

ON ITERATED FUNCTION SYSTEMS WITH PLACE-DEPENDENT PROBABILITIES

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ABSTRACT. In this paper we study a family of invariant measures of parameterized iterated function systems where the corresponding probabilities are place-dependent. We prove that the Hausdorff dimension of the measure is equal to entropy/Lyapunov exponent whenever it is less than 1 and the measure is absolute continuous w.r.t. the Lebesgue measure if entropy/Lyapunov exponent is greater than 1 for Lebesgue almost every parameters.

1. INTRODUCTION AND STATEMENTS

Let X be a compact interval on the real line and let $\{\psi_i\}_{i=1}^k$ be a family of contractive maps mapping X into itself. We call the set $\{\psi_i\}_{i=1}^k$ an *iterated function system* (IFS) on X . It is well known that there exists a unique non-empty compact set $\Lambda \subseteq X$ such that it is invariant w.r.t. the IFS, that is, $\Lambda = \bigcup_{i=1}^k \psi_i(\Lambda)$. We call the set Λ the *attractor* of the IFS. Moreover, for any probability weights $\{p_i\}_{i=1}^k$ such that $0 < p_i < 1$ and $\sum_{i=1}^k p_i = 1$, there exists a unique probability measure μ that satisfies $\text{supp}\mu = \Lambda$ and

$$(1.1) \quad \mu = \sum_{i=1}^k p_i \mu \circ \psi_i^{-1}.$$

The measure μ is called the *invariant measure* of the IFS (see e.g. Hutchinson [5], Falconer [2]).

In this paper we focus on an extended class of invariant measures. We consider the probability measures on the set Λ that satisfy the equation

$$(1.2) \quad \int f(x) d\mu(x) = \sum_{i=1}^k \int p_i(x) f(\psi_i(x)) d\mu(x) \text{ for every } f \in C(X),$$

where $p_i : X \mapsto (0, 1)$ are Hölder continuous for every $i = 1 \dots k$ and $\sum_{i=1}^k p_i(x) \equiv 1$. Fan and Lau proved that there exists a unique probability measure which satisfies equation (1.2) and $\text{supp}\mu = \Lambda$; see [3]. Let us call the measure μ *place-dependent invariant measure*.

Place-dependent invariant measures were studied in several papers; see e.g. [4, 6–8, 19]. Our goal is to determine the *Hausdorff dimension* of such measures and to give a sufficient condition for the absolute continuity w.r.t. the Lebesgue

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measure. For the basic properties and definition of the Hausdorff dimension see for example [2].

For further analysis let us introduce two notations. Let us define the *entropy* h_μ of the measure μ as

$$(1.3) \quad h_\mu := - \int \sum_{i=1}^k p_i(x) \log p_i(x) d\mu(x).$$

Moreover, denote χ_μ the *Lyapunov exponent* of the IFS w.r.t. μ . That is,

$$(1.4) \quad \chi_\mu := - \int \sum_{i=1}^k p_i(x) \log |\psi'_i(x)| d\mu(x).$$

Fan and Lau showed that if the IFS $\{\psi_i\}_{i=1}^k$ satisfies the so-called *open set condition* (i.e. there exists an open set U such that $\psi_i(U) \cap \psi_j(U) = \emptyset$ for every $i \neq j$), then

$$\dim_H \mu = \frac{h_\mu}{\chi_\mu},$$

where $\dim_H \mu$ denotes the Hausdorff dimension of the measure μ ; see [3, Corollary 3.5]. Jaroszewska and Rams proved that without any separation condition the entropy divided by the Lyapunov exponent is always an upper bound for the Hausdorff dimension; see [7, Theorem 1]. Furthermore, Lau, Ngai and Wang constructed absolutely continuous place-dependent invariant measures defined by non-linear IFSs with overlaps; see [9, Section 7]. However, sufficient condition for absolute continuity is not known in general.

In the case of ordinary (not place-dependent) invariant measures (see (1.1)), to prove absolute continuity in general, essentially the only approach is the so-called *transversality method* which was first introduced by Pollicott and Simon [15]. Simon, Solomyak and Urbański considered parameterized families of iterated function systems and proved that the Hausdorff dimension of the invariant measure is the minimum of the entropy divided by the Lyapunov exponent and 1. Further, the measure is absolute continuous if the entropy/Lyapunov exponent is strictly greater than 1 for Lebesgue almost every parameters whenever the IFS satisfies the *transversality condition*; see [18]. Our main theorem establishes this result for place-dependent invariant measures.

Let $U \subset \mathbb{R}^d$ be an open, bounded set. Let us consider a family of IFSs $\Psi_\lambda = \{\psi_i^\lambda\}_{i=1}^k$, $\lambda \in \bar{U}$. Denote $\mathcal{S} = \{1, \dots, k\}$ the set of symbols and $\Sigma = \mathcal{S}^{\mathbb{N}}$ the *symbolic space*. Let us define the *natural projection* as

$$(1.5) \quad \pi_\lambda(\mathbf{i}) := \lim_{n \rightarrow \infty} \psi_{i_0}^\lambda \circ \dots \circ \psi_{i_n}^\lambda(0), \text{ for } \mathbf{i} = (i_0 i_1 \dots) \in \Sigma.$$

Let us suppose that Ψ_λ and the weights $\{p_i(x)\}_{i \in \mathcal{S}}$ satisfy the following conditions.

Principal Assumptions:

- (A1) CONTINUITY: the maps $\lambda \mapsto \psi_i^\lambda$ are continuous from \bar{U} to $C^{1+\theta}(X)$ for every $i \in \mathcal{S}$.
- (A2) HYPERBOLICITY: there exist $0 < \gamma < \kappa < 1$ such that $\gamma < |(\psi_i^\lambda)'(x)| < \kappa$ for every $\lambda \in \bar{U}$, $x \in X$ and $i \in \mathcal{S}$.
- (A3) TRANSVERSALITY: there exists a constant C_1 such that for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$

$$\mathcal{L}_d \{ \lambda \in U : |\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})| < r \} < C_1 r \text{ for all } r > 0.$$

(A4) PROBABILITIES: the maps $p_i : X \mapsto (0, 1)$ are Hölder continuous and bounded away from zero for every $i \in \mathcal{S}$. Moreover, $\sum_{i=1}^k p_i(x) \equiv 1$.

Theorem 1.1. *Suppose that the family $\{\Psi_\lambda\}_{\lambda \in \bar{U}}$ of iterated function systems satisfies the assumptions (A1), (A2) and (A3). Moreover, let $\{p_i(x)\}_{i \in \mathcal{S}}$ be place-dependent probability weights satisfying (A4) and let μ_λ be the place-dependent invariant measure according to the weights $\{p_i(x)\}_{i \in \mathcal{S}}$. Then*

(1) for Lebesgue almost every $\lambda \in U$

$$\dim_H \mu_\lambda = \min \left\{ \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}}, 1 \right\},$$

(2) $\mu_\lambda \ll \mathcal{L}_1$ for Lebesgue almost every $\lambda \in \left\{ \lambda \in U : \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}} > 1 \right\}$.

As an easy consequence we have the following corollary.

Corollary 1.2. *The measure μ_λ is equivalent with $\mathcal{L}_1|_{\Lambda_\lambda}$ for Lebesgue almost every $\lambda \in \left\{ \lambda \in U : \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}} > 1 \right\}$.*

Proof. The statement follows immediately from Theorem 1.1(2) and [4, Theorem 1.1]. \square

Remark 1.3. The statement of Theorem 1.1 is also valid when the probabilities depend on the parameters continuously. Precisely, when the function $\lambda \mapsto p_i^\lambda$ is continuous from \bar{U} to $C^0(X)$ for every $i \in \mathcal{S}$.

Remark 1.4. In the assumption (A4) the Hölder continuity implies the uniqueness of the measure. Jaroszevska showed that if the probability weights are only continuous, then the place-dependent invariant measure is not necessarily unique; see [6, Theorem 4].

The organization of the paper is as follows: Section 2 is devoted to preliminaries, and we prove our main theorem in Section 3. The proof follows the idea and the method of [18]. Simon, Solomyak and Urbański [18] investigated the dimension theory of the push-down measures of fixed left-shift invariant measures on the symbolic space. Since our measure is place-dependent, the induced measure on the symbolic space will depend on the natural projection and therefore on the parameters. So it is not possible to apply the known methods directly. To avoid the difficulty that the induced measure on the symbolic space depends also on the parameters, we apply the method introduced by Persson [13]. In Section 4 we apply our results for place-dependent Bernoulli convolutions.

2. PRELIMINARIES

First of all, we introduce some standard notation. Denote σ the left-shift operator on Σ , that is,

$$\sigma(i_0 i_1 \dots) = (i_1 i_2 \dots).$$

It is easy to see from the definition of the natural projection (1.5) that

$$(2.1) \quad \pi_\lambda(\mathbf{i}) = \psi_{i_0}^\lambda(\pi_\lambda(\sigma \mathbf{i})).$$

Let us define the set $\mathcal{S}^* = \cup_{n=0}^\infty \mathcal{S}^n$ as the set of finite length words of symbols, and for $\mathbf{i} = (i_0 i_1 \dots) \in \Sigma$ let $\mathbf{i}|_n = (i_0 i_1 \dots i_{n-1})$. Moreover, for a finite length word $(i_0 \dots i_n)$ let $[i_0 \dots i_n]$ be the corresponding cylinder set,

$$[i_0 \dots i_n] := \{\mathbf{j} = (j_0 j_1 \dots) \in \Sigma : j_0 = i_0, \dots, j_n = i_n\}.$$

Furthermore, let $\mathbf{i} \wedge \mathbf{j} = \min\{n : i_n \neq j_n\}$.

Let us consider a set of place-dependent probability weights $\{p_i(x)\}_{i \in \mathcal{S}}$ and a family of parameterized IFS $\Psi_\lambda = \{\psi_i^\lambda\}_{\lambda \in \mathcal{S}}$ and suppose they satisfy conditions (A1), (A2) and (A4). Then, without loss of generality, we can assume that there exist constants $C, q > 0$ and $0 < \theta < 1$ such that

$$(2.2) \quad |p_i(x) - p_i(y)| \leq C|x - y|^\theta \text{ and } p_i(x) > q > 0 \text{ for every } i \in \mathcal{S} \text{ and } x, y \in X,$$

and

$$(2.3) \quad |(\psi_i^\lambda)'(x) - (\psi_i^\lambda)'(y)| \leq C|x - y|^\theta \text{ for every } x, y \in X, i \in \mathcal{S} \text{ and } \lambda \in \bar{U}.$$

Define the corresponding Ruelle operator $T_\lambda : C(X) \mapsto C(X)$, where $C(X)$ is the set of continuous functions and

$$(2.4) \quad (T_\lambda f)(x) = \sum_{i=1}^k p_i(x) f(\psi_i^\lambda(x)),$$

and let $T_\lambda^* : M(X) \mapsto M(X)$ be the adjoint operator, where $M(X)$ denotes the set of Borel probability measures on X .

Proposition 2.1 (Fan, Lau). *Suppose that $\{p_i(x)\}_{i \in \mathcal{S}}$ satisfies (2.2) and let $\{\psi_i^\lambda\}_{i=1}^k$ be an IFS on X satisfying (A2). Then for every $\lambda \in \bar{U}$ there exists a unique probability measure μ_λ such that*

$$T_\lambda^* \mu_\lambda = \mu_\lambda.$$

Moreover, for every $f \in C(X)$, $T_\lambda^n f$ converges uniformly to $\int f(x) d\mu_\lambda(x)$.

Proof. The proposition follows from [3, Theorem 1.1]. □

The uniqueness of the measure μ_λ implies that it is necessarily in *pure type*, that is, the measure μ_λ is either singular or absolutely continuous w.r.t. the Lebesgue measure.

An important characterization of the measure μ_λ is that μ_λ is a push-down measure of a Gibbs measure on the symbolic space. To show this phenomena, we need the following lemma.

Lemma 2.2. *There exist a unique, ergodic, left-shift invariant probability measure ν_λ on Σ and a constant $c > 1$ independent of λ such that for every $\mathbf{i} \in \Sigma$ and $n \geq 1$*

$$(2.5) \quad c^{-1} \leq \frac{\nu_\lambda([\mathbf{i}|_n])}{\prod_{m=0}^{n-1} p_{i_m}(\pi_\lambda(\sigma^{m+1}\mathbf{i}))} \leq c.$$

Moreover, the entropy h_{ν_λ} of ν_λ satisfies

$$(2.6) \quad h_{\nu_\lambda} = - \int_\Sigma \sum_{i=1}^k p_i(\pi_\lambda(\mathbf{i})) \log p_i(\pi_\lambda(\mathbf{i})) d\nu_\lambda(\mathbf{i}).$$

Proof. Let $\varphi_\lambda(\mathbf{i}) := \log p_{i_0}(\pi_\lambda(\sigma\mathbf{i}))$ be the potential function and let $\mathcal{T}_\lambda : C(\Sigma) \mapsto C(\Sigma)$ be the corresponding operator

$$(2.7) \quad (\mathcal{T}_\lambda f)(\mathbf{i}) := \sum_{i=1}^k e^{\varphi_\lambda(i\mathbf{i})} f(i\mathbf{i}) = \sum_{i=1}^k p_i(\pi_\lambda(\mathbf{i})) f(i\mathbf{i}).$$

It is easy to see that there are constants $b > 0$ and $0 < \alpha < 1$ such that

$$|\varphi_\lambda(\mathbf{i}) - \varphi_\lambda(\mathbf{j})| \leq b\alpha^{i \wedge j} \text{ for every } \mathbf{i}, \mathbf{j} \in \Sigma.$$

Then the statement of the lemma follows by standard argument applying [1, Theorem 1.4, Proposition 1.14, Theorem 1.22]. \square

Lemma 2.3. *The measure μ_λ is the push-down measure of ν_λ . That is,*

$$\mu_\lambda = (\pi_\lambda)_* \nu_\lambda = \nu_\lambda \circ \pi_\lambda^{-1}.$$

Proof. Let $f : X \mapsto \mathbb{R}$ continuous. Then

$$\int_X f(x) d(\pi_\lambda)_* \nu_\lambda(x) = \int_\Sigma f(\pi_\lambda(\mathbf{i})) d\nu_\lambda(\mathbf{i}).$$

Applying the operator \mathcal{T}_λ defined in (2.7), we have

$$\begin{aligned} \int_\Sigma f(\pi_\lambda(\mathbf{i})) d\nu_\lambda(\mathbf{i}) &= \int_\Sigma \sum_{i=1}^k p_i(\pi_\lambda(\mathbf{i})) f(\pi_\lambda(i\mathbf{i})) d\nu_\lambda(\mathbf{i}) \\ &= \int_\Sigma \sum_{i=1}^k p_i(\pi_\lambda(\mathbf{i})) f(\psi_i^\lambda(\pi_\lambda(\mathbf{i}))) d\nu_\lambda(\mathbf{i}) = \int_X \sum_{i=1}^k p_i(x) f(\psi_i^\lambda(x)) d(\pi_\lambda)_* \nu_\lambda(x) \\ &= \int_X (\mathcal{T}_\lambda f)(x) d(\pi_\lambda)_* \nu_\lambda(x). \end{aligned}$$

In the second equality we used the identity (2.1). This implies that $(\pi_\lambda)_* \nu_\lambda = \mathcal{T}_\lambda^*(\pi_\lambda)_* \nu_\lambda$. The statement of the lemma follows from Proposition 2.1. \square

For simplicity, let us introduce the following notation. For an $\mathbf{i} \in \Sigma$ let

$$\psi_{\mathbf{i}_n}^\lambda(x) := \psi_{i_0}^\lambda \circ \dots \circ \psi_{i_{n-1}}^\lambda(x).$$

Lemma 2.4. *There exists a constant $C_2 > 1$ such that for every $n \geq 1$, $\mathbf{i} \in \Sigma$, $\lambda \in \overline{U}$ and $x, y \in X$ we have*

$$(2.8) \quad C_2^{-1} \leq \frac{(\psi_{\mathbf{i}_n}^\lambda)'(x)}{(\psi_{\mathbf{i}_n}^\lambda)'(y)} \leq C_2,$$

Proof. For the proof we refer to [17, Lemma 5.8]. \square

Finally, we prove the continuity of the entropy and Lyapunov exponent.

Proposition 2.5. *The maps $\lambda \mapsto h_{\mu_\lambda}$ and $\lambda \mapsto \chi_{\mu_\lambda}$ are continuous on \overline{U} .*

Proof. Let us observe that

$$h_{\mu_\lambda} = - \int \varphi_\lambda(\mathbf{i}) d\nu_\lambda(\mathbf{i}) \text{ and } \chi_{\mu_\lambda} = - \int \log \left| (\psi_{i_0}^\lambda)'(\pi_\lambda(\sigma\mathbf{i})) \right| d\nu_\lambda(\mathbf{i}).$$

Since the potential function φ_λ depends continuously on λ , the Gibbs measure ν_λ depends continuously on $\lambda \in \overline{U}$ in the weak topology. Moreover, the potentials φ_λ and $\log \left| (\psi_{i_0}^\lambda)'(\pi_\lambda(\sigma\mathbf{i})) \right|$ depend continuously on λ ; hence the entropy and Lyapunov-exponent depend continuously on λ as well. \square

3. PROOF OF THEOREM 1.1

During the proof of Theorem 1.1 we follow the method of Simon, Solomyak and Urbański [18] with a relevant modification based on the idea of Persson [13]. The method introduced by Pollicott and Simon [15] and later extended by Peres and Solomyak [12] and Simon, Solomyak and Urbański [18] is applied for the push-down measures of left-shift invariant ergodic measures on the symbolic space. More precisely, for a fixed ergodic σ -invariant measure on the symbolic space, the dimension of its push-down measure is the minimum of the entropy/Lyapunov exponent and 1 whenever the transversality condition holds. That is, the measure on the symbolic space is independent of the parameters. In our case, the Gibbs measure on the symbolic space depends on the parameters of the IFS as well. To avoid the difficulties of this fact we need the following lemma according to [13, Lemma 3].

Lemma 3.1. *Let $\mathbf{i}, \mathbf{j} \in \Sigma$ be such that $i_0 \neq j_0$. Then for every $r > 0$ there exists a function $G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda})$ such that*

$$(3.1) \quad \mathbb{1}_{\{|\pi_{\boldsymbol{\lambda}}(\mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\mathbf{j})| < r\}} \leq G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda})$$

and

$$(3.2) \quad \int G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda}) d\mathcal{L}_d(\boldsymbol{\lambda}) \leq 2C_1 r,$$

where $\mathbb{1}$ denotes the indicator function and C_1 is the constant from (A3). Moreover, the function $G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda})$ is constant on cylinders $[\mathbf{i}]_{N+1} \times [\mathbf{j}]_{N+1}$, where $N = \left\lceil \frac{\log(r/(2\text{diam}X))}{\log \kappa} \right\rceil$.

Proof. Let us suppose that $\mathbf{i} \in [i_0 \dots i_N]$ and $\mathbf{j} \in [j_0 \dots j_N]$. Let $\mathbf{i}^* := (i_0 \dots i_N \mathbf{1})$ and $\mathbf{j}^* := (j_0 \dots j_N \mathbf{1})$, where $\mathbf{1}$ denotes the word $(11 \dots)$. Define $G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda})$ as

$$G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda}) := \mathbb{1}_{\{|\pi_{\boldsymbol{\lambda}}(\mathbf{i}^*) - \pi_{\boldsymbol{\lambda}}(\mathbf{j}^*)| < 2r\}}.$$

Then by (2.1) and the Lagrange mean value theorem,

$$\begin{aligned} & \left| |\pi_{\boldsymbol{\lambda}}(\mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\mathbf{j})| - |\pi_{\boldsymbol{\lambda}}(\mathbf{i}^*) - \pi_{\boldsymbol{\lambda}}(\mathbf{j}^*)| \right| \leq |(\pi_{\boldsymbol{\lambda}}(\mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\mathbf{i}^*)) - (\pi_{\boldsymbol{\lambda}}(\mathbf{j}) - \pi_{\boldsymbol{\lambda}}(\mathbf{j}^*))| \\ & \leq \left| \left(\psi_{\mathbf{i}|_{N+1}}^{\boldsymbol{\lambda}} \right)'(\xi_1) \right| |\pi_{\boldsymbol{\lambda}}(\sigma^{N+1}\mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\mathbf{1})| + \left| \left(\psi_{\mathbf{j}|_{N+1}}^{\boldsymbol{\lambda}} \right)'(\xi_2) \right| |\pi_{\boldsymbol{\lambda}}(\sigma^{N+1}\mathbf{j}) - \pi_{\boldsymbol{\lambda}}(\mathbf{1})| \\ & \leq 2\text{diam}X \kappa^{N+1} \leq r. \end{aligned}$$

This implies the inequality (3.1). The other inequality follows from (A3). \square

Now we recall some classical results. By the Shannon-McMillan-Breiman Theorem,

$$-\frac{1}{n} \log \nu_{\boldsymbol{\lambda}}([\mathbf{i}]_n) \rightarrow h_{\nu_{\boldsymbol{\lambda}}} \text{ for } \nu_{\boldsymbol{\lambda}} \text{ a.e. } \mathbf{i} \in \Sigma,$$

and by Birkhoff's Ergodic Theorem,

$$-\frac{1}{n} \log \left| \left(\psi_{\mathbf{i}|_n}^{\boldsymbol{\lambda}} \right)'(\pi_{\boldsymbol{\lambda}}(\sigma^n \mathbf{i})) \right| \rightarrow \chi_{\nu_{\boldsymbol{\lambda}}} \text{ for } \nu_{\boldsymbol{\lambda}} \text{ a.e. } \mathbf{i} \in \Sigma.$$

Lemma 3.2. *For every $\varepsilon > 0$ and $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exist a set $J \subseteq U$ and a constant $\tilde{C} > 1$ such that $\mathcal{L}_d(U \setminus J) < \varepsilon_2$ and for every $\boldsymbol{\lambda} \in U$ there exists a set $\Omega_{\boldsymbol{\lambda}} \subseteq \Sigma$ such that $\nu_{\boldsymbol{\lambda}}(\Omega_{\boldsymbol{\lambda}}) > 1 - \varepsilon_1$,*

$$(3.3) \quad \tilde{C}^{-1} e^{-n(h_{\mu_{\boldsymbol{\lambda}}} + \varepsilon)} \leq \nu_{\boldsymbol{\lambda}}([\mathbf{i}]_n) \leq \tilde{C} e^{-n(h_{\mu_{\boldsymbol{\lambda}}} - \varepsilon)} \text{ and}$$

$$(3.4) \quad \tilde{C}^{-1}e^{-n(\chi_{\mu_\lambda} + \varepsilon)} \leq \left| \left(\psi_{\mathbf{i}|_n}^\lambda \right)' (\pi_\lambda(\sigma^{n+1}\mathbf{i})) \right| \leq \tilde{C}e^{-n(\chi_{\mu_\lambda} - \varepsilon)} \text{ for every } n \geq 1,$$

$\lambda \in J \text{ and } \mathbf{i} \in \Omega_\lambda.$

Proof. By Egorov’s Theorem, for every $\varepsilon > 0$, $\varepsilon_1 > 0$ and $\lambda \in U$ there exists a set $\Omega_\lambda \subseteq \Sigma$ such that $\nu_\lambda(\Omega_\lambda) > 1 - \varepsilon_1$,

$$\tilde{C}_\lambda^{-1}e^{-n(h_{\mu_\lambda} + \varepsilon)} \leq \nu_\lambda([\mathbf{i}|_n]) \leq \tilde{C}_\lambda e^{-n(h_{\mu_\lambda} - \varepsilon)} \text{ and}$$

$$\tilde{C}_\lambda^{-1}e^{-n(\chi_{\mu_\lambda} + \varepsilon)} \leq \left| \left(\psi_{\mathbf{i}|_n}^\lambda \right)' (\pi_\lambda(\sigma^{n+1}\mathbf{i})) \right| \leq \tilde{C}_\lambda e^{-n(\chi_{\mu_\lambda} - \varepsilon)} \text{ for every } n \geq 1 \text{ and}$$

$\mathbf{i} \in \Omega_\lambda.$

An application of Lusin’s Theorem shows that for every $\varepsilon_2 > 0$ there exist a set $J \subseteq U$ and a constant \tilde{C} such that $\mathcal{L}_d(U \setminus J) < \varepsilon_2$ and $\tilde{C}_\lambda \leq \tilde{C}$ for every $\lambda \in J$. \square

Denote $\tilde{\nu}_\lambda$ the restriction of the measure ν_λ to Ω_λ and $\tilde{\mu}_\lambda$ its push-down measure. That is,

$$(3.5) \quad \tilde{\nu}_\lambda := \nu_\lambda|_{\Omega_\lambda} \text{ and } \tilde{\mu}_\lambda := \tilde{\nu}_\lambda \circ \pi_\lambda^{-1},$$

where Ω_λ is defined in Lemma 3.2. For a finite length word $(l_0 \dots l_{n-1}) \in \mathcal{S}^*$ let us define

$$A_{(l_0 \dots l_{n-1})}^\lambda = \{(\mathbf{i}, \mathbf{j}) \in \Omega_\lambda^2 : \mathbf{i} \wedge \mathbf{j} = n \text{ and } i_m = j_m = l_m \text{ for } 0 \leq m \leq n - 1\}.$$

We note that for an empty word $A_\emptyset^\lambda = \{(\mathbf{i}, \mathbf{j}) \in \Omega_\lambda^2 : i_0 \neq j_0\}$.

Frostman’s Theorem (see [2, Theorem 4.13]) implies

$$(3.6) \quad \dim_H \mu_\lambda \geq \sup \left\{ s > 0 : \iint_{\mathbb{R}^2} |x - y|^{-s} d\mu_\lambda(x) d\mu_\lambda(y) < \infty \right\}.$$

Lemma 3.3. *For every $0 < s < 1$ and $\mathbf{i}, \mathbf{j} \in \Sigma$ such that $i_0 \neq j_0$ we have*

$$|\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})|^{-s} \leq \sum_{n=0}^\infty \frac{2^{s(n+1)}}{\text{diam} X^s} G_{\frac{\text{diam} X}{2^n}}(\mathbf{i}, \mathbf{j}, \lambda).$$

Proof. If $|\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})| = 0$, then the right hand side of the inequality is divergent. Otherwise, if $|\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})| > 0$, then

$$\begin{aligned} |\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})|^{-s} &\leq \sum_{n=0}^\infty \frac{2^{s(n+1)}}{\text{diam} X^s} \mathbb{1}_{\left\{ \frac{2^{sn}}{\text{diam} X^s} < |\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})|^{-s} \leq \frac{2^{s(n+1)}}{\text{diam} X^s} \right\}} \\ &= \sum_{n=0}^\infty \frac{2^{s(n+1)}}{\text{diam} X^s} \mathbb{1}_{\left\{ \frac{\text{diam} X}{2^{n+1}} \leq |\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})| < \frac{\text{diam} X}{2^n} \right\}} \leq \sum_{n=0}^\infty \frac{2^{s(n+1)}}{\text{diam} X^s} \mathbb{1}_{\left\{ |\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})| < \frac{\text{diam} X}{2^n} \right\}}. \end{aligned}$$

The statement follows from (3.1). \square

Proposition 3.4. *For every $\lambda_0 \in U$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$(3.7) \quad \dim_H \mu_\lambda \geq \min \left\{ \frac{h_{\mu_{\lambda_0}} - 2\varepsilon}{\chi_{\mu_{\lambda_0}} + 2\varepsilon}, 1 - \varepsilon \right\} \text{ for Lebesgue-a.e. } \lambda \in B_\delta(\lambda_0),$$

where $B_\delta(\lambda_0)$ denotes the ball with center at λ_0 and radius δ .

Proof. Fix an $\varepsilon > 0$ and a $\lambda_0 \in U$. By Proposition 2.5 let $\delta > 0$ be such that if $|\lambda_0 - \lambda| < \delta$, then

$$(3.8) \quad |h_{\mu_{\lambda_0}} - h_{\mu_\lambda}| < \varepsilon \text{ and } |\chi_{\mu_{\lambda_0}} - \chi_{\mu_\lambda}| < \varepsilon.$$

Let $s < \min \left\{ 1 - \varepsilon, \frac{h_{\mu_\lambda} - 2\varepsilon}{\chi_{\mu_\lambda} + 2\varepsilon} \right\}$ and let $\varepsilon_2 > 0$ and J be as in Lemma 3.2. For simplicity, $J_\delta(\lambda_0) := B_\delta(\lambda_0) \cap J$. Since $\dim_H \mu_\lambda \geq \dim_H \tilde{\mu}_\lambda$ and $\tilde{\mu}_\lambda$ is the push-down measure of $\tilde{\nu}_\lambda$, to prove (3.7) it is enough to show according to (3.6) that

$$\mathcal{I} := \int_{J_\delta(\lambda_0)} \iint_{\Omega_\lambda^2} |\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})|^{-s} d\tilde{\nu}_\lambda(\mathbf{i}) d\tilde{\nu}_\lambda(\mathbf{j}) d\mathcal{L}_d(\lambda) < \infty.$$

For simplicity, denote $\tilde{\nu}_\lambda \times \tilde{\nu}_\lambda$ by ν_λ^2 . Then

$$\mathcal{I} = \sum_{n=0}^\infty \sum_{\bar{\tau} \in \mathcal{S}^n} \int_{J_\delta(\lambda_0)} \iint_{A_{\bar{\tau}}^\lambda} |\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})|^{-s} d\nu_\lambda^2(\mathbf{i}, \mathbf{j}) d\mathcal{L}_d(\lambda).$$

By (2.8) we have

$$\begin{aligned} \mathcal{I} &\leq \\ &\sum_{n=0}^\infty \sum_{\bar{\tau} \in \mathcal{S}^n} \int_{J_\delta(\lambda_0)} \iint_{A_{\bar{\tau}}^\lambda} C_2^s \left| \left(\psi_{\hat{1}_n}^\lambda \right)' \left(\pi_\lambda(\sigma^n \mathbf{i}) \right) \right|^{-s} |\pi_\lambda(\sigma^n \mathbf{i}) - \pi_\lambda(\sigma^n \mathbf{j})|^{-s} d\nu_\lambda^2(\mathbf{i}, \mathbf{j}) d\mathcal{L}_d(\lambda). \end{aligned}$$

By Lemma 3.2, (3.4) and (3.8)

$$\mathcal{I} \leq C_2^s \tilde{C} \sum_{n=0}^\infty e^{n(\chi_{\mu_{\lambda_0}} + 2\varepsilon)s} \sum_{\bar{\tau} \in \mathcal{S}^n} \int_{J_\delta(\lambda_0)} \iint_{A_{\bar{\tau}}^\lambda} |\pi_\lambda(\sigma^n \mathbf{i}) - \pi_\lambda(\sigma^n \mathbf{j})|^{-s} d\nu_\lambda^2(\mathbf{i}, \mathbf{j}) d\mathcal{L}_d(\lambda).$$

By applying Lemma 3.3 and Fatou's Lemma we get

$$\begin{aligned} \mathcal{I} &\leq C_2^s \tilde{C} \sum_{n=0}^\infty e^{n(\chi_{\mu_{\lambda_0}} + 2\varepsilon)s} \sum_{\bar{\tau} \in \mathcal{S}^n} \sum_{m=0}^\infty \frac{2^{s(m+1)}}{\text{diam} X} \\ &\quad \times \int_{J_\delta(\lambda_0)} \iint_{A_{\bar{\tau}}^\lambda} G_{\frac{\text{diam} X}{2^m}}(\sigma^n \mathbf{i}, \sigma^n \mathbf{j}, \lambda) d\nu_\lambda^2(\mathbf{i}, \mathbf{j}) d\mathcal{L}_d(\lambda). \end{aligned}$$

Lemma 3.1 implies that

$$\begin{aligned} &\iint_{A_{\bar{\tau}}^\lambda} G_{\frac{\text{diam} X}{2^m}}(\sigma^n \mathbf{i}, \sigma^n \mathbf{j}, \lambda) d\nu_\lambda^2(\mathbf{i}, \mathbf{j}) \\ &= \sum_{\bar{\mathbf{j}}_1, \bar{\mathbf{j}}_2 \in \mathcal{S}^{N(m)}} \sum_{\substack{l, p \in \mathcal{S} \\ l \neq p}} G_{\frac{\text{diam} X}{2^m}}(l\bar{\mathbf{j}}_1 \mathbf{1}, p\bar{\mathbf{j}}_2 \mathbf{1}, \lambda) \nu_\lambda^2([\bar{l}\bar{\mathbf{j}}_1] \times [\bar{p}\bar{\mathbf{j}}_2]), \end{aligned}$$

where $N(m) = \left\lceil (m+1) \frac{\log 2}{-\log \kappa} \right\rceil$ according to Lemma 3.1. Therefore,

$$\begin{aligned}
& \sum_{\bar{i} \in \mathcal{S}^n} \sum_{\bar{j}_1, \bar{j}_2 \in \mathcal{S}^{N(m)}} \sum_{\substack{l, p \in \mathcal{S} \\ l \neq p}} \int_{J_\delta(\lambda_0)} G_{\frac{\text{diam} X}{2^m}}(l\bar{j}_1 \mathbf{1}, p\bar{j}_2 \mathbf{1}, \lambda) \nu_\lambda^2([\bar{l}\bar{j}_1] \times [\bar{i}p\bar{j}_2]) d\mathcal{L}_d(\lambda) \\
& \leq \sum_{\bar{i} \in \mathcal{S}^n} \sum_{\bar{j}_1, \bar{j}_2 \in \mathcal{S}^{N(m)}} \sum_{\substack{l, p \in \mathcal{S} \\ l \neq p}} \max_{h_1, h_2 \in \mathcal{S}^{N(m)}} \\
& \quad \times \int_{J_\delta(\lambda_0)} G_{\frac{\text{diam} X}{2^m}}(lh_1 \mathbf{1}, ph_2 \mathbf{1}, \lambda) \nu_\lambda^2([\bar{l}\bar{j}_1] \times [\bar{i}p\bar{j}_2]) d\mathcal{L}_d(\lambda) \\
& = \max_{h_1, h_2 \in \mathcal{S}^{N(m)}} \int_{J_\delta(\lambda_0)} G_{\frac{\text{diam} X}{2^m}}(lh_1 \mathbf{1}, ph_2 \mathbf{1}, \lambda) \sum_{\bar{i} \in \mathcal{S}^n} \nu_\lambda^2(A_{\bar{i}}) d\mathcal{L}_d(\lambda) \\
& \leq \sup_{\lambda \in J_\delta(\lambda_0)} \sum_{\bar{i} \in \mathcal{S}^n} \nu_\lambda^2(A_{\bar{i}}) \max_{h_1, h_2 \in \mathcal{S}^{N(m)}} \int_{J_\delta(\lambda_0)} G_{\frac{\text{diam} X}{2^m}}(lh_1 \mathbf{1}, ph_2 \mathbf{1}, \lambda) d\mathcal{L}_d(\lambda) \\
& \leq \frac{2C_1 \text{diam} X}{2^m} \sup_{\lambda \in J_\delta(\lambda_0)} \sum_{\bar{i} \in \mathcal{S}^n} \nu_\lambda^2(A_{\bar{i}}),
\end{aligned}$$

used in the last inequality (3.2). Hence,

$$\mathcal{I} \leq C_2^s \tilde{C} \sum_{n=0}^{\infty} e^{n(\chi_{\mu_{\lambda_0}} + 2\varepsilon)s} \sum_{m=0}^{\infty} \frac{2^{s(m+1)}}{\text{diam} X^s} \frac{2C_1 \text{diam} X}{2^m} \sup_{\lambda \in J_\delta(\lambda_0)} \sum_{\bar{i} \in \mathcal{S}^n} \nu_\lambda^2(A_{\bar{i}}).$$

Applying Lemma 3.2(3.3) and (3.8) we get

$$\nu_\lambda^2(A_{\bar{i}}) \leq \tilde{C} e^{-n(h_{\mu_{\lambda_0}} - 2\varepsilon)} \nu_\lambda([\bar{i}])$$

and this implies that

$$\mathcal{I} \leq C_2^s \tilde{C}^2 C_1 \text{diam} X^{1-s} 2^{s+1} \sum_{n=0}^{\infty} e^{n((\chi_{\mu_{\lambda_0}} + 2\varepsilon)s - (h_{\mu_{\lambda_0}} - 2\varepsilon))} \sum_{m=0}^{\infty} 2^{(s-1)m}.$$

Since $s < \min \left\{ 1 - \varepsilon, \frac{h_{\mu_{\lambda_0}} - 2\varepsilon}{\chi_{\mu_{\lambda_0}} + 2\varepsilon} \right\}$ the right hand side of the inequality is finite.

Moreover, $\varepsilon_2 > 0$ was arbitrary and therefore $\dim_H \mu_\lambda \geq \min \left\{ 1 - \varepsilon, \frac{h_{\mu_{\lambda_0}} - 2\varepsilon}{\chi_{\mu_{\lambda_0}} + 2\varepsilon} \right\}$ for Lebesgue almost every $\lambda \in B_\delta(\lambda_0)$ which was to be proven. \square

Proof of Theorem 1.1 (1). Since $\dim_H \mu_\lambda \leq \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}}$ for all $\lambda \in \bar{U}$ by [7, Theorem 1], we only need to establish the estimate from below.

Let us argue by contradiction. Suppose that there exist an $\varepsilon > 0$ and a positive measure set \hat{U} of parameters such that

$$\dim_H \mu_\lambda < \min \left\{ 1, \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}} \right\} - \varepsilon \text{ for almost every } \lambda \in \hat{U}.$$

Let λ_0 be a density point of \hat{U} . Then there exists a $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$,

$$\mathcal{L}_d \left(\lambda \in B_\delta(\lambda_0) : \dim_H \mu_\lambda < \min \left\{ 1, \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}} \right\} - \varepsilon \right) > 0.$$

Since the entropy and the Lyapunov exponent are continuous functions (Proposition 2.5), for sufficiently small $\delta > 0$,

$$\mathcal{L}_d \left(\boldsymbol{\lambda} \in B_\delta(\boldsymbol{\lambda}_0) : \dim_H \mu_{\boldsymbol{\lambda}} < \min \left\{ 1, \frac{h_{\mu_{\boldsymbol{\lambda}_0}}}{\chi_{\mu_{\boldsymbol{\lambda}_0}}} \right\} - \frac{\varepsilon}{2} \right) > 0.$$

This contradicts the statement of Proposition 3.4. □

Proof of Theorem 1.1(2). Let $U' = U \cap \left\{ \boldsymbol{\lambda} \in U : \frac{h_{\mu_{\boldsymbol{\lambda}}}}{\chi_{\mu_{\boldsymbol{\lambda}}}} > 1 \right\}$, which is, by Proposition 2.5, open. Fix an arbitrary $\boldsymbol{\lambda}_0 \in U'$.

Let $\varepsilon > 0$ be such that $\frac{h_{\mu_{\boldsymbol{\lambda}_0}} - 2\varepsilon}{\chi_{\mu_{\boldsymbol{\lambda}_0}} + 2\varepsilon} > 1$. By Proposition 2.5 let $\delta > 0$ be such that if $|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}| < \delta$, then

$$(3.9) \quad |h_{\mu_{\boldsymbol{\lambda}_0}} - h_{\mu_{\boldsymbol{\lambda}}}| < \varepsilon \text{ and } |\chi_{\mu_{\boldsymbol{\lambda}_0}} - \chi_{\mu_{\boldsymbol{\lambda}}}| < \varepsilon.$$

Let $\varepsilon_1, \varepsilon_2 > 0$ and let J and $\Omega_{\boldsymbol{\lambda}}$ be as in Lemma 3.2 and $J_\delta(\boldsymbol{\lambda}_0) := B_\delta(\boldsymbol{\lambda}_0) \cap J$. We are going to prove that $\tilde{\mu}_{\boldsymbol{\lambda}}$ is absolutely continuous for Lebesgue almost every $\boldsymbol{\lambda} \in J_\delta(\boldsymbol{\lambda}_0)$ with density in L^2 . Letting $\varepsilon_2 \rightarrow 0$ we get that $\tilde{\mu}_{\boldsymbol{\lambda}}$ is absolutely continuous for Lebesgue-a.e. $\boldsymbol{\lambda} \in B_\delta(\boldsymbol{\lambda}_0)$ with density in L^2 . Then letting $\varepsilon_1 \rightarrow 0$ along a countable set we get that $\mu_{\boldsymbol{\lambda}}$ is abs. cont. for almost every $B_\delta(\boldsymbol{\lambda}_0)$, but the L^2 property may disappear.

Let

$$D(\tilde{\mu}_{\boldsymbol{\lambda}}, x) := \liminf_{r \rightarrow 0} \frac{\tilde{\mu}_{\boldsymbol{\lambda}}((x - r, x + r))}{2r}$$

be the lower density of the measure $\tilde{\mu}_{\boldsymbol{\lambda}}$ at the point x . By [10, Theorem 2.12], if $D(\tilde{\mu}_{\boldsymbol{\lambda}}, x) < \infty$ for $\tilde{\mu}_{\boldsymbol{\lambda}}$ -a.e. x , then the measure is absolutely continuous. To prove that, it is sufficient to show that

$$\mathcal{J} := \int_{J_\delta(\boldsymbol{\lambda}_0)} \int_{\mathbb{R}} D(\tilde{\mu}_{\boldsymbol{\lambda}}, x) d\tilde{\mu}_{\boldsymbol{\lambda}}(x) d\mathcal{L}_d(\boldsymbol{\lambda}) < \infty.$$

Applying (3.5) and Fubini's Lemma we get

$$\mathcal{J} \leq \liminf_{r \rightarrow 0} \frac{1}{2r} \int_{J_\delta(\boldsymbol{\lambda}_0)} \iint_{\Omega_{\boldsymbol{\lambda}}^2} \mathbb{1}_{\{|\pi_{\boldsymbol{\lambda}}(\mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\mathbf{j})| < r\}} d\nu_{\boldsymbol{\lambda}}^2(\mathbf{i}, \mathbf{j}) d\mathcal{L}_d(\boldsymbol{\lambda}),$$

where $\nu_{\boldsymbol{\lambda}}^2 = \tilde{\nu}_{\boldsymbol{\lambda}} \times \tilde{\nu}_{\boldsymbol{\lambda}}$. Then

$$\mathcal{J} \leq \liminf_{r \rightarrow 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{\bar{i} \in \mathcal{S}^n} \int_{J_\delta(\boldsymbol{\lambda}_0)} \iint_{A_{\bar{i}}} \mathbb{1}_{\{|\pi_{\boldsymbol{\lambda}}(\mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\mathbf{j})| < r\}} d\nu_{\boldsymbol{\lambda}}^2(\mathbf{i}, \mathbf{j}) d\mathcal{L}_d(\boldsymbol{\lambda}),$$

and by (2.8), (3.4) and (3.9)

$$\begin{aligned} \mathcal{J} &\leq \liminf_{r \rightarrow 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{\bar{i} \in \mathcal{S}^n} \\ &\quad \times \int_{J_\delta(\boldsymbol{\lambda}_0)} \iint_{A_{\bar{i}}} \mathbb{1}_{\{|\pi_{\boldsymbol{\lambda}}(\sigma^n \mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\sigma^n \mathbf{j})| < C_2 \tilde{C} r e^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}} + 2\varepsilon)}\}} d\nu_{\boldsymbol{\lambda}}^2(\mathbf{i}, \mathbf{j}) d\mathcal{L}_d(\boldsymbol{\lambda}). \end{aligned}$$

Applying Lemma 3.1 we get

$$\begin{aligned} \mathcal{J} &\leq \liminf_{r \rightarrow 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{\bar{i} \in \mathcal{S}^n} \sum_{\substack{p, l \in \mathcal{S} \\ p \neq l}} \sum_{\bar{j}_1, \bar{j}_2 \in \mathcal{S}^{N(n,r)}} \\ &\quad \times \int_{J_\delta(\lambda_0)} G_{C_2 \tilde{C} r e^{n(\chi_{\mu_{\lambda_0}} + 2\varepsilon)}}(l\bar{j}_1 \mathbf{1}, p\bar{j}_2 \mathbf{1}, \boldsymbol{\lambda}) \nu_{\lambda}^2([\bar{l}\bar{j}_1] \times [\bar{i}p\bar{j}_2]) d\mathcal{L}_d(\boldsymbol{\lambda}), \end{aligned}$$

where $N(n, r) = \left\lceil \frac{\log C_2 \tilde{C} r e^{n(\chi_{\mu_{\lambda_0}} + 2\varepsilon)}}{\log \kappa} / 2 \text{diam} X \right\rceil$ according to Lemma 3.1. Similarly, as in the proof of Proposition 3.4, one can get

$$\begin{aligned} \sum_{\bar{i} \in \mathcal{S}^n} \sum_{\substack{p, l \in \mathcal{S} \\ p \neq l}} \sum_{\bar{j}_1, \bar{j}_2 \in \mathcal{S}^{N(n,r)}} \int_{J_\delta(\lambda_0)} G_{C_2 \tilde{C} r e^{n(\chi_{\mu_{\lambda_0}} + 2\varepsilon)}}(l\bar{j}_1 \mathbf{1}, p\bar{j}_2 \mathbf{1}, \boldsymbol{\lambda}) \nu_{\lambda}^2([\bar{l}\bar{j}_1] \times [\bar{i}p\bar{j}_2]) d\mathcal{L}_d(\boldsymbol{\lambda}) \\ \leq 2C_1 C_2 \tilde{C} r e^{n(\chi_{\mu_{\lambda_0}} + 2\varepsilon)} \sup_{\lambda \in J_\delta(\lambda_0)} \sum_{\bar{i} \in \mathcal{S}^n} \nu_{\lambda}^2(A_{\bar{i}}) \end{aligned}$$

and

$$\sup_{\lambda \in J_\delta(\lambda_0)} \sum_{\bar{i} \in \mathcal{S}^n} \nu_{\lambda}^2(A_{\bar{i}}) \leq \tilde{C} e^{-n(h_{\mu_{\lambda_0}} - 2\varepsilon)}.$$

Therefore,

$$\mathcal{J} \leq 2C_1 C_2 \tilde{C}^2 \sum_{n=0}^{\infty} e^{n(\chi_{\mu_{\lambda_0}} - h_{\mu_{\lambda_0}} + 4\varepsilon)}.$$

Since $\chi_{\mu_{\lambda_0}} - h_{\mu_{\lambda_0}} + 4\varepsilon < 0$ the right hand side is finite. This completes the proof. \square

4. AN EXAMPLE: PLACE-DEPENDENT BERNOULLI CONVOLUTIONS

In this section we show an application for our main theorem. Let $0 < \rho < \frac{1}{2}$ and $0.5 < \lambda < 1$ and let us consider the following function $f : [-1, 1] \times [0, 1] \mapsto [-1, 1] \times [0, 1]$:

$$f(x, y) = \begin{cases} (\lambda x - (1 - \lambda), \frac{2y}{1+2\rho x}) & \text{if } 0 \leq y < \frac{1}{2} + \rho x \\ (\lambda x + (1 - \lambda), \frac{2y-2\rho x-1}{1-2\rho x}) & \text{if } \frac{1}{2} + \rho x \leq y \leq 1. \end{cases}$$

For the action of f on the rectangle $[-1, 1] \times [0, 1]$ see Figure 1.

It follows by [14] that there exists an f -invariant measure μ_{SBR} which is called the SBR-measure such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \bar{\mathcal{L}}_2 \circ f^{-k} \rightarrow \mu_{\text{SBR}} \text{ weakly,}$$

where $\bar{\mathcal{L}}_2$ is the normalized Lebesgue measure to the rectangle. Our aim is to find a set of parameters such that $\mu_{\text{SBR}} \ll \mathcal{L}_2$. It is easy to see that

$$d\mu_{\text{SBR}}(x, y) = d\mu_{\lambda, \rho}(x) d\mathcal{L}_1(y),$$

where $\mu_{\lambda, \rho}$ is the place-dependent invariant measure of the IFS

$$\Psi_{\lambda} = \{ \psi_0^{\lambda}(x) = \lambda x - (1 - \lambda), \psi_1^{\lambda}(x) = \lambda x + (1 - \lambda) \}$$

with probabilities $\{p_0(x) = \frac{1}{2} + \rho x, p_1(x) = \frac{1}{2} - \rho x\}$. Then the property $\mu_{\text{SBR}} \ll \mathcal{L}_2$ is equivalent to $\mu_{\lambda, \rho} \ll \mathcal{L}_1$. It was proven by Peres and Solomyak [11] that if $\rho = 0$, then $\mu_{\lambda, 0}$ is absolutely continuous w.r.t. Lebesgue measure for Leb-a.e. $\lambda \in (0.5, 1)$.

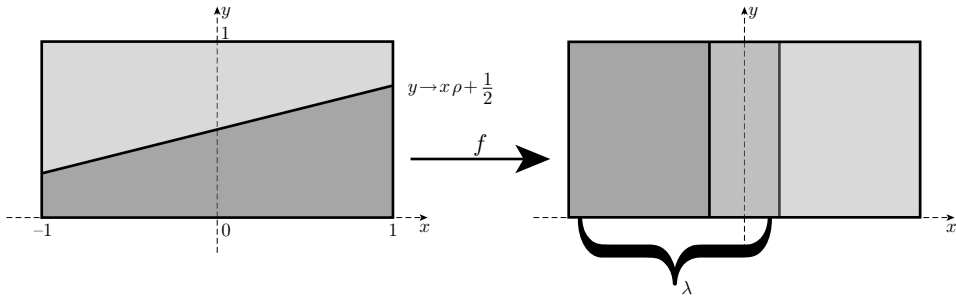


FIGURE 1. The map f acting on the rectangle $[-1, 1] \times [0, 1]$.

According to (1.3) and (1.4) we have

$$\chi_{\mu_{\lambda,\rho}} = -\log \lambda \quad \text{and}$$

$$h_{\mu_{\lambda,\rho}} = -\int_{\mathbb{R}} \left(\frac{1}{2} + \rho x\right) \log \left(\frac{1}{2} + \rho x\right) + \left(\frac{1}{2} - \rho x\right) \log \left(\frac{1}{2} - \rho x\right) d\mu_{\lambda,\rho}(x).$$

To simplify the entropy let us observe that $|2\rho x| < 1$; therefore,

$$(4.1) \quad h_{\mu_{\lambda,\rho}} = \log 2 + \frac{1}{2} \int_{\mathbb{R}} (1 + 2\rho x) \sum_{n=1}^{\infty} \frac{(-2\rho x)^n}{n} + (1 - 2\rho x) \sum_{n=1}^{\infty} \frac{(2\rho x)^n}{n} d\mu_{\lambda,\rho}(x)$$

$$= \log 2 - \sum_{n=1}^{\infty} \frac{(2\rho)^{2n}}{2n(2n-1)} \int_{\mathbb{R}} x^{2n} d\mu_{\lambda,\rho}(x).$$

For brevity, let $F_n = \int_{\mathbb{R}} x^{2n} d\mu_{\lambda,\rho}(x)$. Using (2.4) one can write an inductive formula for the series $\{F_n\}_{n=0}^{\infty}$. Precisely,

$$F_n = (1 - \lambda)^{2n} + \sum_{m=1}^n 2m(1 - \lambda)^{2n-2m} \lambda^{2m-1} \binom{2n}{2m} \left(\frac{\lambda}{2m} - \frac{2\rho(1 - \lambda)}{2n - 2m + 1}\right) F_m,$$

and therefore

$$F_n = \frac{(1 - \lambda)^{2n}}{1 + \lambda^{2n-1}(4n\rho(1 - \lambda) - \lambda)}$$

$$+ \sum_{m=1}^{n-1} \frac{2m(1 - \lambda)^{2n-2m} \lambda^{2m-1}}{1 + \lambda^{2n-1}(4n\rho(1 - \lambda) - \lambda)} \binom{2n}{2m} \left(\frac{\lambda}{2m} - \frac{2\rho(1 - \lambda)}{2n - 2m + 1}\right) F_m.$$

For example,

$$F_1 = \frac{(1-\lambda)^2}{1+\lambda(4\rho(1-\lambda)-\lambda)},$$

$$F_2 = \frac{(1-\lambda)^4}{1+\lambda^3(8\rho(1-\lambda)-\lambda)} + \frac{2(1-\lambda)^4\lambda(3\lambda-4\rho(1-\lambda))}{(1+\lambda^3(8\rho(1-\lambda)-\lambda))(1+\lambda(4\rho(1-\lambda)-\lambda))}, \text{ etc.}$$

It follows from (4.1) that for every $N \geq 1$,

$$(4.2) \quad \log 2 - \sum_{n=1}^N \frac{(2\rho)^{2n}}{2n(2n-1)} F_n - \frac{(2\rho)^{N+1}}{(2N+2)(2N+1)(1-(2\rho)^2)}$$

$$\leq h_{\mu_{\lambda,\rho}} \leq \log 2 - \sum_{n=1}^N \frac{(2\rho)^{2n}}{2n(2n-1)} F_n.$$

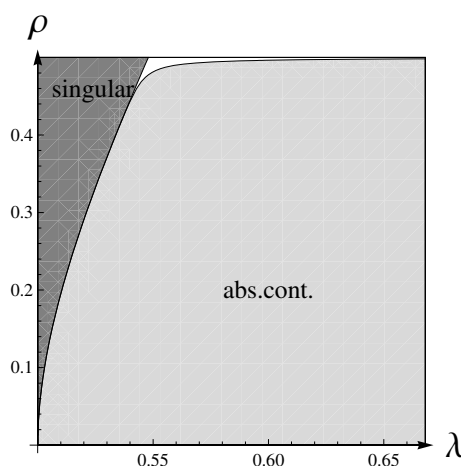


FIGURE 2. The singularity and absolute continuity region of the measure $\mu_{\lambda,\rho}$.

Theorem 4.1. For every $0 \leq \rho < 0.5$ and Lebesgue almost every $\lambda \in (0.5, 0.6684755)$,

$$\frac{\log 2 - \frac{2\rho^2(1-\lambda)^2}{1+\lambda(4\rho(1-\lambda)-\lambda)} - \frac{\rho^2}{3(1-4\rho^2)}}{-\log \lambda} \leq \dim_H \mu_{\lambda,\rho} \leq \frac{\log 2 - \frac{2\rho^2(1-\lambda)^2}{1+\lambda(4\rho(1-\lambda)-\lambda)}}{-\log \lambda}.$$

Moreover, $\mu_{\lambda,\rho}$ is absolutely continuous for Lebesgue almost every

$$\lambda \in \left\{ \lambda \in (0.5, 0.6684755) : \log 2 - \frac{2\rho^2(1-\lambda)^2}{1+\lambda(4\rho(1-\lambda)-\lambda)} - \frac{\rho^2}{3(1-4\rho^2)} > -\log \lambda \right\}.$$

We note that if $0 < \lambda < 0.5$, then the attractor of the IFS is a Cantor set with dimension strictly less than one. Moreover, it satisfies the open set condition trivially. Therefore, without loss of generality we may assume that $\lambda > 0.5$.

Proof. The system Ψ_λ and the probabilities $\{\frac{1}{2} + \rho x, \frac{1}{2} - \rho x\}$ satisfy the conditions (A1), (A2) and (A4) trivially.

It follows from [12, Section 5] and [16, Theorem 2.6] that Ψ_λ satisfies the condition (A3) on the interval $(0.5, 0.6684755)$. The theorem now is an easy consequence of Theorem 1.1 and the bound (4.2) with $N = 1$. \square

For a more precise characterization of the measure $\mu_{\lambda,\rho}$, see Figure 2.

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