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AFFINE VARIETIES WITH EXOTIC MODELS

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Abstract. We show that for every $n \geq 7$ there is a smooth rational affine variety with exotic model. Moreover, we show that for every $n \geq 6$ there are Zariski open subsets U_1, U_2 of $\mathbb{C}^n$, such that they are holomorphically but not algebraically equivalent.

1. Introduction

Given any smooth complex affine variety X, one can ask if there exists smooth affine varieties Y non-isomorphic to X but which are biholomorphic to X when equipped with their underlying structures of complex analytic manifolds. When such exist, these varieties Y could be called exotic models of X.

Examples of affine varieties with exotic models was found in dimensions two and three (see [2], [3], [4], [10], [11]). The aim of this note is to give another method of construction of such examples (in higher dimensions).

Our idea (similar to that of our paper [5]) is as follows. Let X be a smooth affine variety. Denote a trivial algebraic vector bundle of rank s on X by $\mathbf{E}_s$. Assume that X admits an algebraic vector bundle $\mathbf{F}$ of rank s which is algebraically non-trivial but holomorphically trivial. Let $\mathcal{F}$ denote the total space of $\mathbf{F}$ and $\mathcal{E}$ denote the total space of the trivial vector bundle $\mathbf{E}_s$. Then $\mathcal{F}$ is holomorphically equivalent with $\mathcal{E}$, but we show that $\mathcal{F}$ is not algebraically isomorphic to $\mathcal{E}$ if X is not $\mathbb{C}$-uniruled.

Accordingly, to find an affine variety with exotic model, it is enough to find a smooth affine non-$\mathbb{C}$-uniruled variety with an algebraically non-trivial vector bundle, which is holomorphically trivial, on it. We do this by modifying a well-known Mohan-Kumar construction (see [8], [9]).

At the end of this note we give examples of two holomorphically equivalent Zariski open subsets of some $\mathbb{C}^n$, which has different algebraic structures.

2. Exotic models

Let us recall the definition of a $\mathbb{C}$-uniruled variety which was introduced in our paper [7]. First recall that *a polynomial curve* in X is the image of the affine line $A^1(\mathbb{C})$ under a non-constant morphism $\phi : A^1(\mathbb{C}) \to X$. Now we have:

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Definition 2.1. An affine variety X is said to be $\mathbb{C}$-*uniruled* if it is of dimension ≥ 1 and there exists a Zariski open, non-empty subset U of X such that for every point $x \in U$ there is a polynomial curve in X passing through x.

We have the following important result:

Proposition 2.2. *Let X be an affine variety and r be a positive integer. There exists an affine non-$\mathbb{C}$-uniruled variety Y of dimension $d = \dim X + r$ such that*
1) $X \subset Y$,
2) *there is a polynomial retraction* $\pi : Y \to X$.
Moreover, if X is smooth and rational, then Y is also smooth and rational.

Proof. Let $\Gamma = X \times \mathbb{C}^r$. Let $x_1, \ldots, x_s$ be generators of the ring $\mathbb{C}[X]$. Hence the mapping $\iota : X \ni x \to (x_1(x), \ldots, x_s(x)) \in \mathbb{C}^s$ is a closed embedding. Put

$$Y := \{(x, z) \in \Gamma : \prod_{i=1}^{r} z_i \neq 0;\ 1 + x_j(\prod_{i=1}^{r} z_i - 1) \neq 0, j = 1, \ldots, s\}.$$

Let $\phi : \mathbb{C} \to Y$ be a regular mapping. Thus $\phi = (a_1(t), \ldots, a_s(t); b_1(t), \ldots, b_r(t))$, where a_i, b_j are polynomials. Since $(\prod_{i=1}^r z_i) \circ \phi = (\prod_{i=1}^r b_i) \neq 0$, we have that $b_1, \ldots, b_r$ are non-zero constants. Similarly $1 + a_i(\prod_{i=1}^r b_j - 1)$ is a constant. Consequently, either $(\prod_{i=1}^r b_i) = 1$ or all a_j are constant. This means that outside the hypersurface $\{\prod_{i=1}^r z_i = 1\}$ there are no polynomial curves in Y. Let us identify X with $X \times \{(1, \ldots, 1)\} \subset Y$. Now we can define the mapping π as

$$\pi : Y \ni (x, z) \to (x, (1, 1, \ldots, 1)) \in X.$$

□

In the sequel we need the following basic theorem, which was proved in our paper [5]:

Theorem 2.3. *Let X be a non-$\mathbb{C}$-uniruled smooth affine variety. Let $\mathbf{F}$ be an algebraic vector bundle on X of* rank r. *If the total space of $\mathbf{F}$ is isomorphic to $X \times \mathbb{C}^r$, then $\mathbf{F}$ is a trivial vector bundle.*

Now we give a new method of construction of smooth rational affine varieties which have exotic models. In [8] (see also [9]) Mohan Kumar has constructed some nice examples of stably free algebraic vector bundles, which are not free. More precisely, he constructed for every prime number p a smooth rational affine variety X_{p+2}, of dimension $p + 2$ and an algebraic vector bundle $\mathbf{A_p}$ on X_{p+2} of rank p, which is stably trivial but not trivial. Moreover, these vector bundles are holomorphically trivial (in fact every such vector bundle of rank greater or equal to $p/2 + 1$ must be holomorphically trivial; see [12]). We use these examples to prove:

Theorem 2.4. *For every $n \geq 7$ there is a smooth affine rational variety $\mathcal{F}$ of dimension n, which has an exotic model $\mathcal{E}$ (which is also affine and rational).*

Proof. As we have mentioned above, Mohan-Kumar has constructed a smooth rational variety X of dimension 4 and an algebraic vector bundle $\mathbf{A}$ of rank 2 on it. This vector bundle $\mathbf{A}$ is algebraically non-trivial but holomorphically trivial.

For a given integer $r > 0$ let Y_{4+r} be a non-uniruled variety of dimension $4 + r$ constructed in Proposition 2.2. Let $\pi : Y_{4+r} \to X$ be a retraction and let $\mathbf{F}_r = \pi^*\mathbf{A}$. Observe that $\mathbf{F}_r|X = \mathbf{A}$, in particular vector bundle $\mathbf{F}_r$, is algebraically non-trivial. Since $\mathbf{A}$ is holomorphically trivial, we also have that its pullback $\mathbf{F}_r$ is holomorphically trivial. Let $\mathbf{E}_r$ be a trivial vector bundle of rank 2 on Y_{4+r}. By

Theorem 2.3 total spaces $\mathcal{F}_r$ and $\mathcal{E}_r$ of vector bundles $\mathbf{F}_r$ and $\mathbf{E}_r$ are not isomorphic as algebraic varieties. However, in an obvious way $\mathcal{F}_r$ and $\mathcal{E}_r$ are biholomorphic as total spaces of the same trivial holomorphic vector bundle. Moreover, dim $\mathcal{F}_r = 6 + r$. □

Remark 2.5. Since the vector bundle $\mathbf{F}_r$ is stably trivial, we have $\mathcal{F} \times \mathbb{C} \cong \mathcal{E} \times \mathbb{C}$ (as algebraic varieties).

Remark 2.6. If we have one space $\mathcal{F}_r$ which is a total space of a suitable vector bundle over $Y \subset \Gamma = X \times \mathbb{C}^r$, we can easily construct infinitely many pairwise non-isomorphic varieties ${\mathcal{F}_r}^l$ of this type. Indeed, choose sufficiently general polynomials $a_i \in \mathbb{C}[\Gamma],\ i = 1, 2, \dots$. Let $Y_l = \{(x, y) \in Y : 1 \neq a_i(\prod_{i=1}^r z_i - 1),\ i = 1, \dots, l\}$. Let ${\mathcal{F}_r}^l$ be a total space of the vector bundle $\pi_l^*(\mathbf{A})$, where $\pi_l : Y_l \to X$ is a suitable retraction. In this way we obtain a strictly descending sequence of varieties ${\mathcal{F}_r}^0 \supset {\mathcal{F}_r}^1 \supset {\mathcal{F}_r}^2 \supset \dots$, which have exotic models. They are pairwise non-isomorphic by the Ax Theorem (see [1]).

We end this note showing that for every $n \geq 16$ there exist Zariski open subsets of $\mathbb{C}^n$, which are biholomorphic but not isomorphic as algebraic varieties. The following theorem was proved in our paper [6], Theorem 5.4:

Theorem 2.7. *Let $X, Y \subset \mathbb{C}^n$ be smooth complex algebraic submanifolds of dimension k. Let $f : X \to Y$ be a biholomorphism. If $n \geq 2k + 2$, then f can be extended to a tame biholomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$. Moreover, there is a smooth family of tame biholomorphisms $F_t : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, such that $F_0 =$ identity and $F_1|_X = f$.*

Now we are in a position to prove:

Theorem 2.8. *Let $n \geq 16$. Then there exist two Zariski open subsets $U_1, U_2 \subset \mathbb{C}^n$ such that U_1 is biholomorphic to U_2 but U_1 is not isomorphic to U_2 as an algebraic variety.*

Proof. Let X, Y be two smooth affine varieties of dimension 7, which are holomorphically equivalent, but not algebraically isomorphic (we have just constructed such varieties in Theorem 2.4). Since $n \geq 16$ we can embed these varieties into $\mathbb{C}^n$. Put $U_1 = \mathbb{C}^n \setminus X$ and $U_2 = \mathbb{C}^n \setminus Y$. By Theorem 2.7 varieties U_1 and U_2 are holomorphically isomorphic. Now we prove that U_1 and U_2 are not algebraically isomorphic. Indeed, let $\phi : U_1 \to U_2$ be such an isomorphism. Since codim $X, Y > 1$ we can extend this isomorphism to a global isomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$. This implies that X is isomorphic to Y—a contradiction. □

Remark 2.9. Since there are known examples of affine surfaces with exotic models (see [4], [10], [11]) the number 16 in Theorem 2.8 can be replaced by a smaller number 6.

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