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WEIGHTED ESTIMATES FOR L_1-VECTOR FIELDS

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(Communicated by Alexander Iosevich)

Dedicated to Professor Jean-Pierre Gossez on the occasion of his 70th birthday

ABSTRACT. In the context of weighted spaces, we study some types of Bourgain-Brezis and Lanzani-Stein inequalities.

INTRODUCTION

In the classical Hodge theory the following a priori estimate holds: if $f \in C_c^\infty(R^n; \Lambda^l)$ and $1 < p < \infty$, one has

$$||\nabla f||_{L^p} \preceq ||df||_{L^p} + ||\delta f||_{L^p}, \tag{0.1}$$

where d is the exterior differential operator and δ is its dual. The estimate (0.1) is known to be false if $p = 1$ or $p = \infty$. The Sobolev imbedding and the estimate (0.1) lead to

$$||f||_{L_{p_\star}} \preceq ||df||_{L^p} + ||\delta f||_{L^p}, \tag{0.2}$$

with $\frac{1}{p_\star} = \frac{1}{p} - \frac{1}{n}$. When $p = 1$, J. Bourgain and H. Brezis [1] have obtained the estimate

$$||f||_{L^{n/(n-1)}} \preceq ||df||_{L^1} + ||\delta f||_{L^1}. \tag{0.3}$$

L. Lanzani and E. Stein [9] have obtained the same result using an argument developed by J. Van Schaftingen [12]. I. Mitrea and M. Mitrea [11] have extended these estimates to homogeneous Besov spaces. Using interpolation theory, they could replace the norm $||f||_{L^{n/(n-1)}}$ by $||f||_{\dot{B}_p^{\gamma,q}}$ with $\frac{1}{p} - \frac{\gamma}{n} = 1 - \frac{1}{n}$, $0 < \gamma < 1$ and $q = \frac{2}{1-\gamma}$. In a recent work [13], Van Schaftingen has extended the result of Mitrea and Mitrea, taking any $q > 1$ and $0 < \gamma < 1$. In [10] we have also obtained some similar results; among them is the following estimate:
Let $\gamma \in \mathbb{R}$, $0 < p < \infty$ and w be in the Muckenhoupt class A_∞. Then

$$||\nabla f||_{\dot{F}_{p,w}^{\gamma,q}} \preceq ||df||_{\dot{F}_{p,w}^{\gamma,q}} + ||\delta f||_{\dot{F}_{p,w}^{\gamma,q}}. \tag{0.4}$$

A consequence of (0.4) is the following:

Let $\gamma,\ \beta \in \mathbb{R}$ s.t. $0 < 1 - \gamma + \beta$, $0 < p < \infty$ and $w \in A_\infty$ satisfying that for some $d > 0$, $w(B(x,t)) = \int_{B(x,t)} w dy \geq Ct^d$ for all x and all $t > 0$. Then

$$||f||_{\dot{F}_{p_\star,w}^{\gamma,q}} \preceq ||df||_{\dot{F}_{p,w}^{\beta,q}} + ||\delta f||_{\dot{F}_{p,w}^{\beta,q}}, \tag{0.5}$$

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with $\frac{1}{p_\star} = \frac{1}{p} - \frac{1-\gamma+\beta}{d}$. In particular we have

$$||f||_{L^{p_\star}(w)} \preceq ||df||_{H^p(w)} + ||\delta f||_{H^p(w)}, \tag{0.6}$$

with $\frac{1}{p_\star} = \frac{1}{p} - \frac{1}{d}$ and p is assumed to satisfy $p > \frac{d}{d+1}$. Here $H^p(w)$ denotes the weighted Hardy space. In particular, if we take $p = 1$, then

$$||f||_{L^{d/(d-1)}(w)} \preceq ||df||_{H^1(w)} + ||\delta f||_{H^1(w)}, \tag{0.7}$$

which is a good substitute for the estimate (0.3). In this work we extend the estimate (0.3) as follows.

Theorem 0.1. *Let γ, $\beta \in \mathbb{R}$ s.t. $0 < 1 - \gamma + \beta$ and $n \leq d < \infty$. Assume $w \in A_d(\mathbb{R} \times \mathbb{R}^{n-1})$ satisfying the condition* (2.2). *If $2 \leq l \leq n-2$, then*

$$||f||_{L^{p'}(w^{1-p'})} \preceq ||df||_{L^1} + ||\delta f||_{L^1}, \tag{0.8}$$

with $p' = \frac{d}{d-1}$.

Remark 0.1. When $l = 1$, a substitute for (0.8) holds with $||\delta f||_{L^1}$ replaced by $||\delta f||_{H^1}$. Similarly for $l = n-1$ when we replace $||df||_{L^1}$ by $||df||_{H^1}$.

Notation. The constant C may have different values even in the same line, but does not depend on f. $A \approx B$ for positive A and B means that there exists $C > 0$ such that $C^{-1}A \leq B \leq CA$ and $A \preceq B$ stands for $A \leq CB$.

1. Some background tools

1.1. **Muckenhoupt class.** We briefly recall some fundamentals on Muckenhoupt classes. Such classes are important tools in many areas of mathematics, including harmonic analysis and partial differential equations. Good references are [6,14,15].

Let $0 < p < \infty$. We say that $f \in L^p(w)$ if and only if

$$||f||_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty,$$

and f is in the weak-L^p space, or $f \in L^{p,\infty}(w)$ if and only if

$$||f||_{L^{p,\infty}(w)} = \sup_{\lambda > 0} \left(\lambda \left[w\left\{ x \in \mathbb{R}^n : |f(x)| > \lambda \right\} \right]^{\frac{1}{p}} \right) < \infty.$$

When $w = 1$ we drop the subscription w.

The Hardy-Littlewood maximal function Mf is defined, for a local integrable function f, by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{1.1}$$

where the supremum is taken over all cubes containing x.

A nonnegative locally integrable function w is said to be in the Muckenhoupt classes A_p if there exists a constant $C_p > 0$ such that for all cube Q,

$$\frac{1}{|Q|}\int_Q w dy \left(\frac{1}{|Q|}\int_Q w^{1-p'} dy\right)^{p-1} \leq C_p$$

when $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, as well as for $p = 1$,

$$\frac{1}{|Q|}\int_Q w dy \leq C_1 w(x),$$

for a.e. $x \in Q$ or, equivalently, $Mw(x) \leq C_1 w(x)$ for a.e. $x \in \mathbb{R}^n$. We set $A_\infty = \bigcup_{p\geq 1} A_p$.

It is well known that the classes A_p characterize the boundedness of the Hardy-Littlewodd maximal operator on the weighted Lebesgue spaces (see [6], ch. IV). Namely,

$$M : L^p(w) \to L^p(w)$$

if and only if $w \in A_p$ when $1 < p < \infty$, and

$$M :\ L^1(w) \to L^{1,\infty}(w)$$

if and only if $w \in A_1$.

1.2. **Weighted Lizorkin-Triebel space.** We denote by $S(\mathbb{R}^n)$ the space of Schwartz functions and by $S_\infty(\mathbb{R}^n)$ the subspace of functions in $S(\mathbb{R}^n)$ with all vanishing moments; i.e.,

$$S_\infty(\mathbb{R}^n) = \left\{\nu \in S : \int_{R^n} x^\beta \nu(x)dx = 0,\ \ \forall \beta \in \mathbb{N}^n\right\}.$$

The weighted homogeneous Lizorkin-Triebel space $\dot{F}^{\gamma,q}_{p,w}$ can be characterized as follows (see for instance [3] or [7]). Let $\gamma \in \mathbb{R}$, $0 < p,\ q < \infty$, $0 < \delta < min(p,q)$ and $w \in A_{p/\delta}$. Assume $\nu \in S_\infty(\mathbb{R}^n)$. Then $f \in \dot{F}^{\gamma,q}_{p,w}$ if and only if

$$||f||_{\dot{F}^{\gamma,q}_{p,w}} = \left|\left| \left(\int_0^\infty t^{-\gamma q}(\nu_t \star f)^q \frac{dt}{t}\right)^{\frac{1}{q}} \right|\right|_{p,w} < \infty, \tag{1.2}$$

with the usual modification if $p = \infty$. Here $\nu_t(x) = t^{-n}\nu(\frac{x}{t})$.

Remark 1.1. Note that

(1) if I_α denotes the Riesz potential defined by $\widehat{I_\alpha f}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi),\ \ \alpha \in \mathbb{R}$, then we have the following lifting property of I_α :

$$||I_\alpha f||_{\dot{F}^{\alpha+\gamma,q}_{p,w}} \approx ||f||_{\dot{F}^{\gamma,q}_{p,w}};$$

(2) if $R_j,\ \ j = 1,\ldots,n$, denotes the Riesz transform defined by $\widehat{R_j f}(\xi) = i\xi_j|\xi|^{-1}\hat{f}(\xi)$, then

$$||R_j f||_{\dot{F}^{\gamma,q}_{p,w}} \approx ||f||_{\dot{F}^{\gamma,q}_{p,w}};$$

(3) for a positive integer N, $f \in \dot{F}^{N,q}_{p,w}$ if and only if $\partial^\sigma f \in \dot{F}^{0,q}_{p,w}$ for all σ such that $|\sigma| = N$. Moreover,

$$||f||_{\dot{F}^{N,q}_{p,w}} \simeq \sum_{|\sigma|=N} ||\partial^\sigma f||_{\dot{F}^{0,q}_{p,w}}.$$

Indeed, the first assertion follows from the fact that $I_\alpha \nu$ behaves as ν, $\nu_t \star (I_\alpha f) = t^\alpha (I_\alpha \nu)_t \star f$ and $I_\alpha I_{-\alpha} = id$. The second assertion can be obtained from the identity $\nu_t \star (R_j f) = (R_j \nu)_t \star f$. Finally the identity

$$\partial^\sigma = (-1)^{|\sigma|} R^\sigma \circ I_{-|\sigma|}, \quad R = (R_1, \dots, R_n),$$

yields the third assertion.

One should also note (see [5, 10]) that

Theorem 1.1. *Assume $\beta \in \mathbb{R}$, $0 < p$, $d < \infty$, and $0 < q$, $r \leq \infty$. Let $\gamma \in \mathbb{R}$ with $0 < \alpha - \gamma + \beta$ and $w \in A_\infty$. If $w(B(x,t)) \geq Ct^d$ for all $0 < t < \infty$ and all x, then*

$$||I_\alpha f||_{\dot{F}^{\gamma,q}_{p_\star,w}} \leq C||f||_{\dot{F}^{\beta,r}_{p,w}}, \tag{1.3}$$

where $p_\star$ is determined by

$$\frac{1}{p_\star} = \frac{1}{p} - \frac{\alpha - \gamma + \beta}{d}.$$

In particular, we have

Theorem 1.2 ([4, 5, 10])**.** *Assume any reals α, γ s.t. $0 < \alpha - \gamma$, $0 < p < \infty$, $0 < q$, $r \leq \infty$ and $d > 0$. Let $w \in A_\infty$ and $0 < p_\star \leq \infty$ with*

$$\frac{1}{p_\star} = \frac{1}{p} - \frac{\alpha - \gamma}{d}.$$

If $w(B(x,t)) \geq Ct^d$ for all $0 < t < \infty$ and all x, then

$$\dot{F}^{\alpha,q}_{p,w} \subseteq \bigcap_{r>0} \dot{F}^{\gamma,r}_{p_\star,w}, \tag{1.4}$$

with the continuous imbedding $\dot{F}^{\alpha,q}_{p,w} \hookrightarrow \dot{F}^{\gamma,r}_{p_\star,w}$, for each $0 < r \leq \infty$.

Remark 1.2. Such weights in Theorem 1.1 and Theorem 1.2 exist. For instance, the function

$$w(x) = |x|^\alpha log^\beta (2 + |x|^{-1})$$

is in A_p, if $-n < \alpha < n(p-1)$ and $\beta \in \mathbb{R}$; see for example [8]. Moreover, w satisfies $w(B(x,t)) \geq Ct^{n+\alpha}$ for all $0 < t < \infty$ and all x whenever $0 \leq \alpha < n(p-1)$ and $\beta \geq 0$.

Another interesting example given in [8] is

$$w(x) = \begin{cases} |x|^\alpha & \text{if } |x| \leq 1, \\ |x|^\beta & \text{if } |x| > 1, \end{cases}$$

which belongs to A_p, $1 < p < \infty$, whenever $-n < \alpha$ and $\beta < n(p-1)$. If, in addition, $-n < \alpha \leq 0$ and $0 \leq \beta < n(p-1)$, then $w(B(x,t)) \geq Ct^n$ for all $0 < t < \infty$ and all x.

2. Weighted estimates for L_1-vector fields

In this section we assume that $w \in A_p(\mathbb{R} \times \mathbb{R}^{n-1})$; i.e., there exists a constant $C > 0$ such that for every interval $I \subset \mathbb{R}$ and every cube $Q \subset \mathbb{R}^{n-1}$,

$$\frac{1}{|I \times Q|} \int\limits_{I \times Q} w dy \left(\frac{1}{|I \times Q|} \int\limits_{I \times Q} w^{1-p'} dy \right)^{p-1} \leq C, \quad 1 < p < \infty. \tag{2.1}$$

We denote by Q_t a cube with radius $t > 0$ and by I_r an interval with radius $r > 0$.

Theorem 2.1. *Let $d \geq n$ and $w \in A_d(\mathbb{R} \times \mathbb{R}^{n-1})$. Assume that*

$$\int_{I_r \times Q_t} w dy \geq C r t^{d-1}, \quad \forall r,\ t > 0. \tag{2.2}$$

Then there exists a constant C such that for each $f, g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} f.g \, dx \right| \leq C \left(||div\, f||_{L_1} ||g||_{L_d(w)} + ||f||_{L_1} ||\nabla g||_{L_d(w)} \right). \tag{2.3}$$

To prove Theorem 2.1, we need the following lemma.

Lemma 2.1. *Let $w \in A_p(\mathbb{R}^m)$, $d \geq m+1$ and assume that $w(B(x,t)) \geq Ct^{d-1}$ for each $x \in \mathbb{R}^m$ and each $t > 0$. Let $g \in C_c^\infty(\mathbb{R}^m)$ and set*

$$g_1(x) = g(x) - \eta_\delta * g(x),\ g_2(x) = \eta_\delta * g(x),$$

where η is a smooth cut-off function supported in the unit ball with $\int \eta(x)dx = 1$ and $\eta_\delta(x) = \delta^{-m}\eta(x/\delta)$, $\delta > 0$. Then we have the following estimates:

$$||g_1||_{L_\infty} \leq C\delta^\gamma ||\nabla g||_{L_p(w)}, \tag{2.4}$$

$$||g_2||_{L_\infty} \leq C\delta^{\gamma-1} ||g||_{L_p(w)}, \tag{2.5}$$

$$||\nabla g_2||_{L_\infty} \leq C\delta^{\gamma-1} ||\nabla g||_{L_p(w)}, \tag{2.6}$$

with $\gamma = 1 - \frac{d-1}{p}$ provided that $p > d-1$.

Proof. We only prove the estimation (2.4). The proof of (2.5) or (2.6) is similar. We have

$$|g_1(x)| = \delta^{-m} \left| \int \eta(\frac{x-y}{\delta})(g(x) - g(y))dy \right| \leq C \int_{B(x,\delta)} |\nabla g(y)||x-y|^{-m+1} dy.$$

Using Hölder's inequality we get

$$|g_1(x)| \leq C \left(\int_{B(x,\delta)} |\nabla g(y)|^p w(y) dy \right)^{1/p} \left(\int_{B(x,\delta)} |x-y|^{(-m+1)p'} w^{-p'/p} dy \right)^{1/p'}. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned} &\int_{B(x,\delta)} |x-y|^{(-m+1)p'} w^{-p'/p} dy \\ &\quad \preceq \sum_{k \geq 0} (2^{-k}\delta)^{(-m+1)p'} \int_{B(x,2^{-k}\delta)} w^{-p'/p} dy \\ &\quad \preceq \sum_{k \geq 0} (2^{-k}\delta)^{(-m+1)p'} (2^{-k}\delta)^{mp'} \left(\int_{B(x,2^{-k}\delta)} w dy \right)^{\frac{-1}{p-1}} \\ &\quad \preceq \sum_{k \geq 0} (2^{-k}\delta)^{\frac{p}{p-1}} (2^{-k}\delta)^{\frac{1-d}{p-1}} \preceq \delta^{\frac{p+1-d}{p-1}}. \end{aligned} \tag{2.8}$$

Estimates (2.7) and (2.8) imply (2.4). □

Remark 2.1. An alternative proof of (2.4) is to use a weighted version of Theorem IX.12 in [2]. Indeed, one can adapt the argument given in [2] to prove that

$$|g(x) - g(y)| \leq C|x-y|^\gamma ||\nabla g(y)||_{L_p(w)}, \quad \text{for all } x,\ y \in \mathbb{R}^m.$$

This leads immediately to the estimate (2.4).

Proof of Theorem 2.1. We follow the argument in Van Schaftingen [12]. Without loss of generality we may assume that $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$.

Note first that if w satisfies (2.1) and (2.2) with $p = d$, then Lebesgue's Differentiation Theorem implies that for almost every $x_1 \in \mathbb{R}$ and every cube $Q_t \subset \mathbb{R}^{n-1}$,

$$\frac{1}{|Q_t|} \int_{Q_t} w(x_1, x')dx' \left(\frac{1}{|Q_t|} \int_{Q_t} w^{1-p'}(x_1, x')dx' \right)^{p-1} \le C$$

and

$$\int_{Q_t} w(x_1, x')dx' \ge Ct^{d-1}.$$

Fix $x_1 \in \mathbb{R}$ and let η be as in Lemma 2.1 with $m = n-1$ and let $\delta = \delta(x_1) > 0$ be chosen later.

Set

$$g_{1,x_1}(x') = g_{x_1}(x') - \eta_\delta * g_{x_1}(x'),$$
$$g_{2,x_1}(x') = \eta_\delta * g_{x_1}(x'),$$

and

$$J(x_1) = \int_{R^{n-1}} f_1(x_1, x')g_{x_1}(x')dx',$$

with $x' = (x_2, \dots, x_n)$ and $g_{x_1}(x') = g(x_1, x')$. Using the decomposition in Lemma 2.1, we get

$$J(x_1) = J(x_1) + J_2(x_1),$$

with

$$J_1(x_1) = \int_{R^{n-1}} f_1(x_1, x')g_{1,x_1}(x')$$

and

$$J_2(x_1) = \int_{R^{n-1}} f_1(x_1, x')g_{2,x_1}(x').$$

From (2.4) we have

$$|J_1(x_1)| \le C\delta^\gamma ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} ||f(x_1, \dots)||_{L_1}. \tag{2.9}$$

On the other hand,

$$\begin{aligned} J_2(x_1) &= \int_{R^{n-1}} \int_{-\infty}^{x_1} \frac{\partial f_1}{\partial t}(t, x')g_{2,x_1}(x')dtdx' \\ &= \int_{R^{n-1}} \int_{-\infty}^{x_1} div\, f(t, x')g_{2,x_1}(x') - \sum_{j=2}^{n} \int_{R^{n-1}} \int_{-\infty}^{x_1} \frac{\partial f_j}{\partial x_j}(t, x')g_{2,x_1}(x')dtdx' \\ &= \int_{-\infty}^{x_1} \int_{R^{n-1}} div\, f(t, x')g_{2,x_1}(x')dx'dt \\ &\quad + \sum_{j=2}^{n} \int_{-\infty}^{x_1} \int_{R^{n-1}} f_j(t, s)\frac{\partial g_{2,x_1}}{\partial x_j}(x')dx'dt. \end{aligned}$$

Using the estimates (2.5) and (2.6) and the last identity we obtain

(2.10)
$$|J_2(x_1)| \le C\delta^{\gamma-1} \left(||div\, f||_{L_1} ||g_{x_1}||_{L_p(w(x_1,\dots))} + ||f||_{L_1} ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} \right).$$

Estimates (2.9) and (2.10) lead to

$$|J(x_1)| \preceq \delta^\gamma ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} ||f(x_1,\dots)||_{L_1} \\ + \delta^{\gamma-1} \left(||div\, f||_{L_1} ||g_{x_1}||_{L_p(w(x_1,\dots))} + ||f||_{L_1} ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} \right). \tag{2.11}$$

Assume $||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} ||f(x_1,\dots)||_{L_1} \neq 0$ and choose δ so that

$$\delta^\gamma ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} ||f(x_1,\dots)||_{L_1} = \delta^{\gamma-1} \left(||div\, f||_{L_1} ||g_{x_1}||_{L_p(w(x_1,\dots))} \right. \\ \left. + ||f||_{L_1} ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} \right).$$

That is,

$$\delta = \frac{||div\, f||_{L_1} ||g_{x_1}||_{L_p(w(x_1,\dots))} + ||f||_{L_1} ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))}}{||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} ||f(x_1,\dots)||_{L_1}}. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\int_{\mathbb{R}} |J(x_1)| dx_1 \preceq \int_{\mathbb{R}} \left(||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} ||f(x_1,\dots)||_{L_1} \right)^{1-1/p} \\ \times \left(||div\, f||_{L_1} ||g_{x_1}||_{L_p(w(x_1,\dots))} + ||f||_{L_1} ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} \right)^{1/p} dx_1.$$

If $||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} ||f(x_1,\dots)||_{L_1} = 0$, the inequality is trivially true. Otherwise, if we use Hölder's inequality twice, we get

$$\int_{\mathbb{R}} |J(x_1)| dx_1 \preceq \left(\int_{\mathbb{R}} \left(||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} ||f(x_1,\dots)||_{L_1} \right)^{\frac{p}{p+1}} dx_1 \right)^{\frac{p^2-1}{p^2}} \\ \times \left(\int_{\mathbb{R}} \left(||div\, f||_{L_1} ||g_{x_1}||_{L_p(w(x_1,\dots))} + ||f||_{L_1} ||\nabla g_{x_1}||_{L_p(w(x_1,\dots))} \right)^p dx_1 \right)^{\frac{1}{p^2}} \\ \preceq \left(||\nabla g||_{L_p(w)} ||f||_{L_1} \right)^{\frac{p-1}{p}} \left(||div\, f||_{L_1} ||g||_{L_p(w)} + ||f||_{L_1} ||\nabla g||_{L_p(w)} \right)^{\frac{1}{p}} \\ \preceq ||div\, f||_{L_1} ||g||_{L_p(w)} + ||f||_{L_1} ||\nabla g||_{L_p(w)}.$$

□

Corollary 2.1. *Let d and w be as in Theorem 2.1. Then, for every $s \geq 1$, $r > 0$ and $q > 0$ with $sr = d$, there exists $C > 0$ such that for every $f, g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, if $div\, f = 0$, then*

$$\left| \int_{\mathbb{R}^n} f.g \, dx \right| \leq C ||f||_{L_1} ||g||_{\dot{F}^{s,q}_{r,w}}.$$

Proof. Theorem 2.1 and Theorem 1.2 lead to

$$\left| \int_{\mathbb{R}^n} f.g \, dx \right| \preceq ||f||_{L_1} ||\nabla g||_{\dot{F}^{0,2}_{d,w}} \simeq ||f||_{L_1} ||g||_{\dot{F}^{1,2}_{d,w}} \preceq ||f||_{L_1} ||g||_{\dot{F}^{s,q}_{r,w}}.$$

□

2.1. **Extension of Sobolev-Gagliardo-Nirenberg inequality to differential forms.** To simplify notation, we denote also by $\dot{F}^{\gamma,q}_{p,w}$, the space of differential forms with coefficients in $\dot{F}^{\gamma,q}_{p,w}$.

Theorem 2.2. *Let $0 < p < \infty$ and $w \in A_\infty$. Then we have*

$$||\partial_j f||_{\dot{F}^{\gamma,q}_{p,w}} \preceq ||df||_{\dot{F}^{\gamma,q}_{p,w}} + ||\delta f||_{\dot{F}^{\gamma,q}_{p,w}}.$$

Proof. We have

$$\partial_j f \star \nu_t \simeq t^{-1} f \star (\partial_j \nu)_t$$

so that

$$||\partial_j f||_{\dot{F}^{\gamma,q}_{p,w}} \simeq ||f||_{\dot{F}^{\gamma+1,q}_{p,w}}. \tag{2.13}$$

Define the Riesz transform in $S'_\infty(\mathbb{R}^n, \Lambda)$ by

$$\mathcal{R} = d \circ I_1 = I_1 \circ d$$

and its adjoint by

$$\mathcal{R}^* = \delta \circ I_1 = I_1 \circ \delta,$$

where $S'_\infty(\mathbb{R}^n, \Lambda)$ is the dual space of $S_\infty(\mathbb{R}^n, \Lambda)$. It can be seen from (2.13) and the lifting property of I_1 that $\mathcal{R}$ and $\mathcal{R}^*$ are bounded on $\dot{F}^{\gamma,q}_{p,w}$. Now the following identity

$$\partial_j = R_j \circ \mathcal{R} \circ \delta + R_j \circ \mathcal{R}^* \circ d$$

implies what we want to prove. $\square$

Corollary 2.2. *Let $0 < p < \infty$, γ, $\beta \in \mathbb{R}$ and $w \in A_\infty$ satisfying for some $d \geq n$, $w(B(x,t)) \geq Ct^d$ for all x and all $t > 0$. Then*

$$||f||_{\dot{F}^{\gamma,q}_{p_\star,w}} \preceq ||df||_{\dot{F}^{\beta,q}_{p,w}} + ||\delta f||_{\dot{F}^{\beta,q}_{p,w}},$$

with $\frac{1}{p_\star} = \frac{1}{p} - \frac{1-\gamma+\beta}{d}$.

Proof. Theorem 1.1 and Theorem 2.2 imply

$$||f||_{\dot{F}^{\gamma,q}_{p_\star,w}} \simeq ||R_j f||_{\dot{F}^{\gamma,q}_{p_\star,w}} \simeq ||I_1(\partial_j f)||_{\dot{F}^{\gamma,q}_{p_\star,w}} \preceq ||\partial_j f||_{\dot{F}^{\beta,q}_{p,w}} \preceq ||df||_{\dot{F}^{\beta,q}_{p,w}} + ||\delta f||_{\dot{F}^{\beta,q}_{p,w}}.$$

$\square$

Proof of Theorem 0.1. We have

$$\begin{aligned}\langle f, g\rangle = \langle f, (d \circ \delta \circ I_2 + \delta \circ d \circ I_2)g\rangle &= \langle \delta f, \delta \circ I_2 g\rangle + \langle df, d \circ I_2 g\rangle \\ &= \langle \delta f, I_1 \circ \mathcal{R}^* g\rangle + \langle df, I_1 \circ \mathcal{R} g\rangle.\end{aligned}$$

Then, by applying Theorem 2.1 and arguing as in [9] we obtain

$$\begin{aligned}|\langle f, g\rangle| &\preceq (||\delta f||_{L^1} + ||df||_{L^1}) \left(||\nabla \circ I_1 \circ \mathcal{R}^* g||_{L^d(w)} + ||\nabla \circ I_1 \circ \mathcal{R} g||_{L^d(w)}\right) \\ &\preceq (||\delta f||_{L^1} + ||df||_{L^1}) \, ||g||_{L^d(w)}.\end{aligned}$$

Using the dual argument we finish the proof. $\square$

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