqslant n-2$.

Let $Z \in \mathcal{S}$ be such that $\mathcal{N}$ is not Z-invariant. Since $\operatorname{rank}(Z_{21}) = 1$, by choosing appropriate bases $\{e_1, \ldots, e_k\}$ for $\mathcal{N}$ and $\{e_{k+1}, \ldots, e_n\}$ for $\mathcal{N}^\perp$, we may assume that only the $(1,1)$-entry of Z_{21} is non-zero.

Consider the matrix $Z^2 \in \mathcal{S}$. Its $(2,1)$-block is equal to $Z_{21}Z_{11} + Z_{22}Z_{21}$. Notice that only the first row of the matrix $Z_{21}Z_{11}$ may contain non-zero entries and only the first column of the matrix $Z_{22}Z_{21}$ may contain non-zero entries. Since the rank of $Z_{21}Z_{11} + Z_{22}Z_{21}$ is assumed to be at most one, we conclude that one of the matrices $Z_{21}Z_{11}$ or $Z_{22}Z_{21}$ must satisfy the property that all its entries, except perhaps the $(1,1)$-entry, are equal to zero. If all but the $(1,1)$-entry of $Z_{21}Z_{11}$ are zero, then Z_{11} (and, hence, Z) leaves invariant the space $\operatorname{span}\{e_2, \ldots, e_k\}$. If all but the $(1,1)$-entry of $Z_{22}Z_{21}$ are zero, then in the first column of Z_{22} only the first entry may be non-zero, so that Z leaves invariant the space $\operatorname{span}\{e_1, \ldots, e_k, e_{k+1}\}$. $\square$

Now we are ready to prove the main theorem of the paper.

Theorem 2.7. *Let $\mathcal{G}$ be a group of unitary $n \times n$ matrices. If $\operatorname{rank}(AB - BA) \leqslant 2$ for all $A, B \in \mathcal{G}$, then there is a subspace $\mathcal{M}$ of $\mathbb{C}^n$ such that $1 \leqslant \dim \mathcal{M} \leqslant 3$ and $\mathcal{G} \subseteq \mathcal{G}_1 \oplus \mathcal{G}_2$ with $\mathcal{G}_2$ abelian, where the direct sum is with respect to the decomposition $\mathbb{C}^n = \mathcal{M} \oplus \mathcal{M}^\perp$.*

Proof. Clearly, there is no loss of generality in assuming that $\mathcal{G}$ is not abelian and $n \geqslant 4$. Moreover, we may also assume that $\mathcal{G} = \overline{\mathbb{T}\mathcal{G}}$, where $\mathbb{T}$ is the unit circle on the complex plane.

Since $\mathcal{G} = \overline{\mathcal{G}}$, it is a compact Lie group, so [1, Theorem 5] implies that $\mathcal{G}$ contains a finite non-abelian subgroup. It follows that $\mathcal{G}$ contains a minimal non-abelian subgroup. By [7, Lemma 4.2.9], every minimal non-abelian finite group admits an invariant subspace $\mathcal{N}$ such that the restriction of the group to $\mathcal{N}$ is, after a similarity, generated by two matrices αA and βS, where A is a non-scalar diagonal matrix, S is the cyclic permutation, and $\alpha, \beta \in \mathbb{T}$. Since $\mathcal{G} = \mathbb{T}\mathcal{G}$, we conclude that $\mathcal{G}$ contains a subgroup $\mathcal{G}_0$ whose restriction to $\mathcal{N}$ is equal (in an appropriate basis) to the group $\mathcal{G}(p, q, A)$.

It follows from Proposition 2.2 and Lemma 2.5 that, without loss of generality, $p = 2$. Since $\mathcal{G}(2, q, A)$ is not abelian, it contains a matrix C of the form $XYX^{-1}Y^{-1}$ different from the identity. By the properties of $\mathcal{G}(p, q, A)$, this matrix is necessarily diagonal, and its diagonal entries are q-roots of the unity. Since $\det(C) = 1$, we have $C = \operatorname{diag}(\omega, \bar{\omega})$, for some $\omega \neq 1$, $\omega^q = 1$.

If $Z \in \mathcal{G}$ is an arbitrary matrix, then, considering the rank of $ZC - CZ$, we conclude that, with respect to the decomposition $\mathbb{C}^n = \mathcal{N} \oplus \mathcal{N}^\perp$, Z is represented as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

where $\operatorname{rank}(Z_{21}) \leqslant 1$. By Lemma 2.6, either Z admits an eigenvector in $\mathcal{N}$ or the space $\operatorname{span}\{\mathcal{N}, Z\mathcal{N}\}$ has dimension 3 and is Z-invariant. Notice that this space contains $\mathcal{N}$ as a subspace of codimension one.

First, we claim that, assuming $\mathcal{N}$ is not $\mathcal{G}$-invariant, $\mathcal{G}$ admits a matrix without an eigenvector in $\mathcal{N}$.

Indeed, let $V \in \mathcal{G}$ be such that $\mathcal{N}$ is not V-invariant. If V does not have eigenvectors in $\mathcal{N}$, we are done. Suppose that V has an eigenvector in $\mathcal{N}$. Write V as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Let $f \in \mathcal{N}$ be an eigenvector of V. Clearly, $\text{span}\{f\} = \ker(V_{21})$ and f is an eigenvector for V_{11}. Since $G(2, q, A)$ is irreducible (see, e.g., [7, Lemma 4.2.8]), there exists $U \in G(2, q, A)$ such that f is not an eigenvector of UV_{11}. There exists a matrix $Z \in \mathcal{G}$ of the form $U \oplus D$, where D is a unitary $(n-2) \times (n-2)$ matrix. Since $\ker(DV_{21}) = \ker(V_{21}) = \text{span}\{f\}$, the matrix ZV does not admit eigenvectors in $\mathcal{N}$.

Let $T \in \mathcal{G}$ be a matrix without eigenvectors in $\mathcal{N}$. Since T is a unitary matrix, every invariant subspace of it is reducing. By Lemma 2.6, there exist an orthonormal basis $\{e_1, e_2\}$ of $\mathcal{N}$ and a unit vector e_3 in $\mathcal{N}^\perp$ such that, relative to the decomposition $\mathbb{C}^n = \text{span}\{e_1\} \oplus \text{span}\{e_2\} \oplus \text{span}\{e_3\} \oplus (\mathcal{N}^\perp \ominus \text{span}\{e_3\})$, T is written in the form

$$T = \begin{bmatrix} p & q & w & 0 \\ r & s & u & 0 \\ 0 & t & v & 0 \\ 0 & 0 & 0 & U \end{bmatrix},$$

where $r \neq 0$, $t \neq 0$, and U is an $(n-3) \times (n-3)$ unitary matrix.

Let $S \in \mathcal{G}$ be arbitrary. Write, relative to the same decomposition,

$$S = \begin{bmatrix} a & b & * & * \\ c & d & * & * \\ e & f & * & * \\ g & h & * & * \end{bmatrix},$$

where a, b, c, d, e, f are complex numbers, g and h are $(n-3)$-vectors, and the symbol $*$ stands for a number or a matrix whose value does not concern us. Multiplying T by S, we get

$$TS = \begin{bmatrix} * & * & * & * \\ ar + cs + eu & br + ds + fu & * & * \\ ct + ev & dt + fv & * & * \\ Ug & Uh & * & * \end{bmatrix}.$$

Recall that

$$\text{rank}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \leqslant 1 \quad \text{and} \quad \text{rank}\left(\begin{bmatrix} ct + ev & dt + fv \\ Ug & Uh \end{bmatrix}\right) \leqslant 1.$$

Suppose that one of the vectors g and h is not zero, say, $g \neq 0$. Then there exists $\alpha \in \mathbb{C}$ such that $h = \alpha g$, $f = \alpha e$ and $dt + fv = \alpha(ct + ev)$. Since $t \neq 0$, we conclude that $d = \alpha c$. It follows that

$$\text{rank}\left(\begin{bmatrix} c & d \\ e & f \\ g & h \end{bmatrix}\right) = 1.$$

Repeating the same argument with the matrix TS replacing the matrix S, we obtain

$$\text{rank}\left(\begin{bmatrix} ar + cs + eu & br + ds + fu \\ ct + ev & dt + fv \\ Ug & Uh \end{bmatrix}\right) = 1.$$

It follows that $br + ds + fu = \alpha(ar + cs + eu)$. Since $r \neq 0$, the only possibility is that $b = \alpha a$. However, this implies that

$$\operatorname{rank}\left(\begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}\right) = 1.$$

This is impossible since the matrix S is unitary, hence invertible.

The case $h \neq 0$ brings us to the same conclusion. Therefore $g = h = 0$. Since S was chosen arbitrarily, this implies that the space $\mathcal{M} = \mathcal{G}\mathcal{N}$ is $\mathcal{G}$-invariant and $\mathcal{M} \perp (\mathcal{N}^{\perp} \ominus \operatorname{span}\{e_3\})$. Under the assumption that $\mathcal{N}$ is not $\mathcal{G}$-invariant, this means that $\mathcal{M} = \operatorname{span}\{e_1, e_2, e_3\}$, a 3-dimensional $\mathcal{G}$-invariant subspace. The rest of the conclusions of the theorem follow from Proposition 2.4. □

Corollary 2.8. *Let X be a Banach space and $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$ be a semigroup of operators on X containing a non-zero compact operator such that the minimal rank of non-zero operators in $\mathcal{S}$ is at least* 4. *If* $\operatorname{rank}(AB - BA) \leqslant 2$ *for all $A, B \in \mathcal{S}$, then $\mathcal{S}$ is reducible.*

Proof. Suppose that $\mathcal{S}$ is irreducible. It is well-known that a non-trivial ideal of an irreducible semigroup is irreducible. Thus, there is no loss of generality in assuming that $\mathcal{S}$ consists of compact operators.

Denote the minimal non-zero rank of operators in $\mathcal{S}$ by r. By [7, Lemma 8.1.15], r is finite and there exists an idempotent $E \in \mathcal{S}$ of rank r. Let $\mathcal{S}_0 = E\mathcal{S}E|_{\operatorname{Range} E}$. Then $\mathcal{S}_0$ is represented as a semigroup of $r \times r$ matrices. Moreover, every member of this semigroup is either invertible or zero, by the minimality of the rank r in $\mathcal{S}$. Also, as a compression of an irreducible semigroup, the semigroup $\mathcal{S}_0$ must be irreducible. By [7, Lemma 3.1.6], $\mathcal{S}_0 \setminus \{0\}$ is a group of matrices. Moreover, there exists a group $\mathcal{G}$ of unitary matrices such that, after a similarity, $\mathcal{S}_0 \setminus \{0\} \subseteq \mathbb{R}^+\mathcal{G}$. Clearly, $\mathcal{G}$ must be irreducible, too. Also, the proof of [7, Lemma 3.1.6] shows that the group $\mathcal{G}$ is, in fact, similar to the group $\{\frac{1}{r(T)}T : T \in \mathcal{G}_0\}$. Hence, the condition $\operatorname{rank}(AB - BA) \leqslant 2$ holds for all $A, B \in \mathcal{G}$. This, obviously, contradicts the conclusion of Theorem 2.7. □

We remark that the condition about the rank in Corollary 2.8 cannot be improved. This is clear if the minimal rank is allowed to be equal to 2 (take, for example, the group of 2×2 unitaries). The following proposition exhibits an example of an irreducible group of 3×3 unitary matrices with the property $\operatorname{rank}(AB - BA) \leqslant 2$ for all A, B in the group.

Proposition 2.9. *Let*

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then the group $\mathcal{G} = \langle T, S \rangle$ is irreducible and satisfies the condition that

$$\operatorname{rank}(AB - BA) \leqslant 2$$

for all $A, B \in \mathcal{G}$.

Proof. By [7, Lemma 4.2.8], the group $\mathcal{G}$ is irreducible. Let us show that

$$\operatorname{rank}(AB - BA) \leqslant 2$$

for all $A, B \in \mathcal{G}$.

Observe that every member of $\mathcal{G}$, being a finite product of matrices T, S, T^{-1} and S^{-1}, can be written in one of the following three forms:

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{bmatrix},$$

with $\alpha, \beta, \gamma \in \{1, -1\}$. Moreover, among the numbers α, β, γ exactly two or none are equal to -1, the rest being equal to 1. For $A \in \mathcal{G}$, let us refer to the particular form of A among the three forms above as the *pattern* of A.

A routine check shows that for all matrices A and $B \in \mathcal{G}$, the patterns of AB and BA are the same. Hence, the difference $AB - BA$ must have the same pattern, too. Now, since there are exactly zero or two elements equal to -1 among the non-zero elements of AB and BA, a quick check shows that there is at least one entry (i, j) such that $(AB)_{ij}$ and $(BA)_{ij}$ are both equal to 1 or to -1 simultaneously. But this means that the difference $AB - BA$ has at most two non-zero entries, so that $\operatorname{rank}(AB - BA) \leqslant 2$. □

3. On the structure of the group $\mathcal{G}(p, q, A)$

For prime numbers p and q, let $\mathcal{G} = \mathcal{G}(p, q, A)$ be the irreducible group with generators A and S as defined before. These groups played a central role in our arguments from Section 2. In the present section, we will further study the structure of these groups in terms of the following parameters:

$$\rho = \min\{\operatorname{rank}(D - I) \neq 0 : D \in \mathcal{G}; D \text{ diagonal}\}, \tag{3.1}$$

$$r = \max\{\operatorname{rank}(XYX^{-1}Y^{-1} - I) : X, Y \in \mathcal{G}\}. \tag{3.2}$$

Note. Clearly, $1 \le \rho \le r \le p$.

Throughout the remainder of the paper, $\mathcal{G} = \mathcal{G}(p, q, A)$ for some p, q, A. If p, q are fixed, we may also write $\mathcal{G}_A$, ρ_A and r_A to denote $\mathcal{G}(p, q, A)$, ρ and r, respectively.

Theorem 3.1. *Let $\mathcal{D}_A$ be the collection of all diagonal matrices in $\mathcal{G}_A = \mathcal{G}(p, q, A)$ and let $\mathfrak{S}$ be the subgroup generated by S. Also, let $\mathcal{C}_A$ be the commutator subgroup of $\mathcal{G}_A$. Then $\mathcal{G}_A = \mathcal{D}_A\mathfrak{S} = \mathfrak{S}\mathcal{D}_A$ and $\mathcal{C}_A \subset \mathcal{D}_A$. Moreover, if $\mathcal{C}_A \neq \mathcal{D}_A$, then one of the following cases holds:*

(i) *$\mathcal{C}_A$ contains no non-scalar matrix. Then $p/2 \le \rho_A \le r_A = p = q$ and $\mathcal{C}_A = \{\eta I : \eta^p = 1\}$.*

(ii) *$\mathcal{C}_A$ contains non-scalar matrices, and for any non-scalar $B \in \mathcal{C}_A$, $\mathcal{C}_B = \mathcal{D}_B$, $2 \le \rho_B \le r_B \le r_A$ and $\rho_A \le \rho_B \le 2\rho_A$.*

Proof. For convenience, we drop the subscript A and will only maintain the subscript B to avoid confusion. Consider the general word

$$G = A^{\alpha_1}S^{\beta_1}A^{\alpha_2}S^{\beta_2}\cdots A^{\alpha_m}S^{\beta_m} \in \mathcal{G} \tag{3.3}$$

for some integers $m, \alpha_1, \beta_1, \cdots, \alpha_m, \beta_m$. Since

$$S^{\beta}A^{\alpha}S^{-\beta} \in \mathcal{D}, \ \forall \alpha, \beta \in \mathbb{Z}, \tag{3.4}$$

it follows that every word of the form (3.3) can be rewritten as

$$G = DS^{\gamma}, \text{ for some } D \in \mathcal{D}, \tag{3.5}$$

where $\gamma = \beta_1 + \beta_2 + \cdots + \beta_m$. Now, G is diagonal if and only if $\gamma = 0 \pmod p$). Then $\mathcal{G} = \mathcal{D}\mathfrak{S}$ and $\mathcal{C} \subset \mathcal{D}$. Since $\mathcal{G}^{-1} = \mathcal{G}$, it follows that $\mathcal{G} = \mathfrak{S}\mathcal{D}$.

To prove (i), assume $\mathcal{C}$ contains no non-scalar matrix. Since $\mathcal{C} \neq \{I\}$, there exists $C = \eta I$ for some complex number $\eta \neq 1$ and some $C \in \mathcal{C}$. It is easy to see that $\omega^q = 1$. Also, $\omega^p = \det(C) = 1$. Hence, $q|p$, and thus $q = p = r$. Since $\mathcal{C}$ is a group, $\mathcal{C} = \{\eta I : \eta^p = 1\}$. Now, if $\operatorname{rank}(D - I) = \rho < p/2$, then D and $SD^{-1}S^{-1}$ each have at most ρ entries different from 1 and, hence, $DSD^{-1}S^{-1} \neq \eta I$ for some $\eta \in \mathbb{C}$, a contradiction.

For (ii), assume there exists a non-scalar $B \in \mathcal{C}$. Then the subgroup $\mathcal{G}_B$ of $\mathcal{G}$ is non-abelian and the relations (3.4) and (3.5) can be sharpened as follows:

$$S^{\beta}B^{\alpha}S^{-\beta} = B^{\alpha}B^{-\alpha}S^{\beta}B^{\alpha}\ S^{-\beta} \in \mathcal{C}_B, \ \forall \alpha, \beta \in \mathbb{Z}, \tag{3.6}$$

$$G = DS^{\gamma}, \text{ for some } D \in \mathcal{C}_B. \tag{3.7}$$

This shows that $\mathcal{D}_B \subset \mathcal{C}_B$, which proves $\mathcal{D}_B = \mathcal{C}_B$. Since $\det(C) = 1$ for all $C \in \mathcal{C}$, it follows that $\operatorname{rank}(D - I) \geq 2$ whenever $I \neq D \in \mathcal{D}$. The inequality $\rho_B \leq 2\rho_A$ follows from the fact that if $\operatorname{rank}(D - I) = \rho$, then $\operatorname{rank}(DSD^{-1}S^{-1} - I) \leq 2\rho$ and the rest of (ii) is clear. □

The next corollary studies the case $\rho = 1$. We continue to use the notation established in the previous paragraphs.

Corollary 3.2. *It is always true that* $2 \leq r \leq p$ *and, if* $I \neq C \in \mathcal{C}$, *then*

$$\operatorname{rank}(C - I) \geq 2.$$

In particular, if $\rho = 1$, *then one of the following cases holds:*

(i) *In this case,* $\mathcal{C} = \{I, -I\} \subset \mathcal{D} = \{I, -I, \operatorname{diag}(1, -1), \operatorname{diag}(-1, 1)\}$.

(ii) $\mathcal{C}$ *contains non-scalar matrices, and for any non-scalar* $B \in \mathcal{C}$, $\rho_B \geq 2$ *and* $\mathcal{C}_B = \mathcal{D}_B$. *The lower bound* 2 *is attained for some* B.

If $p = q = 2$, *then* $\rho{=}1$, $\mathcal{D}{=}\{I, -I, \operatorname{diag}(1, -1), -\operatorname{diag}(1, -1)\}$ *and* $\mathcal{C} = \{I, -I\}$.

Proof. Observe that if $\operatorname{rank}(X^{-1}Y^{-1}XY - I) = 1$, then $1 \neq \det(X^{-1}Y^{-1}XY) = 1$, a contradiction. Thus, $2 \leq r \leq p$. Now, if $D \in \mathcal{D}$ and $\operatorname{rank}(D - I) = 1$, then $\det(D) \neq 1$ and, hence, $D \notin \mathcal{C}$. In particular, if $\rho = 1$, then $\mathcal{D} \neq \mathcal{C}$ and, in view of Theorem 3.1, one of the following cases holds.

Case 1. $p/2 \leq 1 \leq r = p = q$, which implies that $r = p = q = 2$ and $\mathcal{D} \neq \mathcal{C} = \{I, -I\}$. Thus $\mathcal{D} = \{I, -I, \operatorname{diag}(1, -1), \operatorname{diag}(-1, 1)\}$ is the only choice left.

Case 2. There exists a non-scalar $B \in \mathcal{C}$, and for any such B, $\mathcal{C}_B = \mathcal{D}_B$ and $\rho_B \geq 2$. Now, if $D \in \mathcal{D}$ has exactly one diagonal entry different from 1, then $DSD^{-1}S^{-1}$ is a commutator with exactly two diagonal entries different from 1.

Conversely, if $p = q = 2$, then $\operatorname{rank}(C - I) = 2$ whenever $I \neq C \in \mathcal{C}$, which implies that $\mathcal{D} \neq \{I, -I\} = \mathcal{C}$. Thus, $\mathcal{D} = \mathcal{C} \cup \{\operatorname{diag}(1, -1), \operatorname{diag}(-1, 1)\}$ and, hence, $\rho = 1$. □

The following theorem studies the case $\rho = 2$.

Theorem 3.3. *If $\rho = 2$, then either*

(i) *$r = p$ and $q > 2$ or*

(ii) *$r = p - 1$, $q = 2$.*

Proof. If $p = 2$, then $r = 2$. Also, $q > 2 = p$ by Corollary 3.2.

So, we assume $p \geq 3$. Let $\mathcal{D}_2$ be the (non-empty) collection of all matrices $D \in \mathcal{D}$ such that exactly $p-2$ entries on the main diagonal of D are equal to 1. We claim there exists $\Delta \in \mathcal{D}_2$ for which exactly the first two diagonal entries are different from 1. Let s be the minimal positive integer for which there exist a positive integer h and a matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p) \in \mathcal{D}_2$ such that $\lambda_h \neq 1$ and $\lambda_{h+s} \neq 1$. Examining $S^{-h+1}DS^{h-1}$ and $S^{-h-s+1}DS^{h+s-1}$ reveals that $1 \leq s < p/2$ and allows us to assume without loss of generality that $h = 1$. Let $p - 1 = ms + t$ for some non-negative integers m, t with $0 \leq t \leq s - 1$, and, in fact, since p is an odd prime, it follows that either $s = 1$ or $0 \leq t \leq s - 2$. Let $\lambda_1 = \omega$ and $\lambda_{s+1} = \omega^a \neq 1$ for some primitive q^{th} root ω of 1 and some positive integer $a < q$. For $1 \leq k \leq m-1$, assume $\Delta_1, \Delta_2, \cdots, \Delta_k \in \mathcal{D}_2$ are constructed such that $\Delta_1 = D$ and the first and the $(ks+1)^{\text{th}}$ diagonal entries of Δ_k are ω^{ϵ_k} and ω^{a^k}, respectively, where $\epsilon_k := (-1)^{k+1}$. Define $\Delta_{k+1} = S^{ks}D^{a^k}S^{-ks}\Delta_k^{-1}$. This finite induction yields $\Delta_m \in \mathcal{D}_2$, whose first diagonal entry is ω^{ϵ_m} and whose $(ms+1)^{\text{th}}$ diagonal entry is ω^{a^m} (necessarily, $\neq 1$). Now, observe that the first and the $(t+2)^{\text{nd}}$ diagonal entries of $S^{t+1}\Delta_m S^{-t-1} \in \mathcal{D}_2$ are ω^{a^m} and ω^{ϵ_m}, respectively. Since all other entries are equal to 1 and $\omega^{\epsilon_m} \neq 1$, it follows that $\omega^{a^m} \neq 1$. By minimality, $t + 2 \geq s + 1$; hence, $s = 1$ and $t = 0$.

Thus, there exists $k \in \{1, 2, \cdots, q-1\}$ such that

$$\Delta = \mathrm{diag}(\omega, \omega^k, 1, 1, \cdots, 1) \in \mathcal{D}_2. \tag{3.8}$$

Let $\Omega := \Gamma S \Gamma^{-1} S^{-1} \in \mathcal{C}$, where

$$\Gamma = \mathrm{diag}(\omega, \omega^k, \omega, \omega^k, \cdots, \omega, \omega^k, 1) = \Pi_{j=0}^{(p-3)/2} S^{2j}\Delta S^{-2j} \in \mathcal{D}. \tag{3.9}$$

Hence

$$\Omega = \mathrm{diag}(\omega, \omega^{k-1}, \omega^{1-k}, \omega^{k-1}, \cdots, \omega^{1-k}, \omega^{k-1}, \omega^{-k}). \tag{3.10}$$

Let us assume $q \geq 3$ and settle the problem in this case. We claim $k \geq 2$; otherwise,

$$S^{-1}\Delta S \Pi_{i=1}^{p-2} S^i \Delta^{(-1)^i} S^{-i} = \mathrm{diag}(\omega^2, 1, 1, \cdots, 1) \in \mathcal{D}$$

and $\mathrm{rank}(D - I) = 1$, a contradiction. Therefore, $k \geq 2$ and the proof of part (i) follows from the fact that $r = \mathrm{rank}(\Omega - I) = p$.

All we have to do now is settle the case $p > q = 2$. In (3.9), $\omega = \omega^k = -1$, and one can deduce that

$$\Delta' := \Delta S \Delta S^{-1} = \mathrm{diag}(-1, 1, -1, 1, 1, \cdots, 1) \in \mathcal{D}_2. \tag{3.11}$$

Choose a positive integer u such that $p = 4u \pm 1$. Define $\Omega' := \Gamma' S (\Gamma')^{-1} S^{-1} \in \mathcal{C}$, where

$$\Gamma' = \mathrm{diag}(-1, 1, -1, 1, \cdots, -1, 1, -1) = \Pi_{j=0}^{u-1} S^{4j}\Delta' S^{-4j} \in \mathcal{D}. \tag{3.12}$$

Hence,

$$\Omega' = \mathrm{diag}(1, -1, -1, -1, \cdots, -1, -1, -1). \tag{3.13}$$

Since $r \geq \mathrm{rank}(\Omega' - I) = p - 1$, it follows that $p - 1 \leq r \leq p$. Also, since $\det(C) = 1$ for all $C \in \mathcal{C}$, it follows that $\mathrm{rank}(C) \neq p$, and we are done. ☐

Based on Theorem 3.3, we can sharpen Corollary 3.2 as follows.

Corollary 3.4. *If $\rho = 1$, then one of the following cases holds:*

(i) $r = p = q = 2$. *In this case,*

$$\mathcal{C} = \{I, -I\} \subset \mathcal{D} = \{I, -I, \operatorname{diag}(1, -1), \operatorname{diag}(-1, 1)\}.$$

(ii) $p = r$ *and* $q > 2$.

(iii) $r = p - 1$ *and* $q = 2$.

Proof. Part (i) is the same as part (i) of Corollary 3.2. Let $B \in \mathcal{C}$ be as in part (ii) of Corollary 3.2 such that $\rho_B = 2$. By Theorem 3.3, we have one of the following cases.

Case 1. $r_B = p$ and $q > 2$. Then $p \le r \le p$, which proves (ii).

Case 2. $r_B = p - 1$ and $q = 2$. Then r_B is even and, hence, p is odd. If r were equal to p, we would have $-I \in \mathcal{C}$, which is impossible since the determinant of every member of $\mathcal{C}$ is equal to one. This proves (iii). □

The following corollary studies the case $r = 2$; its easy proof is left to the interested reader.

Corollary 3.5. *If $r = 2$, then one of the following cases holds:*

(i) $\rho = 1$ *and* $p = q = 2$. *In this case,*

$$\mathcal{C} = \{I, -I\} \subset \mathcal{D} = \{I, -I, \operatorname{diag}(1, -1), \operatorname{diag}(-1, 1)\}.$$

(ii) $\rho = 1$, $p = 2$ *and* $q > 2$. *In this case,*

$$\mathcal{C} = \{\operatorname{diag}(\omega, \bar{\omega}) : \omega^q = 1\} \subset \mathcal{D} = \{\operatorname{diag}(\omega, \eta) : \omega^q = \eta^q = 1\}.$$

(iii) $\rho = 1$, $p = 3$ *and* $q = 2$. *In this case,*

$$\mathcal{C} = \{I, \operatorname{diag}(1, -1, -1), \operatorname{diag}(-1, 1, -1), \operatorname{diag}(-1, -1, 1)\}, \tag{3.14}$$

$$\mathcal{D} = \mathcal{C} \cup \{-I, \operatorname{diag}(-1, 1, 1), \operatorname{diag}(1, -1, 1), \operatorname{diag}(1, 1, -1)\}. \tag{3.15}$$

(iv) $\rho = 2$, $p = 2$ *and* $q > 2$. *In this case,* $\mathcal{C} = \mathcal{D} = \{\operatorname{diag}(\omega, \eta) : \omega^q = \eta^q = 1\}$.

(v) $\rho = 2$, $p = 3$ *and* $q = 2$. *In this case,*

$$\mathcal{C} = \mathcal{D} = \{\operatorname{diag}(\omega, \bar{\omega}) : \omega^q = 1\}.$$

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