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COUNTEREXAMPLES TO CONVEXITY OF k-INTERSECTION BODIES

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ABSTRACT. It is a well-known result due to Busemann that the intersection body of an origin-symmetric convex body is also convex. Koldobsky introduced the notion of k-intersection bodies. We show that the k-intersection body of an origin-symmetric convex body is not necessarily convex if $k > 1$.

1. INTRODUCTION

A *body* in $\mathbb{R}^n$ is a compact set with a non-empty interior. We say that a body K is a *star body* if it is star-shaped about the origin and its radial function defined by

$$\rho_K(\xi) = \max\{\lambda > 0 : \lambda\xi \in K\}, \qquad \text{for } \xi \in S^{n-1},$$

is positive and continuous.

The *Minkowski functional* of a star body K is given by

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}, \qquad \text{for } x \in \mathbb{R}^n.$$

It is easy to see that the latter is a homogeneous function of degree 1 on $\mathbb{R}^n$ and $\|\xi\|_K = \rho_K^{-1}(\xi)$, when $\xi \in S^{n-1}$.

The notion of the intersection body of a star body was introduced by Lutwak [L] in 1988 and has played an important role in Convex Geometry since then. The *intersection body of a star body* K is defined to be a star body IK whose radial function is given by

$$\rho_{IK}(\xi) = \text{vol}_{n-1}(K \cap \xi^\perp), \qquad \text{for } \xi \in S^{n-1},$$

where $\xi^\perp$ stands for the hyperplane $\{x \in \mathbb{R}^n : \langle x, \xi\rangle = 0\}$.

Intersection bodies were a key ingredient in the solution of the celebrated Busemann-Petty problem. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\text{vol}_{n-1}(K \cap \xi^\perp) \leq \text{vol}_{n-1}(L \cap \xi^\perp), \quad \text{for all } \xi \in S^{n-1}.$$

Does it necessarily follow that

$$\text{vol}_n(K) \leq \text{vol}_n(L)?$$

The answer to the problem is affirmative if $n \leq 4$ and negative if $n \geq 5$; see [GKS], [K2], [Z2] for historical details.

A generalization of the original Busemann-Petty problem to sections of other dimensions is often called the lower-dimensional Busemann-Petty problem. Let

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$1 \le k \le n-1$ be an integer and let $G(n,k)$ be the Grassmannian of all k-dimensional subspaces of $\mathbb{R}^n$. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_k(K\cap H) \le \mathrm{vol}_k(L\cap H), \quad \text{for all } H \in G(n,k).$$

Is it true that

$$\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)?$$

It is shown by Bourgain and Zhang [BZ] that the answer to this problem is negative if $k \ge 4$. Another proof is given in [K1]. The cases $k=2$ and $k=3$ are still open in dimensions $n \ge 5$.

Related to this problem are certain classes of bodies that generalize the notion of the intersection body. One generalization is due to Zhang [Z1] and another to Koldobsky [K1]. In this paper we will only discuss Koldobsky's k-intersection bodies. For the relation between the two generalizations and other results see Milman's works [M1] and [M2]. Let us emphasize that the study of these classes of bodies is important for the understanding of the open cases of the lower-dimensional Busemann-Petty problem.

Let $1 \le k \le n-1$ and let K and L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is *the k-intersection body of L* if for every $(n-k)$-dimensional subspace $H \subset \mathbb{R}^n$ we have

$$\mathrm{vol}_k(K\cap H^\perp) = \mathrm{vol}_{n-k}(L\cap H).$$

One can see that 1-intersection bodies coincide (up to a scaling factor) with Lutwak's intersection bodies.

Koldobsky has shown that if K is the k-intersection body of L, then the following relation holds (see e.g. [K2, Lemma 4.5]):

$$\|x\|_K^{-k} = \frac{k}{(n-k)(2\pi)^k}(\|\cdot\|_L^{-n+k})^\wedge(x), \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{1}$$

where in the right-hand side we have a Fourier transform in the sense of distributions; see next section for details. The latter formula implies, in particular, that for some bodies L the corresponding k-intersection bodies may not exist when $k>1$.

Formula (1) can be used as a definition of k-intersection bodies in the case when k is not necessarily an integer (and in fact, this is related to the concept of a space being embedded in L_{-k}; see [K2, Section 6.3]). For example, formula (1) for $0<k<1$ can be written as follows:

$$\|x\|_K^{-k} = c_{n,k}\int_{S^{n-1}} |\langle x,\theta\rangle|^{-k}\|\theta\|_L^{-n+k}d\theta.$$

Such bodies naturally arise in the theory of valuations, and in the following form

$$\|x\|_K^{p} = \int_L |\langle x,\theta\rangle|^{p}d\theta, \quad (p>-1,\ p\neq 0),$$

they are usually called the L_p-intersection bodies (see [H], [HL]). Note that with a different normalization these bodies are also known as polar p-centroid bodies (see e.g. [GG], [LYZ], [LZ], [YY]).

There is a natural notion of the complex intersection body. Such bodies were introduced and studied by Koldobsky, Paouris and Zymonopoulou, [KPZ2]. (See also [KKZ], where these studies were initiated.) It is shown that an origin-symmetric complex star body K in $\mathbb{R}^{2n}$ (which is naturally identified with $\mathbb{C}^n$) is a complex intersection body if and only if it is a 2-intersection body in $\mathbb{R}^{2n}$ and has certain rotational symmetries.

It is a classical theorem of Busemann (see [G, Theorem 8.1.10] for example) that the intersection body of an origin-symmetric convex body is also convex. There are various generalizations and modifications of this result; see e.g. [MP], [B], [KYZ]. In particular, Berck [B] has shown that the L_p-intersection bodies of origin-symmetric convex bodies are convex for $p > -1$, $p \neq 0$. Until now it was unknown whether k-intersection bodies of origin-symmetric convex bodies are convex. The question was raised during discussions at various conferences; most recently Bernig asked this question at the Oberwolfach workshop on Convex Geometry and its Applications (December, 2012). In this paper we answer it in the negative for all $k = 2, 3, \dots, n-1$.

It is worth noting that, in contrast with our result, Koldobsky, Paouris and Zymonopoulou [KPZ2] have shown that the complex intersection body of a complex convex body is convex (in other words, the 2-intersection body of a convex body in $\mathbb{R}^{2n}$ with certain symmetries is necessarily convex). Thus the complex structure in fact plays a crucial role in preserving convexity.

For other properties of k-intersection bodies the reader is referred to [K2], [KPZ1], [KY], [S], [Y].

2. Tools and auxiliary results

Let K be a star body. We say that K is origin-symmetric if $\rho_K(\xi) = \rho_K(-\xi)$ for all $\xi \in S^{n-1}$. We would like to compute the Fourier transform of powers of $\|\cdot\|_K$, the Minkowski functional of K. Recall that, given a function $f \in L_1(\mathbb{R}^n)$, its Fourier transform $\widehat{f}$ is defined as follows:

$$\widehat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-i\langle x,y\rangle}\, dy.$$

Unfortunately, no power of $\|\cdot\|_K$ belongs to $L_1(\mathbb{R}^n)$. However, it is still possible to compute the desired Fourier transforms in the sense of distributions. Here we describe a basic idea; for details see [GS], [K2].

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Elements of this space are referred to as test functions. Distributions are the elements of the dual space, $\mathcal{S}'(\mathbb{R}^n)$, of linear continuous functionals on $\mathcal{S}(\mathbb{R}^n)$. The action of a distribution f on a test function ϕ is denoted by $\langle f, \phi\rangle$. The Fourier transform of a distribution f is defined to be a distribution $\hat{f}$ (we also use the notation $(f)^\wedge$) satisfying

$$\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$$

for every test function ϕ from the space $\mathcal{S}(\mathbb{R}^n)$.

Let K be a convex body and let $\xi \in S^{n-1}$. The *parallel section function* $A_{K,\xi}(t)$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap (\xi^\perp + t\xi)), \quad t \in \mathbb{R}.$$

There is a remarkable connection between the derivatives of the parallel section function of a body K and the Fourier transform of the powers of its Minkowski functional.

Theorem 2.1 ([GKS], Theorem 1)**.** *Let K be an origin-symmetric convex body in $\mathbb{R}^n$ with C^∞ boundary, k a non-negative integer, $k \neq n-1$, and $\xi \in S^{n-1}$.*

(a) *If k is even, then*

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_{K,\xi}^{(k)}(0).$$

(b) *If k is odd, then*

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{(k+1)/2} 2(n-1-k)k! \times$$
$$\times \int_0^\infty \frac{A_{K,\xi}(z) - A_{K,\xi}(0) - A''_{K,\xi}(0)\frac{z^2}{2} - \cdots - A_{K,\xi}^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}} dz,$$

where $A_{K,\xi}^{(k)}$ stands for the derivative of the order k and the Fourier transform is considered in the sense of distributions.

In particular, it follows that for infinitely smooth bodies the Fourier transform of $\|x\|_K^{-n+k+1}$ restricted to the unit sphere is a continuous function (see also [K2, Lemma 3.16]).

We also note that the previous theorem together with Brunn's theorem implies that for every origin-symmetric convex body K the Fourier transforms of $\|x\|_K^{-n+2}$ and $\|x\|_K^{-n+3}$ are non-negative functions on the sphere (see [K2, Corollary 4.9]).

3. Main Result

Theorem 3.1. *Let k be an integer, $2 \le k \le n-1$. There is an origin-symmetric convex body L in $\mathbb{R}^n$ such that its k-intersection body K exists and is not convex.*

Proof. We will consider three cases according to the value of k: $4 \le k \le n-1$, $k=2$, $k=3$. The reader might have already noticed that the cases $k=2$ and $k=3$ usually differ from the rest. In our proof, the reason why we need a different construction for $k=2$ and $k=3$ is that the example used in Case 1 does not yield a convex body L when $k=2$ or $k=3$.

Case 1. Let $4 \le k \le n-1$. For a small $\epsilon > 0$ define an origin-symmetric star body $L = L_\epsilon$ by the formula:

$$\|x\|_L^{-n+k} = |x|_2^{-n+k} - \epsilon^{k-1}(1-\epsilon)^{-n+k+1}\|x\|_E^{-n+k}, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

where $|x|_2$ is the Euclidean norm and E is the ellipsoid given by

$$\|x\|_E = \left(\frac{x_1^2 + \cdots + x_{n-1}^2}{(1-\epsilon)^2} + \frac{x_n^2}{\epsilon^2}\right)^{1/2}.$$

Since $\|x\|_E^{-1} < |x|_2^{-1}$, it follows that $\|x\|_L^{-1}$ is positive for all $\epsilon > 0$ small enough, and so the body L is well defined.

We claim that the body L is convex for small enough ε. This is a standard perturbation argument; cf. [K2, p. 96]. By construction, the body L is obtained as a small perturbation of the Euclidean ball. Since the latter has strictly positive curvature, it is enough to control the first and second derivatives of the function $\epsilon^{k-1}(1-\epsilon)^{-n+k+1}\|x\|_E^{-n+k}$. One can see that these are of the order $O(\epsilon^{k-3})$, which is small for small enough ϵ (since $k \ge 4$). Therefore, L also has strictly positive curvature.

We now construct K, the k-intersection body of L. If it exists, then by formula (1) we have

$$\|x\|_K^{-k} = B_{n,k}(\|\cdot\|_L^{-n+k})^\wedge(x),$$

where

$$B_{n,k} = \frac{k}{(n-k)(2\pi)^k};$$

see [K2, Lemma 4.5].

Recall that the Fourier transform of $|\cdot|_2^{-n+k}$, $0<k<n$, equals (see [GS, p. 363])

$$(|\cdot|_2^{-n+k})^\wedge(x) = C_{n,k}|x|_2^{-k},$$

where

$$C_{n,k} = \frac{2^k \pi^{n/2}\Gamma\left(k/2\right)}{\Gamma\left((n-k)/2\right)}.$$

In order to compute the Fourier transform for the norms of ellipsoids, we will use the previous formula and the following connection between the Fourier transform and the linear transformations. Let T be an invertible linear transformation on $\mathbb{R}^n$; then

$$(|Tx|_2^{-n+k})^\wedge(y) = C_{n,k}|\det T|^{-1}|(T^*)^{-1}y|_2^{-k}.$$

Therefore,

$$\begin{aligned}(\|\cdot\|_L^{-n+k})^\wedge(x) &= C_{n,k}\Big(|x|_2^{-k} - \epsilon^k(1-\epsilon)^k\left((1-\epsilon)^2(x_1^2+\cdots+x_{n-1}^2)+\epsilon^2x_n^2\right)^{-k/2}\Big)\\ &= C_{n,k}\left(|x|_2^{-k} - \left(\frac{x_1^2+\cdots+x_{n-1}^2}{\epsilon^2}+\frac{x_n^2}{(1-\epsilon)^2}\right)^{-k/2}\right).\end{aligned}$$

The latter is strictly positive for all $x\in\mathbb{R}^n\setminus\{0\}$. Therefore, the star body K is well defined and is given by

$$\|x\|_K^{-k} = B_{n,k}C_{n,k}\left(|x|_2^{-k} - \left(\frac{x_1^2+\cdots+x_{n-1}^2}{\epsilon^2}+\frac{x_n^2}{(1-\epsilon)^2}\right)^{-k/2}\right).$$

It remains to show that the body K is not convex. To this end, let us compute the distance from the origin to the boundary of K in the directions $\xi_1=(0,0,\dots,0,1)$, $\xi_2=(\sqrt{2}/2,0,\dots,0,\sqrt{2}/2)$, and $\xi_3=(-\sqrt{2}/2,0,\dots,0,\sqrt{2}/2)$. One has

$$\rho_K(\xi_1) = (B_{n,k}C_{n,k})^{1/k}\left(1-(1-\epsilon)^k\right)^{1/k},$$

which can be made as close to zero as we wish by choosing ϵ sufficiently small.

On the other hand,

$$\begin{aligned}\rho_K(\xi_2)=\rho_K(\xi_3) &= (B_{n,k}C_{n,k})^{1/k}\left(1-\left(\frac{1}{2\epsilon^2}+\frac{1}{2(1-\epsilon)^2}\right)^{-k/2}\right)^{1/k}\\ &> (B_{n,k}C_{n,k})^{1/k}\left(1-\left(2\epsilon^2\right)^{k/2}\right)^{1/k}.\end{aligned}$$

The latter does not tend to zero as ϵ gets small. Thus, the body K is not convex.

Case 2. Let $k=2$. Here $L=L_\epsilon$ will be a "smoothened version" of the cube $B_\infty^n=\{x\in\mathbb{R}^n : \max\limits_{1\le i\le n}|x_i|\le 1\}$. For simplicity, one can think of $(1-\epsilon)B_\infty^n+\epsilon B_2^n$. However, the latter is not a C^∞ body. Thus, we will define $L=L_\epsilon$ to be an origin-symmetric convex body with C^∞ boundary that satisfies the following two conditions:

$$(1-\epsilon)B_\infty^n\subset L\subset B_\infty^n$$

and

$$A_{L,e_1}(z)=A_{L,e_1}(0),\qquad \text{for } |z|\le 1-\epsilon, \tag{3}$$

where e_1 is the basis vector $(1,0,...,0)$.

We now define a body K as follows:

$$\|\xi\|_K^{-2} = \frac{2}{(n-2)(2\pi)^2}(\|\cdot\|_L^{-n+2})^\wedge(\xi), \quad \xi \in S^{n-1}.$$

By part (b) of Theorem 2.1 this means

$$\|\xi\|_K^{-2} = -\frac{4}{(2\pi)^2}\int_0^\infty \frac{A_{L,\xi}(z) - A_{L,\xi}(0)}{z^2}dz.$$

Since the latter integral is strictly negative (and convergent, due to the smoothness and origin-symmetry of the body L) for every $\xi \in S^{n-1}$, it follows that the star body K is well defined. Moreover, K is the 2-intersection body of L.

In order to show that K is not convex, we will compute $\rho_K(\xi)$ in the directions $\xi_1 = (1, 0, 0, \dots, 0)$, $\xi_2 = (\sqrt{2}/2, \sqrt{2}/2, 0, \dots, 0)$ and $\xi_3 = (\sqrt{2}/2, -\sqrt{2}/2, 0, \dots, 0)$. By virtue of (3), we have

$$\rho_K^2(\xi_1) = \frac{4}{(2\pi)^2}\int_0^\infty \frac{A_{L,\xi_1}(0) - A_{L,\xi_1}(z)}{z^2}dz = \frac{4}{(2\pi)^2}\int_{1-\epsilon}^\infty \frac{A_{L,\xi_1}(0) - A_{L,\xi_1}(z)}{z^2}dz$$
$$\leq \frac{4A_{L,\xi_1}(0)}{(2\pi)^2}\int_{1-\epsilon}^\infty \frac{1}{z^2}dz \leq \frac{2^{n+1}}{(2\pi)^2(1-\epsilon)}.$$

Here we used the assumption that $L \subset B_\infty^n$ and therefore its central sections must not exceed those of the cube: i.e., $A_{L,\xi_1}(0) \leq 2^{n-1}$. Thus, $\rho_K(\xi_1)$ is bounded above by an absolute constant for all small ϵ.

Now consider

$$\rho_K^2(\xi_2) = \frac{4}{(2\pi)^2}\int_0^\infty \frac{A_{L,\xi_2}(0) - A_{L,\xi_2}(z)}{z^2}dz \geq \frac{4}{(2\pi)^2}\int_\epsilon^{\sqrt{2}} \frac{A_{L,\xi_2}(0) - A_{L,\xi_2}(z)}{z^2}dz.$$

Since $(1-\epsilon)B_\infty^n \subset L$, we have

$$A_{L,\xi_2}(0) \geq A_{(1-\epsilon)B_\infty^n,\xi_2}(0) = \sqrt{2}\ 2^{n-1}(1-\epsilon)^{n-1}.$$

Similarly, $L \subset B_\infty^n$ implies

$$A_{L,\xi_2}(z) \leq A_{B_\infty^n,\xi_2}(z) = 2^{n-1}(\sqrt{2} - z), \qquad \text{when } |z| \leq \sqrt{2}.$$

Thus,

$$\rho_K^2(\xi_2) \geq \frac{2^{n+1}}{(2\pi)^2}\int_\epsilon^{\sqrt{2}} \frac{\sqrt{2}(1-\epsilon)^{n-1} - (\sqrt{2} - z)}{z^2}dz$$
$$= \frac{2^{n+1}}{(2\pi)^2}\left(\frac{(1-\epsilon)^{n-1} - 1}{\epsilon}\sqrt{2} + 1 - (1-\epsilon)^{n-1} + \ln\sqrt{2} - \ln\epsilon\right).$$

The latter is large when ϵ is small.

We also have the same bound for $\rho_K^2(\xi_3)$. Thus, we have proved that the radius of K can be made as large as we want in the directions $(\sqrt{2}/2, \sqrt{2}/2, 0, \dots, 0)$ and $(\sqrt{2}/2, -\sqrt{2}/2, 0, \dots, 0)$, while staying bounded in the direction $(1, 0, 0, \dots, 0)$. Thus, the body K cannot be convex.

Case 3. Consider $k = 3$. For a small $\epsilon > 0$ we define an auxiliary origin-symmetric body $M = M_\epsilon \subset \mathbb{R}^n$ as follows:

$$M = \{x \in \mathbb{R}^n : x_1^4 + \epsilon x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 \leq 1\}.$$

One can see that the boundary surface of M is obtained by rotating the curve $x_1^4 + \epsilon x_1^2 + x_2^2 = 1$ about the x_1-axis. It is easy to check that this curve has strictly positive curvature. (This can be done by solving for x_1 in terms of x_2, as well as solving for x_2 in terms of x_1, and finding the second derivatives of these two functions.) Therefore, the body M also has strictly positive curvature (and thus is convex).

Now we will compute the parallel section function of M in the direction of the basis vector e_1.

$$A_{M,e_1}(z) = \kappa_{n-1}(1 - \epsilon z^2 - z^4)^{(n-1)/2},$$

where κ_{n-1} is the volume of the $(n-1)$-dimensional Euclidean ball B_2^{n-1}.

Therefore,

$$A''_{M,e_1}(0) = -\epsilon(n-1)\kappa_{n-1}.$$

We claim that for the body $M_0 = \{x \in \mathbb{R}^n : x_1^4 + x_2^2 + x_3^2 + \cdots + x_n^2 \leq 1\}$ (which is the limiting case of M_ϵ when $\epsilon \to 0$) there is a direction $\xi_0 \notin e_1^\perp$ such that

$$A''_{M_0,\xi_0}(0) = -\alpha < 0,$$

for some number $\alpha > 0$.

Indeed, the body M_0 has an infinitely smooth norm and therefore (as noted after Theorem 2.1) the Fourier transform $(\|\cdot\|_{M_0}^{-n+3})^\wedge$ is a non-negative continuous function on the sphere. If we had $A''_{M_0,\xi}(0) = 0$ for all $\xi \in S^{n-1}$ outside the equator $e_1^\perp$, then $(\|\cdot\|_{M_0}^{-n+3})^\wedge$ would be zero on the sphere, and therefore $\|\cdot\|_{M_0}^{-n+3}$ would also be identically zero.

Now we will show that $A''_{M,\xi_0}(0)$ is close to $-\alpha$ when ϵ is sufficiently small. Since

$$(\|x\|_M^{-n+3})^\wedge(\xi) = -\pi(n-3)A''_{M,\xi}(0),$$

by part (a) of Theorem 2.1, it is enough to show that $(\|x\|_M^{-n+3})^\wedge(\xi_0)$ is close to $(\|x\|_{M_0}^{-n+3})^\wedge(\xi_0)$ when ϵ is sufficiently small. One can find explicitly a formula for the norm of M:

$$\|x\|_M = \sqrt{\epsilon x_1^2 + x_2^2 + \cdots + x_n^2 + \sqrt{(\epsilon x_1^2 + x_2^2 + \cdots + x_n^2) + 4x_1^4}}.$$

Observe that $\|\cdot\|_M$ and its first and second derivatives are continuous functions of (x, ϵ) on $S^{n-1} \times [0, \epsilon_0]$ for some small ϵ_0, and are therefore uniformly continuous. Thus, $\|\cdot\|_M$ converges to $\|\cdot\|_{M_0}$ in $C^2(S^{n-1})$ as $\epsilon \to 0$. By [K2, Corollary 3.17] it follows that $(\|\cdot\|_M^{-n+3})^\wedge$ converges to $(\|\cdot\|_{M_0}^{-n+3})^\wedge$ in $C(S^{n-1})$.

Thus, for small enough ϵ we have that $(\|\cdot\|_M^{-n+3})^\wedge(\xi_0)$ is close to $\pi(n-3)\alpha$ and $(\|\cdot\|_M^{-n+3})^\wedge(e_1)$ is close to zero. We now fix ϵ so small that

$$(\|\cdot\|_M^{-n+3})^\wedge(\xi_0) > |\langle \xi_0, e_1\rangle|^{-3}(\|\cdot\|_M^{-n+3})^\wedge(e_1).$$

For a small $\lambda > 0$ define an origin-symmetric body L as follows:

$$\|x\|_L^{-n+3} = \|x\|_M^{-n+3} + \lambda |x|_2^{n+3}.$$

Since M has strictly positive curvature, a small perturbation will not affect this property. Thus L is convex for small enough λ. We will also require that λ be small enough to guarantee that

$$(\|\cdot\|_M^{-n+3})^\wedge(\xi_0) + \lambda(|\cdot|_2^{-n+3})^\wedge(\xi_0) > |\langle \xi_0, e_1\rangle|^{-3}\left((\|\cdot\|_M^{-n+3})^\wedge(e_1) + \lambda(|\cdot|_2^{-n+3})^\wedge(e_1)\right). \tag{4}$$

Now define a star body K as follows:

$$\|x\|_K^{-3} = \frac{3}{(n-3)(2\pi)^3}(\|\cdot\|_L^{-n+3})^\wedge(x), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Since $(\|\cdot\|_M^{-n+3})^\wedge(x) \geq 0$ and $(|\cdot|_2^{-n+3})^\wedge(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, it follows that

$$(\|\cdot\|_L^{-n+3})^\wedge(x) = (\|\cdot\|_M^{-n+3})^\wedge(x) + \lambda(|\cdot|_2^{-n+3})^\wedge(x) > 0, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

and therefore the star body K is well defined. Also observe that K is the 3-intersection body of L.

Finally we show that K is not convex. First, note that K is a body of revolution about the x_1-axis, since M was such. Secondly, condition (4) implies that

$$|\langle \xi_0, e_1\rangle| \rho_K(\xi_0) > \rho_K(e_1).$$

This means that the projection of the vector $\rho_K(\xi_0)\xi_0 \in K$ onto the axis of revolution of K lies outside of K. Thus, K is not convex. □

References

[B] Gautier Berck, *Convexity of L_p-intersection bodies*, Adv. Math. **222** (2009), no. 3, 920–936, DOI 10.1016/j.aim.2009.05.009. MR2553373 (2010m:52008)

[BZ] Jean Bourgain and Gaoyong Zhang, *On a generalization of the Busemann-Petty problem*, Convex geometric analysis (Berkeley, CA, 1996), Math. Sci. Res. Inst. Publ., vol. 34, Cambridge Univ. Press, Cambridge, 1999, pp. 65–76. MR1665578 (99m:52011)

[G] Richard J. Gardner, *Geometric tomography*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006. MR2251886 (2007i:52010)

[GG] R. J. Gardner and A. A. Giannopoulos, *p-cross-section bodies*, Indiana Univ. Math. J. **48** (1999), no. 2, 593–613, DOI 10.1512/iumj.1999.48.1689. MR1722809 (2000i:52002)

[GKS] R. J. Gardner, A. Koldobsky, and T. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. (2) **149** (1999), no. 2, 691–703, DOI 10.2307/120978. MR1689343 (2001b:52011)

[GS] I. M. Gel′fand and G. E. Shilov, *Generalized functions. Vol. 1, Properties and operations*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977]. Translated from the Russian by Eugene Saletan. MR0435831 (55 #8786a)

[H] Christoph Haberl, *L_p intersection bodies*, Adv. Math. **217** (2008), no. 6, 2599–2624, DOI 10.1016/j.aim.2007.11.013. MR2397461 (2009a:52001)

[HL] Christoph Haberl and Monika Ludwig, *A characterization of L_p intersection bodies*, Int. Math. Res. Not., posted on 2006, Art. ID 10548, 29, DOI 10.1155/IMRN/2006/10548. MR2250020 (2007k:52007)

[K1] A. Koldobsky, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal. **10** (2000), no. 6, 1507–1526, DOI 10.1007/PL00001659. MR1810751 (2001m:52007)

[K2] Alexander Koldobsky, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs, vol. 116, American Mathematical Society, Providence, RI, 2005. MR2132704 (2006a:42007)

[KKZ] A. Koldobsky, H. König, and M. Zymonopoulou, *The complex Busemann-Petty problem on sections of convex bodies*, Adv. Math. **218** (2008), no. 2, 352–367, DOI 10.1016/j.aim.2007.12.006. MR2407938 (2009c:52016)

[KPZ1] A. Koldobsky, G. Paouris, and M. Zymonopoulou, *Isomorphic properties of intersection bodies*, J. Funct. Anal. **261** (2011), no. 9, 2697–2716, DOI 10.1016/j.jfa.2011.07.011. MR2826412 (2012j:52010)

[KPZ2] A. Koldobsky, G. Paouris, and M. Zymonopoulou, *Complex intersection bodies*, J. Lond. Math. Soc. (2) **88** (2013), no. 2, 538–562, DOI 10.1112/jlms/jdt014. MR3106735

[KY] Alexander Koldobsky and Vladyslav Yaskin, *The interface between convex geometry and harmonic analysis*, CBMS Regional Conference Series in Mathematics, vol. 108, Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 2008. MR2365157 (2008j:52008)

[KYZ] Jaegil Kim, Vladyslav Yaskin, and Artem Zvavitch, *The geometry of p-convex intersection bodies*, Adv. Math. **226** (2011), no. 6, 5320–5337, DOI 10.1016/j.aim.2011.01.011. MR2775903 (2012e:44002)

[L] Erwin Lutwak, *Intersection bodies and dual mixed volumes*, Adv. in Math. **71** (1988), no. 2, 232–261, DOI 10.1016/0001-8708(88)90077-1. MR963487 (90a:52023)

[LYZ] Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), no. 1, 111–132. MR1863023 (2002h:52011)

[LZ] Erwin Lutwak and Gaoyong Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), no. 1, 1–16. MR1601426 (2000c:52011)

[M1] Emanuel Milman, *Generalized intersection bodies*, J. Funct. Anal. **240** (2006), no. 2, 530–567, DOI 10.1016/j.jfa.2006.04.004. MR2261694 (2007h:52007)

[M2] Emanuel Milman, *Generalized intersection bodies are not equivalent*, Adv. Math. **217** (2008), no. 6, 2822–2840, DOI 10.1016/j.aim.2007.11.007. MR2397468 (2009k:52012)

[MP] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geometric aspects of functional analysis (1987–88), Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, pp. 64–104, DOI 10.1007/BFb0090049. MR1008717 (90g:52003)

[S] Jared Schlieper, *A note on k-intersection bodies*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2081–2088 (electronic), DOI 10.1090/S0002-9939-07-08774-6. MR2299484 (2008d:52011)

[Y] Vladyslav Yaskin, *On strict inclusions in hierarchies of convex bodies*, Proc. Amer. Math. Soc. **136** (2008), no. 9, 3281–3291, DOI 10.1090/S0002-9939-08-09424-0. MR2407094 (2009h:52010)

[YY] V. Yaskin and M. Yaskina, *Centroid bodies and comparison of volumes*, Indiana Univ. Math. J. **55** (2006), no. 3, 1175–1194, DOI 10.1512/iumj.2006.55.2761. MR2244603 (2007f:52008)

[Z1] Gaoyong Zhang, *Sections of convex bodies*, Amer. J. Math. **118** (1996), no. 2, 319–340. MR1385280 (97f:52015)

[Z2] Gaoyong Zhang, *A positive solution to the Busemann-Petty problem in $\mathbf{R}^4$*, Ann. of Math. (2) **149** (1999), no. 2, 535–543, DOI 10.2307/120974. MR1689339 (2001b:52010)

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