WEIGHTED ESTIMATES FOR $L_1$-VECTOR FIELDS

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Dedicated to Professor Jean-Pierre Gossez on the occasion of his 70th birthday

ABSTRACT. In the context of weighted spaces, we study some types of Bourgain-Brezis and Lanzani-Stein inequalities.

INTRODUCTION

In the classical Hodge theory the following a priori estimate holds: if $f \in C^\infty_c(R^n; \Lambda^l)$ and $1 < p < \infty$, one has

\begin{equation}
\| \nabla f \|_{L^p} \leq \| df \|_{L^p} + \| \delta f \|_{L^p},
\end{equation}

where $d$ is the exterior differential operator and $\delta$ is its dual. The estimate (0.1) is known to be false if $p = 1$ or $p = \infty$. The Sobolev imbedding and the estimate (0.1) lead to

\begin{equation}
\| f \|_{L^{p_\ast}} \leq \| df \|_{L^p} + \| \delta f \|_{L^p},
\end{equation}

with $\frac{1}{p_\ast} = \frac{1}{p} - \frac{1}{n}$. When $p = 1$, J. Bourgain and H. Brezis [1] have obtained the estimate

\begin{equation}
\| f \|_{L^{n/(n-1)}} \leq \| df \|_{L^1} + \| \delta f \|_{L^1}.
\end{equation}

L. Lanzani and E. Stein [9] have obtained the same result using an argument developed by J. Van Schaftingen [12]. I. Mitrea and M. Mitrea [11] have extended these estimates to homogeneous Besov spaces. Using interpolation theory, they could replace the norm $\| f \|_{L^{n/(n-1)}}$ by $\| f \|_{\dot{B}^{\gamma,q}_{p,w}}$ with $\frac{1}{p} - \frac{\gamma}{n} = 1 - \frac{1}{n}$, $0 < \gamma < 1$ and $q = \frac{2}{1-\gamma}$. In a recent work [13], Van Schaftingen has extended the result of Mitrea and Mitrea, taking any $q > 1$ and $0 < \gamma < 1$. In [10] we have also obtained some similar results; among them is the following estimate:

Let $\gamma \in \mathbb{R}$, $0 < p < \infty$ and $w$ be in the Muckenhoupt class $A_\infty$. Then

\begin{equation}
\| \nabla f \|_{\dot{F}^{\gamma,q}_{p,w}} \leq \| df \|_{\dot{F}^{\gamma,q}_{p,w}} + \| \delta f \|_{\dot{F}^{\gamma,q}_{p,w}}.
\end{equation}

A consequence of (0.4) is the following:

Let $\gamma$, $\beta \in \mathbb{R}$ s.t. $0 < 1 - \gamma + \beta$, $0 < p < \infty$ and $w \in A_\infty$ satisfying that for some $d > 0$, $w(B(x,t)) = \int_{B(x,t)} wdy \geq C t^d$ for all $x$ and all $t > 0$. Then

\begin{equation}
\| f \|_{\dot{F}^{\gamma,q}_{p,w}} \leq \| df \|_{\dot{F}^{\beta,q}_{p,w}} + \| \delta f \|_{\dot{F}^{\beta,q}_{p,w}},
\end{equation}

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with $\frac{1}{p^\ast} = \frac{1}{p} - \frac{1-\gamma+\beta}{d}$. In particular we have

\[(0.6) \quad \|f\|_{L^{p^\ast}(w)} \leq \|df\|_{H^p(w)} + \|\delta f\|_{H^p(w)},\]

with $\frac{1}{p^\ast} = \frac{1}{p} - \frac{1}{d}$ and $p$ is assumed to satisfy $p > \frac{d}{d+1}$. Here $H^p(w)$ denotes the weighted Hardy space. In particular, if we take $p = 1$, then

\[(0.7) \quad \|f\|_{L^{d/(d-1)}(w)} \leq \|df\|_{H^1(w)} + \|\delta f\|_{H^1(w)},\]

which is a good substitute for the estimate (0.3). In this work we extend the estimate (0.3) as follows.

**Theorem 0.1.** Let $\gamma, \beta \in \mathbb{R}$ s.t. $0 < 1 - \gamma + \beta$ and $n \leq d < \infty$. Assume $w \in A_d(\mathbb{R} \times \mathbb{R}^{n-1})$ satisfying the condition (2.2). If $2 \leq l \leq n - 2$, then

\[(0.8) \quad \|f\|_{L^{p'}(w^{1-p'})} \leq \|df\|_{L^1} + \|\delta f\|_{L^1},\]

with $p' = \frac{d}{d-1}$.

**Remark 0.1.** When $l = 1$, a substitute for (0.8) holds with $\|\delta f\|_{L^1}$ replaced by $\|\delta f\|_{H^1}$. Similarly for $l = n - 1$ when we replace $\|df\|_{L^1}$ by $\|df\|_{H^1}$.

**Notation.** The constant $C$ may have different values even in the same line, but does not depend on $f$. $A \approx B$ for positive $A$ and $B$ mean that there exists $C > 0$ such that $C^{-1}A \leq B \leq CA$ and $A \preceq B$ stands for $A \leq CB$.

1. **Some background tools**

1.1. **Muckenhoupt class.** We briefly recall some fundamentals on Muckenhoupt classes. Such classes are important tools in many areas of mathematics, including harmonic analysis and partial differential equations. Good references are [6,14,15].

Let $0 < p < \infty$. We say that $f \in L^p(w)$ if and only if

$$\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty,$$

and $f$ is in the weak-$L^p$ space, or $f \in L^{p,\infty}(w)$ if and only if

$$\|f\|_{L^{p,\infty}(w)} = \sup_{\lambda > 0} \left( \lambda \left[ w \{x \in \mathbb{R}^n : |f(x)| > \lambda \} \right]^{\frac{1}{p}} \right) < \infty.$$

When $w = 1$ we drop the subscription $w$.

The Hardy-Littlewood maximal function $Mf$ is defined, for a local integrable function $f$, by

\[(1.1) \quad Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| dy,\]

where the supremum is taken over all cubes containing $x$. 

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A nonnegative locally integrable function \( w \) is said to be in the Muckenhoupt classes \( A_p \) if there exists a constant \( C_p > 0 \) such that for all cube \( Q \),

\[
\frac{1}{|Q|} \int_Q w \, dy \left( \frac{1}{|Q|} \int_Q w^{1-p'} \, dy \right)^{p-1} \leq C_p
\]

when \( 1 < p < \infty \), \( \frac{1}{p} + \frac{1}{p'} = 1 \), as well as for \( p = 1 \),

\[
\frac{1}{|Q|} \int_Q w \, dy \leq C_1 w(x),
\]

for a.e. \( x \in Q \) or, equivalently, \( Mw(x) \leq C_1 w(x) \) for a.e. \( x \in \mathbb{R}^n \). We set \( A_{\infty} = \bigcup_{p \geq 1} A_p \).

It is well known that the classes \( A_p \) characterize the boundedness of the Hardy-Littlewood maximal operator on the weighted Lebesgue spaces (see [6], ch. IV). Namely,

\[
M : L^p(w) \rightarrow L^p(w)
\]

if and only if \( w \in A_p \) when \( 1 < p < \infty \), and

\[
M : L^1(w) \rightarrow L^{1,\infty}(w)
\]

if and only if \( w \in A_1 \).

1.2. Weighted Lizorkin-Triebel space. We denote by \( S(\mathbb{R}^n) \) the space of Schwartz functions and by \( S_{\infty}(\mathbb{R}^n) \) the subspace of functions in \( S(\mathbb{R}^n) \) with all vanishing moments; i.e.,

\[
S_{\infty}(\mathbb{R}^n) = \left\{ \nu \in S : \int_{\mathbb{R}^n} x^\beta \nu(x) \, dx = 0, \ \forall \beta \in \mathbb{N}^n \right\}.
\]

The weighted homogeneous Lizorkin-Triebel space \( \dot{F}^{\gamma,q}_{p,w} \) can be characterized as follows (see for instance [3] or [7]). Let \( \gamma \in \mathbb{R}, \ 0 < p, q < \infty, \ 0 < \delta < \min(p,q) \) and \( w \in A_{p/\delta} \). Assume \( \nu \in S_{\infty}(\mathbb{R}^n) \). Then \( f \in \dot{F}^{\gamma,q}_{p,w} \) if and only if

\[
||f||_{\dot{F}^{\gamma,q}_{p,w}} = \left\| \left( \int_0^\infty t^{-\gamma q} (\nu_t \ast f)^q \, dt \right)^{\frac{1}{q}} \right\|_{p,w} < \infty,
\]

with the usual modification if \( p = \infty \). Here \( \nu_t(x) = t^{-n} \nu(\frac{x}{t}) \).

Remark 1.1. Note that

1. If \( I_\alpha \) denotes the Riesz potential defined by \( \hat{I}_\alpha f(\xi) = |\xi|^{-\alpha} \hat{f}(\xi), \ \alpha \in \mathbb{R} \), then we have the following lifting property of \( I_\alpha \):

\[
||I_\alpha f||_{\dot{F}^{\alpha+\gamma,q}_{p,w}} \approx ||f||_{\dot{F}^{\gamma,q}_{p,w}};
\]

2. If \( R_j, \ j = 1, \ldots, n \), denotes the Riesz transform defined by \( \hat{R}_j f(\xi) = i \xi_j |\xi|^{-1} \hat{f}(\xi) \), then

\[
||R_j f||_{\dot{F}^{\gamma,q}_{p,w}} \approx ||f||_{\dot{F}^{\gamma,q}_{p,w}};
\]

3. For a positive integer \( N \), \( f \in \dot{F}^{N,q}_{p,w} \) if and only if \( \partial^\sigma f \in \dot{F}^{0,q}_{p,w} \) for all \( \sigma \) such that \( |\sigma| = N \). Moreover,

\[
||f||_{\dot{F}^{N,q}_{p,w}} \approx \sum_{|\sigma|=N} ||\partial^\sigma f||_{\dot{F}^{0,q}_{p,w}}.
\]
Indeed, the first assertion follows from the fact that $I_\alpha \nu$ behaves as $\nu$, $\nu \ast (I_\alpha f) = t^n (I_\alpha \nu) \ast f$ and $I_\alpha I_{-\alpha} = \text{id}$. The second assertion can be obtained from the identity $\nu \ast (R_j f) = (R_j \nu) \ast f$. Finally the identity

$$\partial^\alpha = (-1)^{[\alpha]} R^\alpha \circ I_{-[\alpha]}, \quad R = (R_1, \ldots, R_n),$$

yields the third assertion.

One should also note (see [5][10]) that

**Theorem 1.1.** Assume $\beta \in \mathbb{R}$, $0 < p$, $d < \infty$, and $0 < q$, $r \leq \infty$. Let $\gamma \in \mathbb{R}$ with $0 < \alpha - \gamma + \beta$ and $w \in A_\infty$. If $w(B(x,t)) \geq Ct^d$ for all $0 < t < \infty$ and all $x$, then

$$\|I_\alpha f\|_{F^{\gamma,q}_{p,w}} \leq C\|f\|_{\dot{F}^{\beta,q}_{p,w}},$$

where $p_\ast$ is determined by

$$\frac{1}{p_\ast} = \frac{1}{p} - \frac{\alpha - \gamma + \beta}{d}.$$

In particular, we have

**Theorem 1.2 ([4][5][10]).** Assume any reals $\alpha, \gamma$ s.t. $0 < \alpha - \gamma$; $0 < p < \infty$, $0 < q$, $r \leq \infty$ and $d > 0$. Let $w \in A_\infty$ and $0 < p_\ast \leq \infty$ with

$$\frac{1}{p_\ast} = \frac{1}{p} - \frac{\alpha - \gamma}{d}.$$

If $w(B(x,t)) \geq Ct^d$ for all $0 < t < \infty$ and all $x$, then

$$\dot{F}^{\alpha,q}_{p,w} \subset \bigcap_{r>0} \dot{F}^{\gamma,r}_{p_\ast,w},$$

with the continuous imbedding $\dot{F}^{\alpha,q}_{p,w} \hookrightarrow \dot{F}^{\gamma,r}_{p_\ast,w}$, for each $0 < r \leq \infty$.

**Remark 1.2.** Such weights in Theorem 1.1 and Theorem 1.2 exist. For instance, the function

$$w(x) = |x|^\alpha \log^\beta(2 + |x|^{-1})$$

is in $A_p$, if $-n < \alpha < n(p-1)$ and $\beta \in \mathbb{R}$; see for example [8]. Moreover, $w$ satisfies $w(B(x,t)) \geq Ct^{n+\alpha}$ for all $0 < t < \infty$ and all $x$ whenever $0 \leq \alpha < n(p-1)$ and $\beta \geq 0$.

Another interesting example given in [8] is

$$w(x) = \begin{cases} |x|^\alpha & \text{if } |x| \leq 1, \\ |x|^\beta & \text{if } |x| > 1, \end{cases}$$

which belongs to $A_p$, $1 < p < \infty$, whenever $-n < \alpha$ and $\beta < n(p-1)$. If, in addition, $-n < \alpha \leq 0$ and $0 \leq \beta < n(p-1)$, then $w(B(x,t)) \geq Ct^n$ for all $0 < t < \infty$ and all $x$.

2. Weighted estimates for $L_1$-vector fields

In this section we assume that $w \in A_p(\mathbb{R} \times \mathbb{R}^{n-1})$; i.e., there exists a constant $C > 0$ such that for every interval $I \subset \mathbb{R}$ and every cube $Q \subset \mathbb{R}^{n-1}$,

$$(2.1) \quad \frac{1}{|I \times Q|} \int_{I \times Q} wdy \left( \frac{1}{|I \times Q|} \int_{I \times Q} w^{1-p'} dy \right)^{p-1} \leq C, \quad 1 < p < \infty.$$

We denote by $Q_t$ a cube with radius $t > 0$ and by $I_r$ an interval with radius $r > 0$. 
Theorem 2.1. Let \( d \geq n \) and \( w \in A_d(\mathbb{R} \times \mathbb{R}^{n-1}) \). Assume that
\[
\int_{I_r \times Q_t} wd y \geq Cr^{d-1}, \quad \forall r, \ t > 0.
\]
Then there exists a constant \( C \) such that for each \( f, g \in C^\infty_c(\mathbb{R}, \mathbb{R}^n) \),
\[
\left| \int_{\mathbb{R}^n} f g dx \right| \leq C \left( \| \text{div} f \|_{L_1} \| g \|_{L_d(w)} + \| f \|_{L_1} \| \nabla g \|_{L_d(w)} \right).
\]

To prove Theorem 2.1 we need the following lemma.

Lemma 2.1. Let \( w \in A_p(\mathbb{R}^m) \), \( d \geq m + 1 \) and assume that \( w(B(x, t)) \geq Ct^{d-1} \) for each \( x \in \mathbb{R}^m \) and each \( t > 0 \). Let \( g \in C^\infty_c(\mathbb{R}^m) \) and set
\[
g_1(x) = g(x) - \eta_\delta * g(x), \quad g_2(x) = \eta_\delta * g(x),
\]
where \( \eta \) is a smooth cut-off function supported in the unit ball with \( \int \eta(x) dx = 1 \) and \( \eta_\delta(x) = \delta^{-m} \eta(x/\delta), \ delta > 0 \). Then we have the following estimates:
\[
\| g_1 \|_{L_\infty} \leq C \delta^{\gamma} \| \nabla g \|_{L_p(w)},
\]
\[
\| g_2 \|_{L_\infty} \leq C \delta^{\gamma - 1} \| g \|_{L_p(w)},
\]
\[
\| \nabla g_2 \|_{L_\infty} \leq C \delta^{\gamma - 1} \| \nabla g \|_{L_p(w)},
\]
with \( \gamma = 1 - \frac{d-1}{p} \) provided that \( p > d - 1 \).

Proof. We only prove the estimation (2.4). The proof of (2.5) or (2.6) is similar. We have
\[
|g_1(x)| = \delta^{-m} \left| \int \eta(x - y/\delta) (g(x) - g(y)) dy \right| \leq C \int_{B(x, \delta)} |\nabla g(y)| |x - y|^{-m+1} dy.
\]
Using Hölder’s inequality we get
\[
|g_1(x)| \leq C \left( \int_{B(x, \delta)} |\nabla g(y)|^p w(y) dy \right)^{1/p} \left( \int_{B(x, \delta)} |x - y|^{-(m+1)p'} w^{-p'/p} dy \right)^{1/p'}.
\]
On the other hand, we have
\[
\int_{B(x, \delta)} |x - y|^{-(m+1)p'} w^{-p'/p} dy
\]
\[
\leq \sum_{k \geq 0} (2^{-k} \delta)^{-(m+1)p'} \int_{B(x, 2^{-k} \delta)} w^{-p'/p} dy
\]
\[
\leq \sum_{k \geq 0} (2^{-k} \delta)^{-(m+1)p'} (2^{-k} \delta)^{mp'} \left( \int_{B(x, 2^{-k} \delta)} w dy \right)^{-\frac{1}{p'}}
\]
\[
\leq \sum_{k \geq 0} (2^{-k} \delta)^{p\tau - p} (2^{-k} \delta)^{\frac{1-d}{p'}} \leq \delta^{\frac{p+1-d}{p'}}.
\]
Estimates (2.7) and (2.8) imply (2.4). \( \square \)

Remark 2.1. An alternative proof of (2.4) is to use a weighted version of Theorem IX.12 in [2]. Indeed, one can adapt the argument given in [2] to prove that
\[
|g(x) - g(y)| \leq C |x - y|^{\gamma} \| \nabla g(y) \|_{L_p(w)}, \quad \text{for all } x, y \in \mathbb{R}^m.
\]
This leads immediately to the estimate (2.4).
Proof of Theorem 2.1 We follow the argument in Van Schaftingen [12]. Without loss of generality we may assume that $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$.

Note first that if $w$ satisfies (2.1) and (2.2) with $p = d$, then Lebesgue’s Differentiation Theorem implies that for almost every $x_1 \in \mathbb{R}$ and every cube $Q_t \subset \mathbb{R}^{n-1}$,

$$
\frac{1}{|Q_t|} \int_{Q_t} w(x_1, x') dx' \left( \frac{1}{|Q_t|} \int_{Q_t} w^{1-p'}(x_1, x') dx' \right)^{p-1} \leq C
$$

and

$$
\int_{Q_t} w(x_1, x') dx' \geq C t^{d-1}.
$$

Fix $x_1 \in \mathbb{R}$ and let $\eta$ be as in Lemma 2.1 with $m = n - 1$ and let $\delta = \delta(x_1) > 0$ be chosen later.

Set

$$
g_{1,x_1}(x') = g_{x_1}(x') - \eta \delta * g_{x_1}(x'),
g_{2,x_1}(x') = \eta \delta * g_{x_1}(x'),
$$

and

$$
J(x_1) = \int_{\mathbb{R}^{n-1}} f_1(x_1, x') g_{x_1}(x') dx',
$$

with $x' = (x_2, \ldots, x_n)$ and $g_{x_1}(x') = g(x_1, x')$. Using the decomposition in Lemma 2.1 we get

$$
J(x_1) = J_1(x_1) + J_2(x_1),
$$

with

$$
J_1(x_1) = \int_{\mathbb{R}^{n-1}} f_1(x_1, x') g_{1,x_1}(x')
$$

and

$$
J_2(x_1) = \int_{\mathbb{R}^{n-1}} f_1(x_1, x') g_{2,x_1}(x').
$$

From (2.4) we have

$$
|J_1(x_1)| \leq C \delta^\gamma ||\nabla g_{x_1}||_{L_p(w(x_1, \ldots))} ||f(x_1, \ldots)||_{L_1}.
$$

On the other hand,

$$
J_2(x_1) = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{x_1} \frac{\partial f_1}{\partial t}(t, x') g_{2,x_1}(x') dt dx' \\
= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{x_1} \text{div} f(t, x') g_{2,x_1}(x') dt dx' - \sum_{j=2}^n \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{x_1} \frac{\partial f_j}{\partial x_j}(t, x') g_{2,x_1}(x') dt dx' \\
= \int_{-\infty}^{x_1} \int_{\mathbb{R}^{n-1}} \text{div} f(t, x') g_{2,x_1}(x') dx' dt \\
+ \sum_{j=2}^n \int_{-\infty}^{x_1} \int_{\mathbb{R}^{n-1}} f_j(t, s) \frac{\partial g_{2,x_1}}{\partial x_j}(x') dx' dt.
$$

Using the estimates (2.5) and (2.6) and the last identity we obtain

$$
|J_2(x_1)| \leq C \delta^\gamma \left( ||\text{div} f||_{L_1} ||g_{x_1}||_{L_p(w(x_1, \ldots))} + ||f||_{L_1} ||\nabla g_{x_1}||_{L_p(w(x_1, \ldots))} \right).
$$
Estimates (2.10) and (2.11) lead to
\[
|J(x_1)| \leq \delta^{|\gamma|} \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \| f(x_1,\ldots) \|_{L_1}
\]
(2.11) \quad + \delta^{-1} \left( \| \text{div} f \|_{L_1} \| g_{x_1} \|_{L_p(w(x_1,\ldots))} + \| f \|_{L_1} \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \right). 
Assume \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \| f(x_1,\ldots) \|_{L_1} \neq 0 \text{ and choose } \delta \text{ so that}
\delta^{|\gamma|} \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \| f(x_1,\ldots) \|_{L_1} = \delta^{-1} \left( \| \text{div} f \|_{L_1} \| g_{x_1} \|_{L_p(w(x_1,\ldots))} + \| f \|_{L_1} \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \right).
That is,
\[
\delta = \frac{\| \text{div} f \|_{L_1} \| g_{x_1} \|_{L_p(w(x_1,\ldots))} + \| f \|_{L_1} \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))}}{\| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \| f(x_1,\ldots) \|_{L_1}}.
\]

It follows from (2.11) and (2.12) that
\[
\int \limits_{\mathbb{R}} |J(x_1)| \, dx_1 \leq \int \limits_{\mathbb{R}} \left( \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \| f(x_1,\ldots) \|_{L_1} \right)^{1-1/p} \times \left( \| \text{div} f \|_{L_1} \| g_{x_1} \|_{L_p(w(x_1,\ldots))} + \| f \|_{L_1} \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \right)^{1/p} \, dx_1.
\]
If \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \| f(x_1,\ldots) \|_{L_1} = 0, \text{ the inequality is trivially true. Otherwise, if we use Hölder’s inequality twice, we get}
\[
\int \limits_{\mathbb{R}} |J(x_1)| \, dx_1 \leq \left( \int \limits_{\mathbb{R}} \left( \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \| f(x_1,\ldots) \|_{L_1} \right)^{\frac{p}{p+1}} \right)^{\frac{p+1}{p}} \times \left( \int \limits_{\mathbb{R}} \left( \| \text{div} f \|_{L_1} \| g_{x_1} \|_{L_p(w(x_1,\ldots))} + \| f \|_{L_1} \| \nabla g_{x_1} \|_{L_p(w(x_1,\ldots))} \right)^{\frac{p}{p}} \, dx_1 \right)^{\frac{p}{p+1}}
\leq \left( \| \nabla g \|_{L_p(w)} \| f \|_{L_1} \right)^{\frac{p-1}{p}} \| \text{div} f \|_{L_1} \| g \|_{L_p(w)} + \| f \|_{L_1} \| \nabla g \|_{L_p(w)} \right)^{\frac{1}{p}}
\leq \| \text{div} f \|_{L_1} \| g \|_{L_p(w)} + \| f \|_{L_1} \| \nabla g \|_{L_p(w)}.
\]

Corollary 2.1. Let \( d \) and \( w \) be as in Theorem 2.1. Then, for every \( s \geq 1, \ r > 0 \) and \( q > 0 \) with \( sr = d \), there exists \( C > 0 \) such that for every \( f, g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \), if \( \text{div} f = 0 \), then
\[
\left| \int_{\mathbb{R}^n} f \, g \, dx \right| \leq C \| f \|_{L_1} \| g \|_{\dot{F}^{s,q}_{r,w}}.
\]

Proof. Theorem 2.1 and Theorem 1.2 lead to
\[
\left| \int_{\mathbb{R}^n} f \, g \, dx \right| \leq \| f \|_{L_1} \| \nabla g \|_{\dot{F}^{0,2}_{d,w}} \simeq \| f \|_{L_1} \| g \|_{\dot{F}^{1,2}_{d,w}} \leq \| f \|_{L_1} \| g \|_{\dot{F}^{s,q}_{r,w}}.
\]

2.1. Extension of Sobolev-Gagliardo-Nirenberg inequality to differential forms. To simplify notation, we denote also by \( \dot{F}^{\gamma,q}_{p,w} \), the space of differential forms with coefficients in \( \dot{F}^{\gamma,q}_{p,w} \).

Theorem 2.2. Let \( 0 < p < \infty \) and \( w \in A_\infty \). Then we have
\[
\| \partial_j f \|_{\dot{F}^{\gamma,q}_{p,w}} \leq \| df \|_{\dot{F}^{\gamma,q}_{p,w}} + \| \delta f \|_{\dot{F}^{\gamma,q}_{p,w}}.
\]

Proof. We have
\[
\partial_j f \ast \nu_t \simeq t^{-1} f \ast (\partial_j \nu)_t
\]
so that
\[
\| \partial_j f \|_{\dot{F}^{\gamma,q}_{p,w}} \simeq \| f \|_{\dot{F}^{\gamma+1,q}_{p,w}}.
\]
Define the Riesz transform in $S_0'((\mathbb{R}^n, \Lambda))$ by
\[ \mathcal{R} = d \circ I_1 = I_1 \circ d \]
and its adjoint by
\[ \mathcal{R}^* = \delta \circ I_1 = I_1 \circ \delta, \]
where $S_0'((\mathbb{R}^n, \Lambda))$ is the dual space of $S_0((\mathbb{R}^n, \Lambda))$. It can be seen from (2.13) and the lifting property of $I_1$ that $\mathcal{R}$ and $\mathcal{R}^*$ are bounded on $F^{\gamma,q}_{p,w}$. Now the following identity
\[ \partial_j = R_j \circ \mathcal{R} \circ \delta + R_j \circ \mathcal{R}^* \circ d \]
iimplies what we want to prove. \hfill \square

**Corollary 2.2.** Let $0 < p < \infty$, $\gamma, \beta \in \mathbb{R}$ and $w \in A_\infty$ satisfying for some $d \geq n$, $w(B(x,t)) \geq C t^d$ for all $x$ and all $t > 0$. Then
\[ \|f\|_{F^{\gamma,q}_{p,w}} \leq \|df\|_{F^{\beta,q}_{p,w}} + \|\delta f\|_{F^{\beta,q}_{p,w}}, \]
with $\frac{1}{p_*} = \frac{1}{p} - \frac{1-\gamma+\beta}{d}$.  

**Proof.** Theorem 1.1 and Theorem 2.2 imply
\[ \|f\|_{F^{\gamma,q}_{p,w}} \simeq \|R_j f\|_{F^{\gamma,q}_{p,w}} \simeq \|I_1(\partial_j f)\|_{F^{\gamma,q}_{p,w}} \lesssim \|\partial_j f\|_{F^{\beta,q}_{p,w}} \lesssim \|df\|_{F^{\beta,q}_{p,w}} + \|\delta f\|_{F^{\beta,q}_{p,w}}. \]
\hfill \square

**Proof of Theorem 1.1**  We have
\[ \langle f, g \rangle = \langle f, (d \circ \delta \circ I_2 + \delta \circ d \circ I_2)g \rangle = \langle \delta f, \delta \circ I_2g \rangle + \langle df, d \circ I_2g \rangle = \langle \delta f, I_1 \circ \mathcal{R}^*g \rangle + \langle df, I_1 \circ \mathcal{R}g \rangle. \]

Then, by applying Theorem 2.2 and arguing as in [2] we obtain
\[ \|\langle f, g \rangle\| \lesssim \|\delta f\|_{L^1} + \|df\|_{L^1} \left(\|\nabla \circ I_1 \circ \mathcal{R}^*g\|_{L^d(w)} + \|\nabla \circ I_1 \circ \mathcal{R}g\|_{L^d(w)} \right) \lesssim \|\delta f\|_{L^1} + \|df\|_{L^1} \|g\|_{L^d(w)}. \]

Using the dual argument we finish the proof. \hfill \square

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**References**


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