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A TREE-FREE GROUP THAT IS NOT ORDERABLE

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ABSTRACT. I. M. Chiswell has asked whether every group that admits a free isometric action (without inversions) on a Λ-tree is orderable. We give an example of a multiple HNN extension Γ which acts freely on a $\mathbb{Z}^2$-tree but which has non-trivial generalised torsion elements. The existence of such elements implies that Γ is not orderable.

Let Λ be an ordered abelian group. A group is Λ-*free* if it admits a free isometric action without inversions on a Λ-tree, and *tree-free* if it is Λ'-free for some Λ'. We refer to the book [4] for a detailed account of the fundamentals of Λ-trees.

In this book Chiswell asks [4, §5.5, Question 3] whether all tree-free groups are orderable, or at least right-orderable. (This question also appeared in his earlier paper [3].) There has recently been some progress made on questions of orderability in tree-free groups. Chiswell himself has shown [6, Theorem 3.8] that $\mathbb{R}^n$-free groups are right-orderable. Kharlampovich, Myasnikov and Serbin have shown [10, Corollary 4] that finitely presented tree-free groups are $\mathbb{R}^n$-free for some n; thus these groups are right-orderable. Chiswell has shown moreover [5, Theorem 4.5] that tree-free groups admit a locally invariant order.

In their recent survey Kharlampovich, Myasnikov and Serbin state [11, Corollary 19] that finitely presented tree-free groups have a finite index subgroup that embeds in a right-angled Artin group: this is a consequence of their result [10, Theorem 2] and the extensive work of Wise (see [16, §16] and [15]) on quasi-convex hierarchies on groups. Since right-angled Artin groups are residually torsion-free nilpotent (see [8, Chapter 3, Theorem 1.1]), it follows that finitely presented tree-free groups are virtually residually torsion-free nilpotent; hence they are *virtually* orderable.

Let us also note that tree-free surface groups admit an embedding in a right-angled Artin group (see [7, Theorem 3]); it follows, using Rips' Theorem (see [4, Chapter 6]), that finitely generated $\mathbb{R}$-free groups admit an embedding in a right-angled Artin group.

The author has recently [13] raised the question of whether $\mathbb{Z}^n$-free groups are residually torsion-free nilpotent. An affirmative answer to this question would have implied that $\mathbb{Z}^n$-free groups are orderable.

Nevertheless the answer to Chiswell's question is negative, even when restricted to finitely presented $\mathbb{Z}^2$-free groups as we will show presently. It follows that the

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word ‘virtually’ cannot be dropped in the discussion above. This suggests an analogy with the situation of braid groups B_n and their finite index subgroups, the pure braid groups P_n: the former are right-orderable but not orderable (see [14, §4]), while the latter are residually torsion-free nilpotent [9]. Moreover finitely presented tree-free groups, like braid groups, are now known to be linear: see [11, Theorem 68], [2] and [12].

Recall that a group G is *orderable* if there is a linear order $\leq$ on G satisfying $x \leq y \Rightarrow gxh \leq gyh$ for $g, h \in G$. (One can define right-orderable by restricting to $g = 1$.) It is well-known and easy to see that in an orderable group G there can be no non-trivial *generalised torsion elements*: these are elements g such that $g^{h_1} g^{h_2} \cdots g^{h_n} = 1$ for some $h_1, \ldots, h_n \in G$ and $n \geq 1$. (Here g^h denotes the conjugate $h^{-1}gh$.)

Let F be the free group on $\{x, y, z\}$, and consider the natural free action of F on the corresponding Cayley graph, viewed as a $\mathbb{Z}$-tree. Observe that xy^{-1}, yz^{-1} and zx^{-1} and their respective inverses belong to distinct conjugacy classes since they are cyclically reduced as elements of F and none is a cyclic permutation of another. Moreover, the translation length of $g \in F$ is equal to the word length of a cyclically reduced conjugate of g. Thus the translation lengths of xy^{-1}, yz^{-1} and zx^{-1} are all equal to 2.

Now taking $s_1 = xy^{-1} = s_2$, $t_1 = yz^{-1}$, $t_2 = zx^{-1}$, $u = u_1$ and $v = u_2$, and applying [1, Proposition 4.19], the multiple HNN extension

$$\begin{aligned} \Gamma &= \langle x, y, z, u_1, u_2 \mid u_1 s_1 u_1^{-1} = t_1,\ u_2 s_2 u_2^{-1} = t_2 \rangle \\ &= \langle u, v, x, y, z \mid u(xy^{-1})u^{-1} = yz^{-1},\ v(xy^{-1})v^{-1} = zx^{-1} \rangle \end{aligned}$$

is seen to be $\mathbb{Z}^2$-free. However,

$$1 = (xy^{-1})(yz^{-1})(zx^{-1}) = xy^{-1} \cdot u(xy^{-1})u^{-1} \cdot v(xy^{-1})v^{-1},$$

whence xy^{-1} is a non-trivial generalised torsion element of Γ, and Γ is not orderable. This gives the promised negative answer to Chiswell's question.

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