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CONSTRUCTION OF PATHOLOGICAL GÂTEAUX DIFFERENTIABLE FUNCTIONS

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Abstract. We prove that for many pairs (X, Y) of classical Banach spaces, there exists a bounded, Lipschitz, Gâteaux differentiable function from X to Y whose derivatives are all far apart.

1. Introduction

Let F be a function between real Banach spaces X and Y. We say that F has the *jump property* if F is Gâteaux differentiable at every point of X and there exists a constant $\alpha > 0$ such that

$$\|F'(x) - F'(y)\| \geq \alpha \text{ whenever } x, y \in X \text{ and } x \neq y.$$

We say that the couple (X, Y) has the *jump property* if there exists a Lipschitz continuous, bounded function $F : X \longrightarrow Y$ with the jump property.

This concept was first considered by Deville and Hájek in [9], where it was shown that the couple $(X, \mathbb{R})$ never has the jump property and that such a property cannot be achieved if we replace Gâteaux by Fréchet differentiability. There, it was also proved that $(\ell^1, \mathbb{R}^2)$ has the jump property and that if $1 \leq p, q < \infty$, then (ℓ^p, ℓ^q) enjoys it if and only if $p \leq q$. Later on, Bayart [5] proved that if X is any separable infinite dimensional Banach space, then (X, c_0) has the jump property.

Notice that a couple of Banach spaces (X, Y) has the jump property if and only if there exists a Lipschitz continuous, bounded and Gâteaux differentiable function $F : X \longrightarrow Y$ such that $\|F'(x) - F'(y)\| \geq 1$ whenever x and y are different elements of X. It is also clear that if the couple (X, Y) has the jump property, then the space $\mathcal{L}(X, Y)$ of bounded linear operators from X into Y is nonseparable, and that if Z is a Banach space that contains an isomorphic copy of Y, then the couple (X, Z) has the jump property as well.

A rather opposite kind of construction was provided in [3], where it was shown that if X and Y are separable Banach spaces, then there exists a continuous Gâteaux differentiable function $F : X \longrightarrow Y$ such that $F'(X) = \mathcal{L}(X, Y)$. Some more results in this direction, in the case of Fréchet differentiability, were obtained in [4], [6], [10] and [11].

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Our main aim in this work is to show that the existence of two operators from X to Y of a given form is a sufficient condition to ensure that the couple (X,Y) has the jump property. This condition, which is formulated in terms of the behavior of an unconditional basic sequence in Y with respect to a biorthogonal system in X, can be applied whenever X and Y are classical Banach spaces, such as spaces of continuous functions on a compact metric space, Orlicz sequence spaces or L^p spaces. We shall also see that such a construction is possible in less classical spaces. Finally, we shall prove that, in some cases, the set of functions between X and Y that enjoy the jump property is lineable. We notice that in all our examples, the space X is separable.

Let us recall some notation and terminology. The symbols S_X and X^* stand for the unit sphere and the topological dual of a Banach space X, respectively. The letter $\mathbb{N}$ denotes the set of positive integers. A Schauder basis $(e_n)_n$ in X is said to be *unconditional* if for every $x \in X$, its expansion $\sum_{n=1}^\infty x_n e_n$ converges unconditionally or, equivalently, if there is a constant $C>0$ such that

$$\left\|\sum_{n=1}^\infty a_n x_n e_n\right\| \le C\|a\|_\infty \left\|\sum_{n=1}^\infty x_n e_n\right\| \quad \text{whenever} \quad a=(a_n)_n \in \ell^\infty.$$

A sequence $(e_n, e_n^*)_n \subset X \times X^*$ is called a *biorthogonal system* of X if $e_i^*(e_j)=\delta_{ij}$ for every $i,j \in \mathbb{N}$. The biorthgonal system is called *total* if the sequence $(e_n^*)_n$ separates the points of X, i.e., if $x=0$ whenever $e_n^*(x)=0$ for all $n \in \mathbb{N}$. If $\sup_n \|e_n\| < \infty$ and $\sup_n \|e_n^*\| < \infty$, then we say that the biorthogonal system is *bounded*. We refer to [8] and [12] for some unexplained notions and basic properties of differentibility and Schauder bases in Banach spaces, respectively.

2. Main theorem

The main result of this paper reads as follows.

Theorem 2.1. *Let X and Y be Banach spaces. Suppose that there exist a total, bounded biorthogonal system $(e_n, e_n^*)_n \subset X \times X^*$ and an unconditional basic sequence $(f_n)_n \subset Y$ such that* $\inf_n \|f_n\| > 0$ *and for each $h \in X$, the series $\sum_{n=1}^\infty e_n^*(h) f_{2n-1}$ and $\sum_{n=1}^\infty e_n^*(h) f_{2n}$ converge in norm. Then the couple (X,Y) has the jump property.*

Let us observe that, under the above assumptions, if

$$L(h) = \sum_{n=1}^\infty e_n^*(h)\left(f_{2n-1}+f_{2n}\right),$$

then, according to the uniform boundedness principle, L is a bounded linear operator from X into Y.

We also notice that the hypotheses of Theorem 2.1 imply that the space $\mathcal{L}(X,Y)$ contains an isomorphic copy of ℓ^∞ (see the remark after the proof of Theorem 2.1). In particular, $\mathcal{L}(X,Y)$ is nonseparable.

An important tool in our construction of functions with the jump property is the following result, which generalizes [9, Lemma 3].

Lemma 2.2. *For each norm $|||\cdot|||$ on $\mathbb{R}^2$, each $p=(q,r) \in \mathbb{R}^2$ with $q<r$ and each $\varepsilon>0$, there exists a continuously differentiable function $\varphi = \varphi_{p,\varepsilon} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that:*

(i) $|||\varphi(s,t)||| \le \varepsilon$ *for all* $(s,t) \in \mathbb{R}^2$,

(ii) $\varphi(s,t) = 0$ *whenever* $s < q$,
(iii) $|||\frac{\partial \varphi}{\partial s}(s,t)||| \leq \varepsilon$ *for all* $(s,t) \in \mathbb{R}^2$,
(iv) $|||\frac{\partial \varphi}{\partial t}(s,t)||| \leq 1$ *for all* $(s,t) \in \mathbb{R}^2$, *and*
(v) $|||\frac{\partial \varphi}{\partial t}(s,t)||| = 1$ *whenever* $s \geq r$.

Proof. The function

$$f(\alpha) = \frac{1}{|||(-\sin\alpha, \cos\alpha)|||}, \quad \alpha \in \mathbb{R},$$

is clearly Lipschitz on $\mathbb{R}$ and satisfies $c \leq f(\alpha) \leq C$ for some constants $C > c > 0$ and all $\alpha \in \mathbb{R}$. Thus, according to the Cauchy-Picard theorem, there exists a (unique) continuously differentiable function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\alpha'(t) = \frac{1}{|||\,(-\sin\alpha(t), \cos\alpha(t))\,|||} \quad \text{and} \quad \alpha(0) = 0.$$

In particular, the mapping $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by the formula

$$\gamma(t) = (\cos\alpha(t), \sin\alpha(t)), \quad t \in \mathbb{R},$$

satisfies $|||\gamma'(t)||| = 1$ for all $t \in \mathbb{R}$.

Let $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ be an infinitely differentiable function such that $0 \leq \beta(s) \leq 1$, $\beta(s) = 0$ for $s \leq q$ and $\beta(s) = 1$ whenever $s \geq r$. If $n > C\varepsilon^{-1}(\|\beta'\|_\infty + 1)$, then the function

$$\varphi(s,t) = \frac{\beta(s)}{n}\gamma(nt)$$

satisfies the required properties. □

Apart from the former lemma, we shall need the following criterion of differentiability, whose proof is left to the reader.

Lemma 2.3. *Let* $(F_n)_n$ *be a sequence of Gâteaux differentiable mappings between Banach spaces* X *and* Y *satisfying the following properties:*

(1) *The series* $\sum_{n=1}^\infty F_n$ *converges pointwise to a function* $F : X \longrightarrow Y$.
(2) *For each* $h \in X$, *the series* $\sum_{n=1}^\infty F_n'(x)(h)$ *converges uniformly with respect to* $x \in X$, *and for all* $n \in \mathbb{N}$ *and* $h \in X$, *the mapping* $X \ni x \mapsto F_n'(x)(h)$ *is continuous.*

Then F *is Gâteaux differentiable on* X, *and for every* $x, h \in X$ *we have*

$$F'(x)(h) = \sum_{n=1}^\infty F_n'(x)(h).$$

Proof of Theorem 2.1. We divide the proof into several steps.

Step 1: *Construction of the function* F. The unconditionality of $(f_n)_n$ yields a constant $C > 0$ such that

$$\left\|\sum_{n=1}^\infty a_n x_n f_n\right\| \leq C\|a\|_\infty \left\|\sum_{n=1}^\infty x_n f_n\right\| \tag{2.1}$$

whenever $a = (a_n)_n \in \ell^\infty$ and $(x_n)_n \subset \mathbb{R}$ is any sequence such that the series $\sum_{n=1}^\infty x_n f_n$ is norm convergent.

Since, by the hypothesis, the sequence $(1/\|f_n\|)_n$ is bounded, thanks to the former inequality we have that for each $h \in X$, the series $\sum_{n=1}^\infty e_n^*(h)\frac{f_{2n-1}}{\|f_{2n-1}\|}$ and $\sum_{n=1}^\infty e_n^*(h)\frac{f_{2n}}{\|f_{2n}\|}$ are norm convergent. Thus, we may assume that $\|f_n\| = 1$ for all $n \in \mathbb{N}$. Let us write $\mathbb{P} = \{(q,r) \in \mathbb{Q}^2 : q < r\}$ and let $k \mapsto (n_k, (q_k, r_k))$ be a bijection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{P}$ such that $n_k \neq k$ for all $k \in \mathbb{N}$. Let $M \geq 1$ be such that $\|e_n\| \leq M$ and $\|e_n^*\| \leq M$ for all $n \in \mathbb{N}$, fix $\varepsilon \in (0,1)$ and let $(\varepsilon_k)_k$ be any sequence of positive numbers such that $\sum_{k=1}^{\infty} \varepsilon_k = \varepsilon/2M$. For each $k \in \mathbb{N}$, the formula

$$\|(s,t)\|_k = \|tf_{2k-1} + sf_{2k}\|, \quad (s,t) \in \mathbb{R}^2,$$

defines a norm on $\mathbb{R}^2$, and bearing in mind that $\|f_n\| = 1$ for all $n \in \mathbb{N}$, from (2.1) we obtain

$$\frac{1}{C} \max\{|s|, |t|\} \leq \|(s,t)\|_k, \quad \text{for all} \quad (s,t) \in \mathbb{R}^2. \tag{2.2}$$

Applying Lemma 2.2 we get a continuously differentiable function $\varphi_k : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$\|\varphi_k(s,t)\|_k \leq \varepsilon_k \text{ for all } \quad (s,t) \in \mathbb{R}^2, \tag{2.3}$$

$$\varphi_k(s,t) = 0 \quad \text{whenever} \quad s < q_k, \tag{2.4}$$

$$\|\frac{\partial \varphi_k}{\partial s}(s,t)\|_k \leq \varepsilon_k \quad \text{for all} \quad (s,t) \in \mathbb{R}^2, \tag{2.5}$$

$$\|\frac{\partial \varphi_k}{\partial t}(s,t)\|_k \leq 1 \quad \text{for all} \quad (s,t) \in \mathbb{R}^2, \text{ and} \tag{2.6}$$

$$\|\frac{\partial \varphi_k}{\partial t}(s,t)\|_k = 1 \quad \text{whenever} \quad s \geq r_k. \tag{2.7}$$

Let us denote

$$z_k(x) = \left(e_{n_k}^*(x), e_k^*(x)\right), \quad x \in X,$$
$$i_k(s,t) = tf_{2k-1} + sf_{2k}, \quad (s,t) \in \mathbb{R}^2,$$

and

$$F_k = i_k \circ \varphi_k \circ z_k.$$

It is clear that F_k is a continuously differentiable function from X into Y, and because of (2.3) we have $\|F_k(x)\| \leq \varepsilon_k$ for all $x \in X$. Since the series $\sum_{k=1}^{\infty} \varepsilon_k$ converges, the formula

$$F(x) = \sum_{k=1}^{\infty} F_k(x), \quad x \in X,$$

defines a continuous, bounded function from X into Y.

Step 2: *F is Gâteaux differentiable and Lipschitz.* Let us fix $h \in X$. According to Lemma 2.3, it is enough to prove that the series $\sum_{k=1}^{\infty} F_k'(x)(h)$ converges uniformly with respect to x. An easy computation shows that

$$F_k'(x)(h) = e_k^*(h) i_k \left(\frac{\partial \varphi_k}{\partial t}(z_k(x))\right) + e_{n_k}^*(h) i_k \left(\frac{\partial \varphi_k}{\partial s}(z_k(x))\right) \tag{2.8}$$

for all $k \in \mathbb{N}$ and $x \in X$.

For the sake of brevity we write $\frac{\partial \varphi_k}{\partial t}(z_k(x)) = (a_{2k}(x), a_{2k-1}(x))$. By (2.6) and (2.2) we have $1 \geq \|(a_{2k}(x), a_{2k-1}(x))\|_k \geq \frac{1}{C} \max\{|a_{2k}(x)|, |a_{2k-1}(x)|\}$. Thus, $|a_k(x)| \leq C$ for every $x \in X$ and every $k \in \mathbb{N}$, i.e., $\|(a_k(x))_k\|_\infty \leq C$ for every $x \in X$. Then, thanks to (2.1), for every $h \in X$ and every $N \in \mathbb{N}$ we have

$$\sup_{x \in X} \left\| \sum_{n=N}^{\infty} e_n^*(h) \left(a_{2n}(x) f_{2n} + a_{2n-1}(x) f_{2n-1}\right) \right\| \leq C^2 \left\| \sum_{n=N}^{\infty} e_n^*(h) \left(f_{2n} + f_{2n-1}\right) \right\|. \tag{2.9}$$

Since, by assumption, the series $\sum_{n=1}^{\infty} e_n^*(h)\,(f_{2n}+f_{2n-1})$ is convergent, for every $h \in X$ we have

$$\sup_{x\in X}\left\|\sum_{n=N}^{\infty} e_n^*(h)\,(a_{2n}(x)f_{2n}+a_{2n-1}(x)f_{2n-1})\right\| \to 0$$

as $N \to \infty$. We thus have proved that for every $h \in X$ the series

$$\sum_{k=1}^{\infty} e_k^*(h) i_k \left(\frac{\partial \varphi_k}{\partial t}(z_k(x))\right)$$

converges uniformly for $x \in X$.

On the other hand, thanks to inequality (2.5) we have

$$\left\|e_{n_k}^*(h) i_k\left(\frac{\partial \varphi_k}{\partial s}\,(z_k(x))\right)\right\| \le \|e_{n_k}^*\|\|h\| \left\|\frac{\partial \varphi_k}{\partial s}(z_k(x))\right\|_k \le M\varepsilon_k\|h\|, \tag{2.10}$$

and hence the series $\sum_{k=1}^{\infty} e_{n_k}^*(h) i_k\left(\frac{\partial\varphi_k}{\partial s}(z_k(x))\right)$ converges uniformly with respect to x as well. Therefore, according to Lemma 2.3, F is Gâteaux differentiable at every point of X, and

$$F'(x)(h) = \sum_{k=1}^{\infty} F_k'(x)(h) \quad \text{whenever} \quad x, h \in X.$$

Bearing in mind that $|a_k(x)| \le C$ for all $k \in \mathbb{N}$, from inequalities (2.1) and (2.9) (with $N = 1$), for all $h \in X$ we have

$$\sup_{x\in X}\left\|\sum_{k=1}^{\infty} e_k^*(h) i_k\left(\frac{\partial\varphi_k}{\partial t}(z_k(x))\right)\right\| \le C^2\|L\|\|h\|,$$

where L is the bounded linear operator defined just after the statement of Theorem 2.1. This inequality and (2.10) imply

$$\sup_{x\in X}\|F'(x)\| \le C^2\|L\| + \varepsilon/2.$$

Thus, the function F is Lipschitz on X.

Step 3: *F has the jump property.* Take any two vectors $x, y \in X$ such that $x \ne y$. Pick $m \in \mathbb{N}$ such that $e_m^*(x) \ne e_m^*(y)$. We can assume that $e_m^*(x) < e_m^*(y)$. Find $q, r \in \mathbb{Q}$ so that $e_m^*(x) < q < r < e_m^*(y)$. Find a natural number k such that $n_k = m$, $q_k = q$ and $r_k = r$. Since $\varphi_k(s,t) = 0$ whenever $s < q_k$, according to (2.4) we have $\frac{\partial\varphi_k}{\partial t}(z_k(x)) = \frac{\partial\varphi_k}{\partial t}(e_{n_k}^*(x), e_k^*(x)) = 0$, and using (2.8) with $h = e_k$ we get $F_k'(x)(e_k) = 0$. On the other hand, from (2.7) we obtain $\|\frac{\partial\varphi_k}{\partial t}(e_{n_k}^*(y), e_k^*(y))\|_k = 1$. Consequently, $\|F_k'(y)(e_k)\| = 1$, and thus,

$$\|F_k'(x)(e_k) - F_k'(y)(e_k)\| = 1.$$

Moreover, from (2.8) and (2.5) it follows that

$$\|F_j'(x)(e_k)\| \le \varepsilon_k \text{ and } \|F_j'(y)(e_k)\| \le \varepsilon_k \text{ for all } j \ne k.$$

Therefore,

$$\|F'(x) - F'(y)\| \ge M^{-1}\|F'(x)(e_k) - F'(y)(e_k)\| \ge M^{-1}\left(1 - 2\sum_{j\ne k}\varepsilon_j\right) > M^{-1}(1-\varepsilon),$$

as we wanted to show. □

Remark 2.4. Under the assumptions of Theorem 2.1 it follows that for every $a = (a_n)_n \in \ell^\infty$ the formula

$$L_a(h) = \sum_{n=1}^{\infty} e_n^*(h)\,(a_{2n}f_{2n} + a_{2n-1}f_{2n-1})\,, \quad h \in X,$$

defines a bounded linear operator from X into Y. More precisely, there exist constants $C_1, C_2 > 0$ such that

$$C_1\|a\|_\infty \le \|L_a\| \le C_2\|a\|_\infty. \tag{2.11}$$

Indeed, thanks to inequality (2.1) we have

$$\|L_a(h)\| \le C\|a\|_\infty \left\|\sum_{n=1}^{\infty} e_n^*(h)\,(f_{2n} + f_{2n-1})\right\| \le C\|a\|_\infty\|L\|\|h\|$$

for every $h \in X$ and some constant $C > 0$. (Here, L denotes the bounded linear operator defined after the statement of Theorem 2.1.)

On the other hand (assuming that $\|f_n\| = 1$ for each $n \in \mathbb{N}$), if we denote $M = \sup\{\|e_1\|, \|e_2\|, \ldots\}$, from inequality (2.2) we get

$$\|L_a\| \ge \frac{1}{M}\|L_a(e_m)\| = \frac{1}{M}\|a_{2m-1}f_{2m-1} + a_{2m}f_{2m}\| \ge \frac{1}{MC}\max\{|a_{2m-1}|, |a_{2m}|\}$$

for every $m \in \mathbb{N}$. Hence, $\|L_a\| \ge \frac{1}{MC}\|a\|_\infty$ for every $a \in \ell^\infty$.

As a consequence of (2.11) it follows that the space ℓ^∞ embeds into $\mathcal{L}(X,Y)$. Thus, our theorem does not apply if the space $\mathcal{L}(X,Y)$ does not contain any isomorphic copy of ℓ^∞. It is not known whether the couple $(X, \mathbb{R}^2)$ has the jump property whenever X is a separable Banach space with a nonseparable dual. An affirmative answer was given in [9] whenever $X = \ell^1$, but the question remains open if X is the James tree space. In this case, $\mathcal{L}(X, \mathbb{R}^2)$ does not contain any isomorphic copy of ℓ^∞.

3. Application to classical Banach spaces

As a particular case of Theorem 2.1, we obtain the aforementioned result by Bayart in [5].

Corollary 3.1. *If X is a separable Banach space, then the couple (X, c_0) has the jump property. In particular, if K is an infinite metric compact space, then $(X, \mathcal{C}(K))$ has the jump property.*

Proof. It is well-known (see e.g. [12, Theorem 1.f.14]) that there exists a bounded, total biorthogonal system $(e_n, e_n^*)_n \subset X \times X^*$ such that $\|e_n^*\| = 1$ for all $n \in \mathbb{N}$, and $X = \overline{\text{span}(e_n)_n}$. In particular, for every $h \in X$ we have $\lim_n e_n^*(h) = 0$, that is, $(e_n^*(h))_n \in c_0$, and if we denote by $(f_n)_n$ the unit vector basis of c_0, then

$$\left\|\sum_{n=1}^{\infty} e_n^*(h) f_{2n-1}\right\| \le \|h\| \quad \text{and} \quad \left\|\sum_{n=1}^{\infty} e_n^*(h) f_{2n}\right\| \le \|h\|.$$

Since $(f_n)_n$ is unconditional and normalized, from Theorem 2.1 we deduce that the couple (X, c_0) has the jump property. The second statement follows from the fact that $\mathcal{C}(K)$ contains an isomorphic copy of c_0. $\square$

A Schauder basis of a Banach space is called *subsymmetric* (cf. [12, p. 114]) if it is unconditional and equivalent to any of its subsequences. It is well-known (see e.g. [12, Proposition 3.a.3]) that a Schauder basis $(e_n)_n$ is subsymmetric if it is *symmetric*; that is, if for any permutation $\pi : \mathbb{N} \to \mathbb{N}$, $(e_n)_n$ is equivalent to $(e_{\pi(n)})_n$ (cf. [12, p. 113]).

Corollary 3.2. *Let X be a Banach space with a bounded Schauder basis $(e_n)_n$, let Y be a Banach space, and assume there exists $U \in \mathcal{L}(X,Y)$ such that $(U(e_n))_n$ is a subsymmetric basic sequence in Y with $\inf_n \|U(e_n)\| > 0$. Then (X,Y) has the jump property.*

Proof. Observe that if $(e_n^*)_n$ is the sequence of biorthogonal functionals to $(e_n)_n$ and if we denote $f_n = U(e_n)$, then $U(h) = \sum_{n=1}^{\infty} e_n^*(h) f_n$ for all $h \in X$.

On the other hand, thanks to the subsymmetry of $(f_n)_n$, the sequences $(f_{2n})_n$ and $(f_{2n-1})_n$ are equivalent to $(f_n)_n$. Thus, the series $\sum_{n=1}^{\infty} e_n^*(h) f_{2n-1}$ and $\sum_{n=1}^{\infty} e_n^*(h) f_{2n}$ converge for all $h \in X$. Since the basic sequence $(f_n)_n$ is unconditional and satisfies $\inf_n \|f_n\| > 0$, we can apply Theorem 2.1. □

The former corollary implies that if X has a subsymmetric basis (in particular, if X has a symmetric basis), then the couple (X,X) satisfies the jump property.

Corollary 3.3. *If $\infty > q \geq p \geq 1$, then the couple (ℓ^p, ℓ^q) has the jump property. More generally, let M and N be two Orlicz functions such that $N(t) \leq k_1 M(k_2 t)$ for some constants $k_1, k_2 > 0$ and all t in a neighbourhood of zero. If h_M and h_N are the corresponding Orlicz sequence spaces, then the couple (h_M, h_N) has the jump property.*

Proof. It is clear that the unit vector basis of h_N is symmetric. On the other hand, using an argument as in [12, Proposition 4.a.5], from the inequality $N(t) \leq k_1 M(k_2 t)$ we deduce that the inclusion mapping $U : h_M \longrightarrow h_N$ is well-defined, linear and continuous. Thus, Corollary 3.2 applies. □

The first assertion in the former corollary, which is a consequence of the result on the Orlicz sequence spaces, is [9, Theorem 3]. There it was also shown that (ℓ^p, ℓ^q) does not have jump property when $p > q$, using the fact that in this case the space $\mathcal{L}(\ell^p, \ell^q)$ is separable.

Another consequence of Theorem 2.1 is the following result.

Corollary 3.4. *Let X be a Banach space with a bounded Schauder basis $(e_n)_n$, let Y be a Banach space, and assume there exists $U \in \mathcal{L}(X,Y)$ such that:*

(1) *$(U(e_n))_n$ is an unconditional basic sequence in Y with $\inf_n \|U(e_n)\| > 0$ and*
(2) *Y is isomorphic to $Y \oplus Y$.*

Then (X,Y) has the jump property.

Proof. Let P be an isomorphism from $Y \oplus Y$ onto Y, and let us define, for $h \in X$, $T(h) = P(U(h), 0)$, $S(h) = P(0, U(h))$, and for $n \in \mathbb{N}$, $f_{2n-1} = T(e_n)$ and $f_{2n} = S(e_n)$. Clearly, $(f_n)_n$ an unconditional basic sequence in Y (with $\inf_n \|f_n\| > 0$).

On the other hand, because of the continuity of T and S, for every $h \in X$ we have

$$T(h) = T\Big(\sum_{n=1}^{\infty} e_n^*(h) e_n\Big) = \sum_{n=1}^{\infty} e_n^*(h) f_{2n-1}$$

and

$$S(h) = \sum_{n=1}^{\infty} e_n^*(h) f_{2n}.$$

So, the above series converge and Theorem 2.1 applies. □

A particular case of the former corollary is the following result.

Corollary 3.5. *If X is a separable Banach space with an unconditional basis and X is isomorphic to $X \oplus X$, then the couple (X, X) has the jump property.*

This result applies when $X = L^p([0,1])$, for $1 < p < \infty$. Indeed, it is well-known that the space $L^p([0,1])$ is isomorphic to its square. On the other hand, if $p \in (1, \infty)$, then the Haar system constitutes an unconditional basis of $L^p([0,1])$ (cf. [13, Theorem 2.c.5]). Thus, the couple $(L^p([0,1]), L^p([0,1]))$ has the jump property if $1 < p < \infty$. As an easy consequence of this fact we get the following result.

Corollary 3.6. *If $2 \geq p \geq q \geq 1$ and $p \neq 1$, then the couple $(L^p([0,1]), L^q([0,1]))$ has the jump property.*

Proof. Since the space $L^q([0,1])$ contains an isomorphic copy of $L^p([0,1])$ (cf. [1, Prop. 11.1.9]) and $\big(L^p([0,1]), L^p([0,1])\big)$ has the jump property, the couple $(L^p([0,1]), L^q([0,1]))$ also satisfies this property. □

Remark 3.7. The couple $\big(L^2([0,1]), L^p([0,1])\big)$ has the jump property for every $p \geq 1$. This follows from the fact that $L^2([0,1])$ is isomorphic to a subspace of $L^p([0,1])$ (see e.g. [12, Theorem 2.b.3]). Notice also that for all $1 < p, q < \infty$, $\mathcal{L}\left(L^p([0,1]), L^q([0,1])\right)$ contains an isomorphic copy of ℓ^∞. This follows easily from the fact that the space generated by the Rademacher functions is complemented both in $L^p([0,1])$ and in $L^q([0,1])$. Therefore, unlike the case of ℓ^p spaces, it is not clear to us whether there exist couples (p, q) such that $(L^p([0,1]), L^q([0,1]))$ fails to have the jump property. In particular, we do not know if $\big(L^1([0,1]), L^1([0,1])\big)$ has this property.

We conclude this section with some more couples of Banach spaces satisfying the jump property and another one which fails this property.

Corollary 3.8. *If $\mathcal{T}$ is the Tsirelson's space, then the couples $(\mathcal{T}, \mathcal{T})$ and $(\mathcal{T}^*, \mathcal{T}^*)$ have the jump property. We notice that the space $\mathcal{T}$ does not have any subsymmetric Schauder basis.*

Proof. Let $(e_n)_n$ be the sequence of unit vectors of $\mathcal{T}$. According to [7, Propositions I.9 and I.12] it follows that $(e_n)_n$ is an unconditional basis of $\mathcal{T}$, which is equivalent to its subsequences $(e_{2n-1})_n$ and $(e_{2n})_n$. In particular, $\mathcal{T}$ is isomorphic to its square, and we can apply Corollary 3.5. A similar argument shows that the couple $(\mathcal{T}^*, \mathcal{T}^*)$ shares this property. □

Corollary 3.9. *If J is the James space, then the couple (J, ℓ^2), and hence (J, J), has the jump property. We observe that the space J does not have any unconditional Schauder basis.*

Proof. Let $(u_n)_n$ and $(f_n)_n$ be the canonical vector bases of J and ℓ^2, respectively. Let us denote by $(s_n)_n$ the summing basis associated to $(u_n)_n$. If $e_n^* = u_n^* - u_{n+1}^*$, then $(e_n^*)_n$ is the sequence of biorthogonal functionals associated to $(s_n)_n$.

Let us define, for each $n \in \mathbb{N}$, $U(s_n) = f_n$, and extend U by linearity. For each $x = (x_n)_n \in J$ with finite support we have

$$\|U(x)\|_2^2 = \sum_{n=1}^{\infty} (x_{n+1} - x_n)^2 \leq \|x\|_J^2.$$

Hence, U extends to a continuous linear operator from J to ℓ^2. As $(f_n)_n$ is a symmetric basis of ℓ^2, Corollary 3.2 ensures that the couple $\big(J, \ell^2\big)$ has the jump property. Since ℓ^2 is isomorphic to a subspace of J, the couple (J, J) also enjoys this property. □

Example 3.10. If X is the Banach space constructed by Argyros and Haydon in [2], then (X, X) fails the jump property.

Proof. This space is separable, has the property that every $T \in \mathcal{L}(X)$ is of the form $\lambda I + K$ where I is the identity operator and K is a compact operator, and satisfies that $\mathcal{L}(X)$ is separable. Hence (X, X) does not have the jump property. □

4. The set of functions satisfying the jump property

Let X and Y be Banach spaces and let $G(X, Y)$ denote the space of all bounded and Lipschitz functions from X to Y which are Gâteaux differentiable at each point of X, endowed with its natural norm

$$\|F\| := \sup\{\|F(x)\|;\, x \in X\} + \sup\{\|F'(x)\|;\, x \in X\}.$$

Let us also write $G_*(X, Y) := \{f \in G(X, Y);\, f \text{ has the jump property}\}$. The couple (X, Y) has the jump property if and only if $G_*(X, Y) \neq \emptyset$. Bayart [5] has shown that if X is a separable Banach space, then $G_*(X, c_0)$ is spaceable; i.e., $G_*(X, c_0) \cup \{0\}$ contains a closed infinite dimensional subspace of $G(X, c_0)$. We notice here the following result.

Proposition 4.1. *Let X be a Banach space with a bounded Schauder basis, and let Y be a Banach space. Assume that there exists $U \in \mathcal{L}(X, Y)$ such that $(U(e_n))_n$ is a subsymmetric basic sequence in Y with $\inf_n \|U(e_n)\| > 0$. Then $G_*(X, Y)$ is lineable; i.e., $G_*(X, Y) \cup \{0\}$ contains an infinite dimensional subspace of $G(X, Y)$.*

Proof. Let $\{I_p\}_{p\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into countably many infinite subsets, and for each $p \in \mathbb{N}$ put $Y_p = \overline{\operatorname{span}\,(U(e_n))_{n\in I_p}}$. Let $\pi_p : \mathbb{N} \longrightarrow I_p$ be an increasing bijection. Since the sequence $(U(e_n))_n$ is subsymmetric, it is equivalent to the sequence $(U(e_{\pi_p(n)}))_n$ which itself is subsymmetric. Therefore, for each p, if we define $U_p(e_n) = U(e_{\pi_p(n)})$, then $(U_p(e_n))_n$ is a subsymmetric basic sequence in Y_p such that $\inf_n \|U_p(e_n)\| > 0$. Applying Corollary 3.2 to the couple (X, Y_p) we get a Gâteaux differentiable function $F_p : X \to Y_p$ such that

$$\|F_p'(x) - F_p'(y)\|_{\mathcal{L}(X,Y_p)} \geq 1 \;\text{ whenever }\; x, y \in X \;\text{ and }\; x \neq y.$$

Notice that $\bigoplus_{p=1}^{\infty}\mathcal{L}(X,Y_p)=\mathcal{L}\left(X,\bigoplus_{p=1}^{\infty}Y_p\right)\subset\mathcal{L}(X,Y)$, so there exists a constant $c>0$ (coming from the unconditionality of $(U(e_n))_n$) such that, if $T_p\in\mathcal{L}(X,Y_p)$ and $T=\sum T_p\in\mathcal{L}(X,Y)$, then $\|T\|\geq c\sup\{\|T_p\|;\,p\in\mathbb{N}\}$. We claim that for each $N\in\mathbb{N}$ and each sequence of scalars $(\beta_1,\dots,\beta_N)$ which is not identically equal to zero, the function $F=\sum_{p=1}^{N}\beta_pF_p$ satisfies the jump property. Indeed, find p_0 such that $\beta_{p_0}\neq 0$. Now, for every $x,y\in X$ with $x\neq y$ we have $F_p'(x)-F_p'(y)\in\mathcal{L}(X,Y_p)$ for all p, and $F'(x)-F'(y)\in\bigoplus_{p=1}^{\infty}\mathcal{L}(X,Y_p)\subset\mathcal{L}(X,Y)$. Therefore,

$$\|F'(x)-F'(y)\|=\left\|\sum_{p=1}^{N}\beta_p\big(F_p'(x)-F_p'(y)\big)\right\|\geq c|\beta_{p_0}|\,\big\|F_{p_0}'(x)-F_{p_0}'(y)\big\|\geq c|\beta_{p_0}|,$$

as we wanted to show. □

We are unable to decide whether, under the assumptions of Proposition 4.1, the set $G_*(X,Y)$ is spaceable.

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