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AN ENTIRE FUNCTION WITH NO FIXED POINTS AND NO INVARIANT BAKER DOMAINS

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ABSTRACT. We show that there exists an entire function which has neither fixed points nor invariant Baker domains. The question whether such a function exists was raised by Buff.

1. Introduction and result

Newton's method of finding the zeros of an entire function g consists of iterating the meromorphic function $f(z) = z - g(z)/g'(z)$. Douady suggested that paths where g tends to the asymptotic value 0 are related to f-invariant domains where the iterates of f tend to infinity. A maximal domain with this property is called an *invariant Baker domain*; cf. [3, §4.7] or [11]. In response to Douady's question it was shown in [8] that under mild additional hypotheses the existence of an invariant Baker domain does indeed imply that 0 is an asymptotic value of g. However, this is not always the case [5].

If g has no zeros at all, then the Newton function f has no fixed points. Moreover, 0 is an asymptotic value of g by Iversen's theorem [10, p. 289]. This led Buff to ask whether there exists an entire function having no fixed points and no invariant Baker domains. We show that such a function exists.

Theorem. *There exists an entire function with no fixed points and no invariant Baker domains.*

A meromorphic function with this property was constructed in [4]. The present construction is based on similar ideas. As in [4], a function f satisfying the conclusion of the Theorem can be given explicitly.

Let (r_k) be a sequence of real numbers tending to ∞ and let (n_k) be a sequence of positive integers satisfying $n_k \geq k$ for all $k \in \mathbb{N}$. Then

$$h(z) = \prod_{k=1}^{\infty} \left(1 + \left(\frac{z}{r_k}\right)^{n_k}\right)$$

defines an entire function h. Indeed, if $|z| \leq R$ and k is so large that $r_k \geq 2R$, then $|z/r_k|^{n_k} \leq 2^{-k}$, implying that the infinite product converges locally uniformly.

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For $k \geq 2$ we put $m_k = \sum_{j=1}^{k-1} n_j$. We shall show that if

$$r_k \geq 2r_{k-1} \geq 4 \quad \text{and} \quad n_k \geq 20 r_k^2 \exp\left(4 r_k^{m_k}\right) \tag{1.1}$$

for $k \geq 2$, then $f(z) = z + e^{h(z)}$ has the required property.

The idea of the proof is as follows. We show (in section 2.2) that close to the zeros of h on the circle of radius r_k there are points where $\operatorname{Re} h$ and thus $|f|$ are large. Thus the Euclidean distance of the images of two nearby points is large, and comparison with the hyperbolic metric (see section 2.1) shows that a hypothetical Baker domain of f cannot contain a large disk around a point on the circle of radius r_k. In fact, the maximal radius of such a disk can be made arbitrarily small by choosing n_k large. Thus the Baker domain is very "thin" near the circle of radius r_k and hence the density of its hyperbolic metric is very large there, provided n_k is large. Moreover, $|h|$ is bounded above and thus $|f(z) - z)|$ is bounded below on the circle of radius r_k by some expression depending only on $r_1, \ldots, r_k$ and $n_1, \ldots, n_{k-1}$, but not on n_k. Together this yields that the hyperbolic distance of z and $f(z)$ can be made large for z on this circle by choosing n_k large. On the other hand, it is well known and easy to see that the hyperbolic distance of z and $f(z)$ is bounded for z on some curve tending to infinity. Choosing z as the point where this curve intersects the circle of radius r_k we obtain a contradiction, provided k is large and n_k is sufficiently large compared to $r_1, \ldots, r_k$ and $n_1, \ldots, n_{k-1}$. The analysis will show that (1.1) suffices to make this argument work.

2. Preliminaries

2.1. **The hyberbolic metric.** We need some standard results about the hyperbolic metric which can be found in, e.g., [9, Section I.4].

We denote the open disk of radius r around a point $c \in \mathbb{C}$ by $D(c, r)$ and put $\mathbb{D} = D(0, 1)$. The density of the hyperbolic metric in a hyperbolic domain U is denoted by λ_U, normalized such that $\lambda_{\mathbb{D}}(z) = 2/(1 - |z|^2)$. The hyperbolic metric is denoted by ρ_U. For $a, b \in U$ we thus have

$$\rho_U(a, b) = \inf_{\gamma} \int_{\gamma} \lambda_U(z) |dz|,$$

where the infimum is taken over all curves γ that connect a and b. Then [9, p. 11]

$$\rho_{\mathbb{D}}(0, z) = \log \frac{1 + |z|}{1 - |z|} \quad \text{for } z \in \mathbb{D}. \tag{2.1}$$

It follows from Schwarz's lemma and the Koebe one quarter theorem that if U is simply connected, then [9, Theorem I.4.3]

$$\frac{1}{2 \operatorname{dist}(z, \partial U)} \leq \lambda_U(z) \leq \frac{2}{\operatorname{dist}(z, \partial U)} \tag{2.2}$$

for all $z \in U$. Here $\operatorname{dist}(z, \partial U) = \inf_{\zeta \in \partial U} |\zeta - z|$.

The following lemma is a simple consequence of (2.2).

Lemma 2.1. *Let U be a simply connected hyperbolic domain, $a, b \in U$ and $c \in \mathbb{C} \backslash U$. Then*

$$\rho_U(a, b) \geq \frac{1}{2} \left| \log \left| \frac{b - c}{a - c} \right| \right|.$$

Proof. Without loss of generality we may assume that $c = 0$. Let γ be a curve from a to b and let L be a branch of the logarithm defined in U. Then (2.2) yields

$$\begin{aligned}\int_\gamma \lambda_U(z)|dz| &\geq \frac{1}{2}\int_\gamma \frac{|dz|}{\operatorname{dist}(z,\partial U)} \geq \frac{1}{2}\int_\gamma \frac{|dz|}{|z|} \geq \frac{1}{2}\left|\int_\gamma \frac{dz}{z}\right| \\ &= \frac{1}{2}\left|L(b)-L(a)\right| \geq \frac{1}{2}\left|\operatorname{Re}(L(b)-L(a))\right| = \frac{1}{2}\left|\log\left|\frac{b}{a}\right|\right|,\end{aligned}$$

from which the conclusion follows. □

The next lemma follows easily from (2.1) and the triangle inequality.

Lemma 2.2. *If $a, b \in D(c, r/2)$, then $\rho_{D(c,r)}(a,b) \leq 2\log 3$.*

Finally we have the following form of Schwarz's lemma [9, Theorem I.4.3].

Lemma 2.3. *Let U, V be hyperbolic domains, $f\colon U \to V$ holomorphic and $a, b \in U$. Then $\rho_V(f(a), f(b)) \leq \rho_U(a,b)$.*

Applying this lemma to $f(z) = z$ yields

$$\rho_V(a,b) \leq \rho_U(a,b) \quad \text{if } U \subset V. \tag{2.3}$$

2.2. **Some growth estimates.** We have to estimate h on the circle of radius r_k from above and at certain points on the circle with radius $s_k = (1 + 1/n_k)r_k$ from below. The estimates needed are summarized in the following lemma.

Lemma 2.4. *For $k \geq 2$ we have*

$$|h(z)| \leq 4r_k^{m_k} \quad \textit{for } |z| = r_k \tag{2.4}$$

and for large k and $\nu \in \{0, 1, \ldots, n_k - 1\}$ there exists $\theta_\nu \in [0,1]$ such that

$$\operatorname{Re} h(s_k e^{2\pi i(\nu+\theta_\nu)/n_k}) \geq s_k. \tag{2.5}$$

Proof. For $k \geq 2$ and $|z| = r_k$ we have

$$\begin{aligned}\log|h(z)| &\leq \sum_{j=1}^{k-1} \log\left(1 + \left(\frac{r_k}{r_j}\right)^{n_j}\right) + \log 2 + \sum_{j=k+1}^{\infty} \log\left(1 + \left(\frac{r_k}{r_j}\right)^{n_j}\right) \\ &\leq \sum_{j=1}^{k-1} \log\left(1 + \frac{1}{2}{r_k}^{n_j}\right) + \log 2 + \sum_{j=k+1}^{\infty} \left(\frac{r_k}{r_j}\right)^{n_j} \\ &\leq \sum_{j=1}^{k-1} \log\left({r_k}^{n_j}\right) + \log 2 + \sum_{j=k+1}^{\infty} 2^{-n_j} \leq m_k \log r_k + 2\log 2,\end{aligned}$$

from which (2.4) follows.

For $t \in [0, 2\pi]$ and $z = s_k e^{it}$ we have

$$h(z) = \prod_{j=1}^{k-1}\left(1 + \left(\frac{z}{r_j}\right)^{n_j}\right) \cdot \left(1 + \left(1 + \frac{1}{n_k}\right)^{n_k} e^{in_k t}\right) \cdot \prod_{j=k+1}^{\infty}\left(1 + \left(\frac{z}{r_j}\right)^{n_j}\right).$$

It is easy to see that the second product tends to 1 as $k \to \infty$. Moreover,

$$1 + \left(\frac{z}{r_j}\right)^{n_j} = \left(\frac{z}{r_j}\right)^{n_j}(1 + \eta_j),$$

where $|\eta_j| = (r_j/s_k)^{n_j} \le 2^{-(k-j)n_j} \le 2^{-(k-j)j} \le 2^{1-k}$, implying that the first product in the above expression for $h(z)$ also tends to 1 as $k \to \infty$. Altogether we find, using the terminology $x_k \sim y_k$ if $x_k/y_k \to 1$, that

$$h(z) \sim \prod_{j=1}^{k-1} \left(\frac{z}{r_j}\right)^{n_j} \left(1 + e \cdot e^{in_k t}\right) = \prod_{j=1}^{k-1} \left(\frac{s_k}{r_j}\right)^{n_j} e^{im_k t} \left(1 + e \cdot e^{in_k t}\right)$$

for $z = s_k e^{it}$ as $k \to \infty$, uniformly for $t \in [0, 2\pi]$. Putting

$$T_k = \prod_{j=1}^{k-1} \left(\frac{s_k}{r_j}\right)^{n_j}$$

we thus have

$$h(s_k e^{it}) \sim T_k e^{im_k t} \left(1 + e \cdot e^{in_k t}\right) \tag{2.6}$$

as $k \to \infty$.

It is not difficult to see that for each $\varphi \in \mathbb{R}$ there exists $\theta = \theta(\varphi) \in [0, 1]$ such that $e^{2\pi i\varphi} \left(1 + e \cdot e^{2\pi i\theta}\right)$ is positive. For $\nu \in \{0, 1, \ldots, n_k - 1\}$ we put

$$\theta_\nu = \theta(\nu m_k/n_k) \quad \text{and} \quad p_\nu = e^{2\pi i\nu m_k/n_k} \left(1 + e \cdot e^{2\pi i\theta_\nu}\right).$$

Then p_ν is positive and thus $p_\nu = |p_\nu| \ge e - 1$. Since $m_k/n_k \to 0$ by (1.1), we deduce from (2.6) that

$$\begin{aligned} h(s_k e^{2\pi i(\nu+\theta_\nu)/n_k}) &\sim T_k e^{2\pi i(\nu+\theta_\nu)m_k/n_k} \left(1 + e \cdot e^{2\pi i(\nu+\theta_\nu)}\right) \\ &= p_\nu T_k e^{2\pi i\theta_\nu m_k/n_k} \sim p_\nu T_k. \end{aligned}$$

Thus

$$\operatorname{Re} h(s_k e^{2\pi i(\nu+\theta_\nu)/n_k}) \ge T_k$$

for large k and all $\nu \in \{0, 1, \ldots, n_k - 1\}$, from which (2.5) follows since we have $T_k \ge (s_k/r_2)^2 \ge s_k$ for large k by the definition of T_k. □

3. Proof of the Theorem

Let f be as defined in the introduction and suppose that f has an invariant Baker domain U. By a result of Baker [1], U is simply connected. Take $z_0 \in U$, connect z_0 and $f(z_0)$ by a curve γ_0 in U and put $\gamma = \bigcup_{j=0}^{\infty} f^j(\gamma_0)$. Then γ is a curve in U connecting z_0 to ∞. As γ_0 is compact, there exists $K > 0$ such that $\rho(f(z), z) \le K$ for all $z \in \gamma_0$. Since every $z \in \gamma$ has the form $z = f^j(\zeta)$ for some $\zeta \in \gamma_0$ and some $j \ge 0$, Lemma 2.3 yields

$$\rho(f(z), z) \le K \quad \text{for } z \in \gamma. \tag{3.1}$$

For large k the curve γ intersects the circle $\{z\colon |z| = r_k\}$. Let z_k be a point of intersection.

We shall show first that if k is large enough, then the disk $D(z_k, 20r_k/n_k)$ is not contained in U; that is,

$$D(z_k, 20r_k/n_k) \cap \partial U \ne \emptyset. \tag{3.2}$$

In order to do so we assume that $D(z_k, 20r_k/n_k) \subset U$. We write $z_k = r_k e^{2\pi i t_k}$ with $t_k \in [0, 1)$ and put $\nu = [n_k t_k]$, where $[x]$ denotes the largest integer not greater than x. Thus $n_k t_k = \nu + \delta$ where $\nu \in \{0, 1, \ldots, n_k - 1\}$ and $\delta \in [0, 1)$. Let

$$a_k = r_k e^{(2\nu+1)\pi i/n_k} \quad \text{and} \quad b_k = s_k e^{2\pi i(\nu+\theta_\nu)/n_k}.$$

Then

$$|a_k - z_k| = r_k \left| e^{(2\nu+1)\pi i/n_k - 2\pi i t_k} - 1 \right| = r_k \left| e^{(1-2\delta)\pi i/n_k} - 1 \right| \sim \frac{|1-2\delta|\pi r_k}{n_k}$$

and

$$\begin{aligned} |b_k - z_k| &\le |b_k - r_k e^{2\pi i(\nu+\theta_\nu)/n_k}| + |r_k e^{2\pi i(\nu+\theta_\nu)/n_k} - z_k| \\ &= s_k - r_k + r_k \left| e^{2\pi i(\nu+\theta_\nu)/n_k - 2\pi i t_k} - 1 \right| \\ &= \frac{r_k}{n_k} + r_k \left| e^{2\pi i(\theta_\nu - \delta)/n_k} - 1 \right| \\ &\sim \frac{(1+2\pi|\theta_\nu - \delta|)r_k}{n_k}, \end{aligned}$$

which imply that

$$a_k \in D(z_k, 10r_k/n_k) \quad \text{and} \quad b_k \in D(z_k, 10r_k/n_k)$$

for large k. Lemma 2.2 and (2.3) now yield

$$\rho_U(a_k, b_k) \le \rho_{D(z_k, 20r_k/n_k)}(a_k, b_k) \le 2\log 3. \tag{3.3}$$

Since $h(a_k) = 0$ by the definition of h and $\operatorname{Re} h(b_k) \ge s_k$ by (2.5), we have

$$|f(a_k)| = |a_k + 1| \le r_k + 1 \quad \text{and} \quad |f(b_k)| \ge e^{s_k} - s_k \ge s_k^2 \ge r_k^2 \tag{3.4}$$

for large k. Fix a point $c \in \partial U$. Lemma 2.1 and (3.4) imply that

$$\rho_U(f(a_k), f(b_k)) \ge \frac{1}{2}\log\left|\frac{f(b_k) - c}{f(a_k) - c}\right| \ge \frac{1}{2}\log\frac{r_k^2 - |c|}{r_k + 1 + |c|} \tag{3.5}$$

for large k. Now a contradiction is obtained from Lemma 2.3, (3.3) and (3.5), provided k is sufficiently large. This contradiction shows that (3.2) holds for large k.

Thus, for large k, there exists $c_k \in D(z_k, 20r_k/n_k) \cap \partial U$. Lemma 2.1 now yields

$$\rho_U(f(z_k), z_k) \ge \frac{1}{2}\log\left|\frac{f(z_k) - c_k}{z_k - c_k}\right| = \frac{1}{2}\log\left|\frac{e^{h(z_k)}}{z_k - c_k} + 1\right|.$$

Since

$$\left|\frac{e^{h(z_k)}}{z_k - c_k}\right| \ge \frac{e^{-|h(z_k)|}}{|z_k - c_k|} \ge \frac{n_k \exp\left(-4r_k^{m_k}\right)}{20r_k} \ge r_k$$

for $k \ge 2$ by (1.1) and (2.4), we obtain

$$\rho_U(f(z_k), z_k) \ge \frac{1}{2}\log(r_k - 1)$$

for large k, contradicting (3.1). □

Remark 1. Buff and Rückert [8] considered *virtual immediate basins* instead of invariant Baker domains. However, for functions for which all Baker domains are simply connected the two concepts coincide; cf. the discussion in [4, p. 431] or [8, p. 4]. By the result of Baker [1] already quoted, this holds in particular for entire functions. By a recent result of Barański, Fagella, Jarque and Karpińska [2], it also holds for Newton maps of entire functions.

Remark 2. The function f constructed in the proof is the Newton function for

$$g(z) = \exp\left(-\int_0^z e^{-h(t)}dt\right).$$

Since g has no zeros, g has a direct singularity over 0; see [10, p. 289] for this result, as well as [6, 12] for the terminology used here and below. As f has no invariant Baker domains, g has no logarithmic singularity over 0 by one of the results obtained by Buff and Rückert in the paper already mentioned in the introduction [8, Theorem 4.1]. Thus g has a direct non-logarithmic singularity over 0. This implies [7, Theorem 5] that g has uncountably many direct non-logarithmic singularities over 0. As g has no critical points, a result of Sixsmith [12, Theorem 1.2] yields that every neighborhood of any of these singularities contains a neighborhood of an indirect or logarithmic singularity of g whose projection is different from 0. Overall we see that the set of singularities of the inverse of g has a quite complicated structure.

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References

[1] I. N. Baker, *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), no. 2, 277–283. MR0402044 (53 #5867)

[2] Krzysztof Barański, Núria Fagella, Xavier Jarque and Bogusława Karpińska, *On the connectivity of the Julia sets of meromorphic functions*, Invent. Math. (to appear in print), electronically published on February 8, 2014, DOI 10.1007/S00222-014-0504-5.

[3] Walter Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 2, 151–188, DOI 10.1090/S0273-0979-1993-00432-4. MR1216719 (94c:30033)

[4] Walter Bergweiler, *Newton's method and Baker domains*, J. Difference Equ. Appl. **16** (2010), no. 5-6, 427–432, DOI 10.1080/10236190903203846. MR2642458 (2011f:37076)

[5] Walter Bergweiler, David Drasin, and James K. Langley, *Baker domains for Newton's method* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **57** (2007), no. 3, 803–814. MR2336830 (2009b:37079)

[6] Walter Bergweiler and Alexandre Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana **11** (1995), no. 2, 355–373, DOI 10.4171/RMI/176. MR1344897 (96h:30055)

[7] Walter Bergweiler and Alexandre Eremenko, *Direct singularities and completely invariant domains of entire functions*, Illinois J. Math. **52** (2008), no. 1, 243–259. MR2507243 (2010c:30041)

[8] Xavier Buff and Johannes Rückert, *Virtual immediate basins of Newton maps and asymptotic values*, Int. Math. Res. Not., posted on 2006, Art. ID 65498, 18, DOI 10.1155/IMRN/2006/65498. MR2211149 (2006m:37062)

[9] Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR1230383 (94h:30033)

[10] Rolf Nevanlinna, *Eindeutige analytische Funktionen* (German), 2te Aufl, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd XLVI, Springer-Verlag, Berlin, 1953. MR0057330 (15,208c)

[11] P. J. Rippon, *Baker domains*, Transcendental dynamics and complex analysis, London Math. Soc. Lecture Note Ser., vol. 348, Cambridge Univ. Press, Cambridge, 2008, pp. 371–395, DOI 10.1017/CBO9780511735233.015. MR2458809 (2009j:30061)

[12] D. J. Sixsmith, *A new characterisation of the Eremenko-Lyubich class*, J. Anal. Math., to appear.

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