

## ON THE REPRESENTATION TYPE OF A PROJECTIVE VARIETY

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ABSTRACT. Let  $X \subset \mathbb{P}^n$  be a smooth arithmetically Cohen-Macaulay variety. We prove that the restriction  $\nu_{3|X}$  to  $X$  of the Veronese 3-uple embedding  $\nu_3 : \mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+3}{3}-1}$  embeds  $X$  as a variety of wild representation type.

### 1. INTRODUCTION

The importance of the existence of arithmetically Cohen-Macaulay bundles (i.e. bundles without intermediate cohomology) on a non-singular projective variety relies on the fact that a natural way to measure the complexity of a non-singular projective variety is to ask for the family of non-isomorphic indecomposable arithmetically Cohen-Macaulay (for short, ACM) bundles that it supports. This problem has a long and interesting history behind it. A seminal result is due to Horrocks (cf. [Hor64]), who asserted that, up to twist, there is only one indecomposable ACM bundle on  $\mathbb{P}^n$ :  $\mathcal{O}_{\mathbb{P}^n}$ . This corresponds with the general idea that a “simple” variety should have associated to it a “simple” category of ACM bundles. Following these lines, a cornerstone result was the classification of ACM varieties of *finite representation type*, i.e., varieties that support (up to twist and isomorphism) only a finite number of indecomposable ACM bundles. It turned out that they fall into a very short list:  $\mathbb{P}^n$ , a smooth hyperquadric  $Q \subset \mathbb{P}^n$ , a cubic scroll in  $\mathbb{P}^4$ , the Veronese surface in  $\mathbb{P}^5$ , a rational normal curve and three or less reduced points in  $\mathbb{P}^2$  (cf. [BGS87, Theorem C] and [EH88, p. 348]).

For the rest of the ACM varieties, it became an interesting problem to give a criterium to split them into a finer classification. Inspired in representation theory, it has been proposed that the classification of ACM varieties should be *finite, tame or wild* (see Definition 2.3) according to the complexity of their associated category of ACM bundles. So far only a few examples of varieties of wild representation type are known: curves of genus  $g \geq 2$  (cf. [DG01]), del Pezzo surfaces and Fano blow-ups of points in  $\mathbb{P}^n$  (cf. [MPLa], the cases of the cubic surface and the cubic threefold have also been handled in [CH]), ACM rational surfaces on  $\mathbb{P}^4$  (cf. [MPLb]), any Segre variety unless the quadric surface is in  $\mathbb{P}^3$  (cf. [CMPL12, Theorem 4.6]), and any rational normal scroll unless  $\mathbb{P}^n$ , the rational normal curve, the quadric

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surface in  $\mathbb{P}^3$  and the cubic scroll in  $\mathbb{P}^4$  which are of finite representation type (cf. [Mir13, Theorem 3.8]).

The representation type of an ACM variety strongly depends on the chosen polarization. For instance, we have mentioned that the quadric surface  $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$  is a variety of finite representation type with respect to the very ample line bundle  $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ . However, the smooth quadric surface  $X$  embedded in  $\mathbb{P}^8$  through the very ample anticanonical divisor  $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)$  is of wild representation type (cf. [MPLa, Theorem 4.10]). This leads to the following problem.

**Problem 1.1.** (a) Given an ACM variety  $X \subset \mathbb{P}^n$ , is there an integer  $N_X$  such that  $X$  can be embedded in  $\mathbb{P}^{N_X}$  as a variety of wild representation type?

(b) If so, what is the smallest possible integer  $N_X$ ?

The goal of this short note is to answer affirmatively Problem (a) and to provide an upper bound for  $N_X$ . In other words, for any smooth projective variety  $X$  there is an embedding of  $X$  into a projective space  $\mathbb{P}^{N_X}$  such that the corresponding homogeneous coordinate ring has arbitrary large families of non-isomorphic indecomposable graded Maximal Cohen-Macaulay modules. Actually, it is proved that such an embedding can be obtained as the composition of the “original” embedding  $X \subset \mathbb{P}^n$  and the Veronese 3-uple embedding  $\nu_3 : \mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+3}{3}-1}$ . The idea will be to construct on any ACM variety  $X \subset \mathbb{P}^n$  of dimension  $d \geq 2$  irreducible families  $\mathcal{F}$  of vector bundles  $\mathcal{E}$  of arbitrarily rank and dimension with the extra feature that any  $\mathcal{E} \in \mathcal{F}$  satisfy  $H^i(X, \mathcal{E}(t)) = 0$  for all  $t \in \mathbb{Z}$ , and  $2 \leq i \leq d - 1$  and  $H^1(X, \mathcal{E}(t)) = 0$  for all  $t \neq -1, -2$ . Therefore,  $X$  embedded in  $\mathbb{P}^{\text{h}^0(\mathcal{O}_X(s))-1}$  through the very ample line bundle  $\mathcal{O}_X(s)$ ,  $s \geq 3$ , is of wild representation type.

Let us outline the structure of this paper. In section 2, we recall the definitions and basic facts on ACM varieties and bundles we will need later. Section 3 is the heart of the paper and contains our main result (cf. Theorem 3.4).

*Notation.* Throughout this paper  $K$  will be an algebraically closed field of characteristic zero,  $R = K[x_0, x_1, \dots, x_n]$ ,  $\mathfrak{m} = (x_0, \dots, x_n)$  and  $\mathbb{P}^n = \text{Proj}(R)$ . Given a non-singular variety  $X$  equipped with an ample line bundle  $\mathcal{O}_X(1)$ , the line bundle  $\mathcal{O}_X(1)^{\otimes l}$  will be denoted by  $\mathcal{O}_X(l)$ . For any coherent sheaf  $\mathcal{E}$  on  $X$  we are going to denote the twisted sheaf  $\mathcal{E} \otimes \mathcal{O}_X(l)$  by  $\mathcal{E}(l)$ . As usual,  $H^i(X, \mathcal{E})$  stands for the cohomology groups,  $h^i(X, \mathcal{E})$  for their dimension and  $H_*^i(X, \mathcal{E}) = \bigoplus_{l \in \mathbb{Z}} H^i(X, \mathcal{E}(l))$ .

## 2. PRELIMINARIES

Here we set up some preliminary notions mainly concerning the definitions and basic results on ACM schemes  $X \subset \mathbb{P}^n$ , as well as on ACM sheaves  $\mathcal{E}$  on  $X$  needed in the sequel.

**Definition 2.1.** A subscheme  $X \subseteq \mathbb{P}^n$  is said to be arithmetically Cohen-Macaulay (briefly, ACM) if its homogeneous coordinate ring  $R_X = R/I_X$  is a Cohen-Macaulay ring, i.e.  $\text{depth}(R_X) = \dim(R_X)$ .

Thanks to the graded version of the Auslander-Buchsbaum formula (for any finitely generated  $R$ -module  $M$ ):

$$\text{pd}(M) = n + 1 - \text{depth}(M),$$

we deduce that a subscheme  $X \subseteq \mathbb{P}^n$  is ACM if and only if  $\text{pd}(R_X) = \text{codim } X$ . Hence, if  $X \subseteq \mathbb{P}^n$  is a codimension  $c$  ACM subscheme, a graded minimal free

$R$ -resolution of  $I_X$  is of the form:

$$0 \longrightarrow F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow R_X \longrightarrow 0$$

with  $F_0 = R$  and  $F_i = \bigoplus_{j=1}^{\beta_i} R(-n_j^i)$ ,  $1 \leq i \leq c$  (in this setting, minimal means that  $\text{im } \varphi_i \subset \mathfrak{m}F_{i-1}$ ).

**Definition 2.2.** Let  $X \subset \mathbb{P}^n$  be a smooth projective variety with a very ample invertible sheaf  $\mathcal{L}$ . A coherent sheaf  $\mathcal{E}$  on  $X$  is *Arithmetically Cohen Macaulay* (ACM for short) with respect to  $\mathcal{L}$  if

$$H^i(X, \mathcal{E} \otimes \mathcal{L}^{\otimes t}) = 0 \quad \text{for all } 1 \leq i \leq \dim X - 1 \text{ and } t \in \mathbb{Z}.$$

Often, when  $\mathcal{L} = \mathcal{O}_X(1)$  we will omit it and we will simply say that  $\mathcal{E}$  is ACM.

A possible way to classify ACM varieties is according to the complexity of the category of ACM sheaves that they support. Recently, inspired by an analogous classification for quivers and for  $K$ -algebras of finite type, it has been proposed the classification of any ACM variety should be of *finite, tame or wild representation type* (cf. [DG01]). Let us introduce these definitions slightly modified with respect to the usual one:

**Definition 2.3.** Let  $X \subseteq \mathbb{P}^N$  be an ACM scheme of dimension  $n$ .

(i) We say that  $X$  is of *finite representation type* if it has, up to twist and isomorphism, only a finite number of indecomposable ACM sheaves.

(ii)  $X$  is of *tame representation type* if either it has, up to twist and isomorphism, an infinite discrete set of indecomposable ACM sheaves or, for each rank  $r$ , the indecomposable ACM sheaves of rank  $r$  form a finite number of families of dimension at most  $n$ .

(iii)  $X$  is of *wild representation type* if there exist  $l$ -dimensional families of non-isomorphic indecomposable ACM sheaves for arbitrarily large  $l$ .

The problem of classifying ACM varieties according to the complexity of the category of ACM sheaves that they support has recently attracted much attention and, in particular, the following problem is still open:

**Problem 2.4.** Is the trichotomy finite representation type, tame representation type and wild representation type exhaustive?

One of the main achievements in this field has been the classification of varieties of finite representation type (cf. [BGS87, Theorem C] and [EH88, p. 348]); it turns out that they fall into a very short list: three or less reduced points on  $\mathbb{P}^2$ , a projective space, a non-singular quadric hypersurface  $X \subseteq \mathbb{P}^n$ , a cubic scroll in  $\mathbb{P}^4$ , the Veronese surface in  $\mathbb{P}^5$  or a rational normal curve. As examples of a variety of tame representation type we have the elliptic curves and the quadric cone in  $\mathbb{P}^3$  (cf. [CH04, Proposition 6.1]). Finally, on the other extreme of complexity lie those varieties that have very large families of ACM sheaves. So far only a few examples of varieties of wild representation type are known: curves of genus  $g \geq 2$  (cf. [DG01]), del Pezzo surfaces and Fano blow-ups of points in  $\mathbb{P}^n$  (cf. [MPLa], the cases of the cubic surface and the cubic threefold have also been handled in [CH]), ACM rational surfaces on  $\mathbb{P}^4$  (cf. [MPLb]), Segre varieties other than the quadric in  $\mathbb{P}^3$  (cf. [CMPL12, Theorem 4.6]), rational normal scrolls other than  $\mathbb{P}^n$ , the rational normal curve in  $\mathbb{P}^n$ , the quadric in  $\mathbb{P}^3$  and the cubic scroll in  $\mathbb{P}^4$  (cf. [Mir13, Theorem 3.8]) and hypersurfaces  $X \subset \mathbb{P}^n$  of degree  $\geq 4$  (cf. [To, Corollary 1]).

As we pointed out in the introduction, the representation type of an ACM variety  $X \subset \mathbb{P}^n$  strongly depends on the chosen polarization and our goal will be to prove that on a projective variety  $X \subset \mathbb{P}^n$  there always exists a very ample line bundle  $\mathcal{L}$  on  $X$  which embeds  $X$  in  $\mathbb{P}^{h^0(X, \mathcal{L})-1}$  as a variety of wild representation type (cf Theorem 3.4). As an immediate consequence, we will have many new examples of ACM varieties of wild representation type.

### 3. THE REPRESENTATION TYPE OF AN ACM VARIETY

In this section,  $X$  will be a smooth ACM variety of dimension  $d \geq 2$  in  $\mathbb{P}^n$  with a minimal free  $R$ -resolution of the following type:

$$(3.1) \quad 0 \longrightarrow F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow R_X \longrightarrow 0$$

with  $c = n - d$ ,  $F_0 = R$  and  $F_i = \bigoplus_{j=1}^{\beta_i} R(-n_j^i)$ ,  $1 \leq i \leq c$ . Our goal is to prove that the Veronese embedding  $\nu_3 : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+3}{3}-1}$  embeds  $X$  as a variety of wild representation type. The idea will be to construct on  $X$  families of indecomposable vector bundles of arbitrary rank and dimension which will be ACM with respect to an  $\mathcal{O}_X(3)$  but not necessarily with respect to  $\mathcal{O}_X(1)$ . The ACM bundles  $\mathcal{E}$  on  $X$  will be constructed as kernels of certain surjective maps between  $\mathcal{O}_X(t)^a$  and  $\mathcal{O}_X(t+1)^b$  for suitable values of  $t, a, b \in \mathbb{Z}$ .

To this end, let us fix some notation as presented in [EH92]. We consider  $K$ -vector spaces  $A$  and  $B$  of dimension  $a$  and  $b$ , respectively. Set  $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$  and let  $M = \text{Hom}(B, A \otimes V)$  be the space of  $(a \times b)$ -matrices of linear forms. It is well-known that there exists a bijection between the elements  $\phi \in M$  and the morphisms

$$\phi : B \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow A \otimes \mathcal{O}_{\mathbb{P}^n}(1).$$

Taking the tensor with  $\mathcal{O}_{\mathbb{P}^n}(1)$  and considering global sections, we have morphisms

$$H^0(\phi(1)) : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^b) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)^a).$$

The following result tells us under which conditions the aforementioned morphisms  $\phi$  and  $H^0(\phi(1))$  are surjective:

**Proposition 3.1.** *For  $a \geq 1$ ,  $b \geq a + n$  and  $2b \geq (n + 2)a$ , the set of elements  $\phi \in M$  such that*

$$\phi : B \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow A \otimes \mathcal{O}_{\mathbb{P}^n}(1) \text{ and } H^0(\phi(1)) : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^b) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)^a)$$

*are surjective forms a non-empty open dense subset that we will denote by  $V_n$ .*

*Proof.* See [EH92, Proposition 4.1]. □

For any  $2 \leq n$  and any  $1 \leq a$ , we denote by  $\mathcal{E}_{n,a}$  any vector bundle on  $\mathbb{P}^n$  given by the exact sequence

$$(3.2) \quad 0 \longrightarrow \mathcal{E}_{n,a} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{(n+2)a} \xrightarrow{\phi(1)} \mathcal{O}_{\mathbb{P}^n}(2)^{2a} \longrightarrow 0$$

where  $\phi \in V_n$ . Note that  $\mathcal{E}_{n,a}$  has rank  $na$ .

**Lemma 3.2.** *With the above notation we have:*

(i)

$$h^0(\mathbb{P}^n, \mathcal{E}_{n,a}(t)) = \begin{cases} 0 & \text{for } t \leq 0, \\ a((n+2)\binom{n+t+1}{n} - 2\binom{n+t+2}{n}) & \text{for } t > 0. \end{cases}$$

(ii)

$$h^1(\mathbb{P}^n, \mathcal{E}_{n,a}(t)) = \begin{cases} 0 & \text{for } t < -2 \text{ or } t \geq 0, \\ an & \text{for } t = -1 \\ 2a & \text{for } t = -2. \end{cases}$$

(iii)  $h^i(\mathbb{P}^n, \mathcal{E}_{n,a}(t)) = 0$  for all  $t \in \mathbb{Z}$  and  $2 \leq i \leq n-1$ .(iv)  $h^n(\mathbb{P}^n, \mathcal{E}_{n,a}(t)) = 0$  for  $t \geq -n-1$ .

*Proof.* Since  $\phi \in V_n$ , by Proposition 3.1,  $H^0(\phi(1))$  is surjective. But, since the  $K$ -vector spaces  $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{(n+2)a})$  and  $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)^{2a})$  have the same dimension,  $H^0(\phi(1))$  is an isomorphism and therefore  $H^0(\mathcal{E}_{n,a}) = 0$ . *A fortiori*,  $H^0(\mathcal{E}_{n,a}(t)) = 0$  for  $t \leq 0$ . On the other hand, again by the surjectivity of  $H^0(\phi(1))$ ,  $H^1(\mathcal{E}_{n,a}) = 0$ . Since it is obvious that  $H^i(\mathcal{E}_{n,a}(1-i)) = 0$  for  $i \geq 2$  it turns out that  $\mathcal{E}_{n,a}$  is 1-regular and in particular,  $H^1(\mathcal{E}_{n,a}(t)) = 0$  for  $t \geq 0$ . The rest of the cohomology groups can be easily deduced from the long exact cohomology sequence associated to the exact sequence (3.2).  $\square$

From now on, for any  $2 \leq n$  and any  $1 \leq a$ , we call  $\mathcal{F}_{n,a}^X$  the family of *general* rank  $na$  vector bundles  $\mathcal{E}$  on  $X \subset \mathbb{P}^n$  sitting in an exact sequence of the following type:

$$(3.3) \quad 0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(1)^{(n+2)a} \xrightarrow{f} \mathcal{O}_X(2)^{2a} \longrightarrow 0.$$

**Proposition 3.3.** *Let  $X \subset \mathbb{P}^n$  be a smooth ACM variety of dimension  $d \geq 2$ . With the above notation, we have:*

(1) *A general vector bundle  $\mathcal{E} \in \mathcal{F}_{n,a}^X$  satisfies*

$$\begin{aligned} H_*^i \mathcal{E} &= 0 && \text{for } 2 \leq i \leq d-1, \\ H^1(X, \mathcal{E}(t)) &= 0 && \text{for } t \neq -1, -2. \end{aligned}$$

(2) *A general vector bundle  $\mathcal{E} \in \mathcal{F}_{n,a}^X$  is simple.*(3)  *$\mathcal{F}_{n,a}^X$  is a non-empty irreducible family of dimension  $a^2(n^2 + 2n - 4) + 1$  of simple (hence indecomposable) rank  $na$  vector bundles on  $X$ .*

*Proof.* (1) Since  $H^i(X, \mathcal{E}(t)) = 0$  for all  $t \in \mathbb{Z}$  and  $2 \leq i \leq d-1$ , and  $H^1(X, \mathcal{E}(t)) = 0$  for  $t \neq -1, -2$  are open conditions, it is enough to exhibit a vector bundle  $\mathcal{E} \in \mathcal{F}_{n,a}^X$  verifying these vanishings. Tensoring the exact sequence (3.2) with  $\mathcal{O}_X$ , we get

$$(3.4) \quad 0 \longrightarrow \mathcal{E} := \mathcal{E}_{n,a} \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{(n+2)a} \longrightarrow \mathcal{O}_X(2)^{2a} \longrightarrow 0.$$

Taking cohomology, we immediately obtain  $H^i(X, \mathcal{E}(t)) = 0$  for all  $t \in \mathbb{Z}$  and  $2 \leq i \leq d-1$ . On the other hand, we tensor with  $\mathcal{E}_{n,a}$  the exact sequence (3.1) sheafified

$$\begin{aligned} 0 \longrightarrow \bigoplus_{j=1}^{\beta_c} \mathcal{O}_{\mathbb{P}^n}(-n_j^c) \xrightarrow{\varphi_c} \bigoplus_{j=1}^{\beta_{c-1}} \mathcal{O}_{\mathbb{P}^n}(-n_j^{c-1}) \xrightarrow{\varphi_{c-1}} \dots \xrightarrow{\varphi_2} \\ \bigoplus_{j=1}^{\beta_1} \mathcal{O}_{\mathbb{P}^n}(-n_j^1) \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\varphi_0} \mathcal{O}_X \longrightarrow 0 \end{aligned}$$

and we get

$$(3.5) \quad \begin{aligned} 0 \longrightarrow \bigoplus_{j=1}^{\beta_c} \mathcal{E}_{n,a}(-n_j^c) \xrightarrow{\varphi_c} \bigoplus_{j=1}^{\beta_{c-1}} \mathcal{E}_{n,a}(-n_j^{c-1}) \xrightarrow{\varphi_{c-1}} \dots \xrightarrow{\varphi_{i+1}} \bigoplus_{j=1}^{\beta_i} \mathcal{E}_{n,a}(-n_j^i) \xrightarrow{\varphi_i} \\ \dots \xrightarrow{\varphi_2} \bigoplus_{j=1}^{\beta_1} \mathcal{E}_{n,a}(-n_j^1) \xrightarrow{\varphi_1} \mathcal{E}_{n,a} \xrightarrow{\varphi_0} \mathcal{E} = \mathcal{E}_{n,a} \otimes \mathcal{O}_X \longrightarrow 0. \end{aligned}$$

Set  $\mathcal{H}_i := \ker(\varphi_i)$ ,  $0 \leq i \leq c-2$ . Cutting the exact sequence (3.5) into short exact sequences and taking cohomology, we obtain

$$\begin{aligned} \cdots &\longrightarrow \mathrm{H}^1(\mathbb{P}^n, \mathcal{E}_{n,a}(t)) \longrightarrow \mathrm{H}^1(X, \mathcal{E}(t)) \longrightarrow \mathrm{H}^2(\mathbb{P}^n, \mathcal{H}_0(t)) \longrightarrow \cdots \\ \cdots &\longrightarrow \mathrm{H}^2(\mathbb{P}^n, \bigoplus_{j=1}^{\beta_1} \mathcal{E}_{n,a}(-n_j^1 + t)) \longrightarrow \mathrm{H}^2(\mathbb{P}^n, \mathcal{H}_0(t)) \longrightarrow \mathrm{H}^3(\mathbb{P}^n, \mathcal{H}_1(t)) \longrightarrow \cdots \\ &\quad \cdots \\ \cdots &\longrightarrow \mathrm{H}^{c-1}(\mathbb{P}^n, \bigoplus_{j=1}^{\beta_{c-2}} \mathcal{E}_{n,a}(-n_j^{c-2} + t)) \longrightarrow \mathrm{H}^{c-1}(\mathbb{P}^n, \mathcal{H}_{c-3}(t)) \\ &\quad \longrightarrow \mathrm{H}^c(\mathbb{P}^n, \mathcal{H}_{c-2}(t)) \longrightarrow \cdots \\ \cdots &\longrightarrow \mathrm{H}^c(\mathbb{P}^n, \bigoplus_{j=1}^{\beta_{c-1}} \mathcal{E}_{n,a}(-n_j^{c-1} + t)) \longrightarrow \mathrm{H}^c(\mathbb{P}^n, \mathcal{H}_{c-2}(t)) \\ &\quad \longrightarrow \mathrm{H}^{c+1}(\mathbb{P}^n, \bigoplus_{j=1}^{\beta_c} \mathcal{E}_{n,a}(-n_j^c + t)) \longrightarrow \cdots \end{aligned}$$

Using Lemma 3.2, we conclude that  $\mathrm{H}^1(X, \mathcal{E}(t)) = 0$  for  $t \neq -1, -2$ .

(2) A general vector bundle  $\mathcal{E} \in \mathcal{F}_{n,a}^X$  sits in an exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{g} \mathcal{O}_X(1)^{(n+2)a} \xrightarrow{f} \mathcal{O}_X(2)^{2a} \longrightarrow 0$$

and checking that  $\mathcal{E}$  is simple is equivalent to checking that  $\mathcal{E}^\vee$  is simple. Notice that the morphism  $f^\vee : \mathcal{O}_X(-2)^{2a} \longrightarrow \mathcal{O}_X(-1)^{(n+2)a}$  appearing in the exact sequence

$$(3.6) \quad 0 \longrightarrow \mathcal{O}_X(-2)^{2a} \xrightarrow{f^\vee} \mathcal{O}_X(-1)^{(n+2)a} \xrightarrow{g^\vee} \mathcal{E}^\vee \longrightarrow 0$$

is a general element of the  $K$ -vector space

$$M := \mathrm{Hom}(\mathcal{O}_X(-2)^{2a}, \mathcal{O}_X(-1)^{(n+2)a}) \cong K^{n+1} \otimes K^{2a} \otimes K^{(n+2)a}$$

because  $\mathrm{Hom}(\mathcal{O}_X(-2), \mathcal{O}_X(-1)) \cong \mathrm{H}^0(\mathcal{O}_X(1)) \cong \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1)) \cong K^{n+1}$ . Therefore,  $f^\vee$  can be represented by an  $(n+2)a \times 2a$  matrix  $A$  with entries in  $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1))$ . Since  $\mathrm{Aut}(\mathcal{O}_X(-1)^{(n+2)a}) \cong \mathrm{GL}((n+2)a)$  and  $\mathrm{Aut}(\mathcal{O}_X(-2)^{2a}) \cong \mathrm{GL}(2a)$ , the group  $\mathrm{GL}((n+2)a) \times \mathrm{GL}(2a)$  acts naturally on  $M$  by

$$\begin{aligned} \mathrm{GL}((n+2)a) \times \mathrm{GL}(2a) \times M &\longrightarrow M \\ (g_1, g_2, A) &\longmapsto g_1^{-1} A g_2. \end{aligned}$$

For all  $A \in M$  and  $\lambda \in K^*$ ,  $(\lambda \mathrm{Id}_{(n+2)a}, \lambda \mathrm{Id}_{2a})$  belongs to the stabilizer of  $A$  and, hence,  $\dim_K \mathrm{Stab}(A) \geq 1$ . Since  $(2a)^2 + (n+2)a^2 - 2a(n+1)(n+2)a < 0$ , it follows from [Kac80, Theorem 4] that  $\dim_K \mathrm{Stab}(A) = 1$ . We will now check that  $\mathcal{E}^\vee$  is simple. Otherwise, there exists a non-trivial morphism  $\phi : \mathcal{E}^\vee \longrightarrow \mathcal{E}^\vee$  and composing with  $g^\vee$  we get a morphism

$$\bar{\phi} = \phi \circ g^\vee : \mathcal{O}_X(-1)^{(n+2)a} \longrightarrow \mathcal{E}^\vee.$$

Applying  $\mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, -)$  to the exact sequence (3.6) and taking into account that

$$\mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{O}_X(-2)^{2a}) = \mathrm{Ext}^1(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{O}_X(-2)^{2a}) = 0$$

we obtain  $\mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{O}_X(-1)^{(n+2)a}) \cong \mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{E}^\vee)$ . Therefore, there is a non-trivial morphism  $\tilde{\phi} \in \mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{O}_X(-1)^{(n+2)a})$  induced by  $\bar{\phi}$  and represented by a matrix  $B \neq \mu \mathrm{Id} \in \mathrm{Mat}_{(n+2)a \times (n+2)a}(K)$  such

that the following diagram commutes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{O}_X(-2)^{2a} & \xrightarrow{f^\vee} & \mathcal{O}_X(-1)^{(n+2)a} & \xrightarrow{g^\vee/\tilde{\phi}} & \mathcal{E} \longrightarrow 0 \\
 & & \downarrow C & & \downarrow B & \searrow & \downarrow \phi \\
 0 & \longrightarrow & \mathcal{O}_X(-2)^{2a} & \xrightarrow{f^\vee} & \mathcal{O}_X(-1)^{(n+2)a} & \xrightarrow{g^\vee} & \mathcal{E}^\vee \longrightarrow 0
 \end{array}$$

where  $C \in \text{Mat}_{2a \times 2a}(K)$  is the matrix associated to  $\tilde{\phi}|_{\mathcal{O}_X(-2)^{2a}}$ . Then the pair  $(C, B) \neq (\mu Id, \mu Id)$  verifies  $AC = BA$ . Let us consider an element  $\alpha \in K$  that does not belong to the set of eigenvalues of  $B$  and  $C$ . Then the pair  $(B - \alpha Id, C - \alpha Id) \in GL((n+2)a) \times GL(2a)$  belongs to  $\text{Stab}(f)$  and therefore  $\dim_K \text{Stab}(f) > 1$  which is a contradiction. Thus,  $\mathcal{E}$  is simple.

(3) It only remains to compute the dimension of  $\mathcal{F}_{n,a}^X$ . Since the isomorphism class of a general vector bundle  $\mathcal{E} \in \mathcal{F}_{n,a}^X$  associated to a morphism  $\phi \in M := \text{Hom}(\mathcal{O}_X^{(n+2)a}, \mathcal{O}_X(1)^{2a})$  depends only on the orbit of  $\phi$  under the action of  $GL((n+2)a) \times GL(2a)$  on  $M$ , we have:

$$\begin{aligned}
 \dim \mathcal{F}_{n,a}^X &= \dim M - \dim \text{Aut}(\mathcal{O}_X^{(n+2)a}) - \dim \text{Aut}(\mathcal{O}_X(1)^{2a}) + 1 \\
 &= 2a^2(n+2)(n+1) - a^2(n+2)^2 - 4a^2 + 1 = a^2(n^2 + 2n - 4) + 1.
 \end{aligned}$$

□

We are now ready to prove the main result of this short paper and give an affirmative answer to Problem 1.1.

**Theorem 3.4.** *Let  $X \subset \mathbb{P}^n$  be a smooth ACM variety of dimension  $d \geq 2$ . The very ample line bundle  $\mathcal{O}_X(s)$ ,  $s \geq 3$ , embeds  $X$  in  $\mathbb{P}^{\text{h}^0(\mathcal{O}_X(s)) - 1}$  as a variety of wild representation type.*

*Proof.* Indeed, given any integer  $p$ , we choose an integer  $a$  such that  $an \geq p$ . By Proposition 3.3, there exists a family  $\mathcal{F}_{n,a}^X$  of dimension  $a^2(n^2 + 2n - 4) + 1$  of simple (hence indecomposable) vector bundles  $\mathcal{E}$  on  $X$  of rank  $an$ . For a general  $\mathcal{E} \in \mathcal{F}_{n,a}^X$ , we have  $H^i(X, \mathcal{E}(t)) = 0$  for all  $t \in \mathbb{Z}$  and  $2 \leq i \leq d - 1$ , and  $H^1(X, \mathcal{E}(t)) = 0$  for  $t \neq -2, -1$ . Therefore, the very ample line bundle  $\mathcal{O}_X(s)$  embeds  $X$  in  $\mathbb{P}^{\text{h}^0(\mathcal{O}_X(s)) - 1}$ , as a variety of wild representation type. □

**Corollary 3.5.** *The smallest possible integer  $N_X$  such that  $X$  embeds as a variety of wild representation type is bounded by  $N_X \leq \binom{n+3}{3} - 1$ .*

We will finish the paper with a final remark concerning the terminology.

*Final Remark 3.6.* In [DG01, Definition 1.4], Drozd and Greuel introduced two definitions of wildness: geometrically wild and algebraically wild. Roughly speaking a projective variety  $X \subset \mathbb{P}^n$  is *geometrically wild* if the corresponding homogeneous coordinate ring  $R_X$  has arbitrarily large families of indecomposable Maximal Cohen-Macaulay  $R_X$ -modules.  $X$  is said to be *algebraically wild* if for every finitely generated  $k$ -algebra  $A$  there exists a family of Maximal Cohen-Macaulay  $R_X$ -modules  $\mathcal{M}$  such that the following conditions hold:

- (1) For every indecomposable  $A$ -module  $L$  the  $R_X$ -module  $\mathcal{M} \otimes_A L$  is indecomposable.
- (2) If  $\mathcal{M} \otimes_A L \cong \mathcal{M} \otimes_A L'$  for some finite dimensional  $A$ -modules  $L$  and  $L'$ , then  $L \cong L'$ .

It is not difficult to check that if  $X$  is algebraically wild, then it is also geometrically wild. It is not known though conjectured whether the converse is true, i.e. if geometrically wild implies algebraically wild.

It is worthwhile to point out that the constructed embedding proves the geometrical wildness of  $X \subset \mathbb{P}^{N \times}$  but it remains open whether it is also algebraically wild.

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