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ON LIFTING THE APPROXIMATION PROPERTY FROM A BANACH SPACE TO ITS DUAL

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Abstract. We prove that if a Banach space is extendably locally reflexive, then it is compactly locally reflexive. Using results by Lima and Lima, this implies that if a Banach space has the approximation property and is extendably locally reflexive, then its dual space has the approximation property.

1. Introduction

For a Banach space X, it is well known that the approximation property (AP) passes from the dual space X^* to X, but not from X to X^*. For Banach spaces which are *extendably locally reflexive* (ELR), Johnson and Oikhberg [2] proved that the bounded approximation property passes from X to X^*, and Oja [7] proved that the weak bounded approximation property passes from X to X^*. Below we show that also the approximation property passes from X to X^* for these spaces.

Let us set our notation and give some definitions. The space of all bounded operators from a Banach space X to a Banach space Y is denoted by $\mathcal{L}(X,Y)$. The subspaces of finite rank operators, compact operators, and weakly compact operators are denoted by $\mathcal{F}(X,Y)$, $\mathcal{K}(X,Y)$, and $\mathcal{W}(X,Y)$ respectively. Let I_X denote the identity operator on X.

We say that a Banach space X has the *approximation property* (AP) if for every compact set $K \subset X$ and every $\varepsilon > 0$ there exists a finite rank operator $T \in \mathcal{F}(X,X)$ such that $\|Tx - x\| < \varepsilon$ for all $x \in K$. If, in addition, there exists $\lambda \in [1,\infty)$ such that we can always choose $T \in \mathcal{F}(X,X)$ with $\|T\| \leq \lambda$, then we say that X has the *λ-bounded approximation property* (λ-BAP). In [5] yet another version of approximation property was introduced. If X is a Banach space and $\lambda \in [1,\infty)$, then we say that X has the *weak λ-bounded approximation property* (weak λ-BAP) if for every Banach space Y and every weakly compact operator $T \in \mathcal{W}(X,Y)$ there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \leq \lambda\|T\|$ such that $S_\alpha \to I_X$ uniformly on compact subsets of X.

In general we have

$$\lambda\text{-BAP} \quad \Rightarrow \quad \text{weak } \lambda\text{-BAP} \quad \Rightarrow \quad \text{AP}.$$

In [5, Corollary 3.3], Lima and Oja proved that if X is complemented in its bidual X^{**} by a projection P and X has the AP, then X has the weak $\|P\|$-BAP. In [6], Oja proved that if X^* has the Radon-Nikodým property and X has the weak λ-BAP,

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then X has the λ-BAP. (See [8] or [4, Theorems 1.3 and 1.4] and [1, Theorem VI.4.8] for alternative proofs.)

Following [9] and [2], we say that a Banach space X is λ-*extendably locally reflexive* (λ-ELR) if for every finite dimensional subspace $E \subset X^{**}$ and $F \subset X^*$, and for every $\varepsilon > 0$, there exists an operator $T : X^{**} \to X^{**}$ such that $T(E) \subset X$, $\|T\| \leq \lambda + \varepsilon$, and $\langle f, e\rangle = \langle f, Te\rangle$ for all $e \in E$ and $f \in F$. We say that X is *extendably locally reflexive* (ELR) if X is λ-ELR for some $\lambda \in [1, \infty)$. Let us cite some known results.

Theorem 1.1 (Johnson and Oikhberg)**.** *Let X be a Banach space.*

(1) *If X is λ-ELR and has the μ-BAP, then X^* has the $\lambda\mu$-BAP.*
(2) *If X^* has the λ-BAP, then X is λ-ELR.*

Theorem 1.2 (Oja)**.** *Let X be a Banach space. If X is ELR and has the weak λ-BAP for some $\lambda \in [1, \infty)$, then X^* has the AP.*

Following [3], we say that a Banach space X is *compactly locally reflexive* (CLR) if for every reflexive Banach space Y, we have ker V = ker W, where $V : X^*\hat{\otimes}_\pi Y \to \mathcal{K}(Y, X)^*$ and $W : X^*\hat{\otimes}_\pi Y \to \mathcal{K}(Y, X^{**})^*$ are the trace mappings.

Theorem 1.3 (Lima and Lima)**.** *Let X be a Banach space. The following statements are equivalent:*

(a) *X^* has the AP.*
(b) *X is CLR and has the AP.*

Let us present the new results here.

Theorem 1.4. *If X is ELR, then X is CLR.*

The proof will be presented in the next section.

Corollary 1.5. *Let X be a Banach space. If X is ELR and has the AP, then X^* has the AP.*

This answers a question of Oja [7].

Our notation is standard. X and Y denote Banach spaces. The dual space of X is denoted by X^* and its bidual space by X^{**}. The closed unit ball of X is denoted by B_X. The projective tensor product of X and Y is denoted by $X\hat{\otimes}_\pi Y$.

2. Proof of Theorem 1.4

Proof of Theorem 1.4. We assume that X is λ-ELR for some $\lambda \in [1, \infty)$. Let Y be a reflexive Banach space. We want to verify that ker V = ker W (see [3, Proposition 2.2 (c)]), where the trace mappings V and W are defined by

$$V : X^*\hat{\otimes}_\pi Y \to \mathcal{K}(Y, X)^*$$

and

$$W : X^*\hat{\otimes}_\pi Y \to \mathcal{K}(Y, X^{**})^*.$$

Clearly ker $W \subset$ ker V.

In order to prove that ker $V \subset$ ker W, let $u = \sum_n x_n^* \otimes y_n \in \ker V \subset X^*\hat{\otimes}_\pi Y$. To show that $u \in \ker W$, it suffices to show that if $T \in \mathcal{K}(Y, X^{**})$ and $\varepsilon > 0$, then $|\langle u, T\rangle| < \varepsilon$.

We want to use induction to find a sequence $(S_n)_{n=1}^{\infty} \subset \mathcal{L}(X^{**}, X^{**})$ such that if we define $T_1 = S_1 T$ and $T_{n+1} = S_{n+1} T_n$, we have

$$|\langle u, T\rangle - \langle u, T_n\rangle| < \varepsilon/2 + \varepsilon/4 + \ldots + \varepsilon/2^n, \tag{2.1}$$

$$T_n(B_Y) \subset X + B_{X^{**}}(0, 1/2^n), \tag{2.2}$$

and

$$\|T_{n+1} - T_n\| < 1/2^n. \tag{2.3}$$

From (2.3), we get that $(T_n)_{n=1}^{\infty} \subset \mathcal{K}(Y, X^{**})$ is a Cauchy sequence, so $T_\infty = \lim_n T_n$ exists. By (2.2) we get that $T_\infty \in \mathcal{K}(Y, X)$. By (2.1), using that $u \in \ker V$, it then follows that $|\langle u, T\rangle| = |\langle u, T\rangle - \langle u, T_\infty\rangle| < \varepsilon$.

Let us construct the sequence of operators $(S_n)_{n=1}^{\infty} \subset \mathcal{L}(X^{**}, X^{**})$.

We may assume $\|y_n\| = 1$ for all n and $\sum_n \|x_n^*\| < \infty$. First choose a strictly increasing sequence (m_n) of natural numbers such that

$$\sum_{k > m_n} \|x_k^*\| < \frac{\varepsilon}{2^n \|T\|(1 + (\lambda + \varepsilon)^n)}.$$

For each $n \geq 1$, define $F_n = \operatorname{span}(x_k^*)_{k=m_{n-1}+1}^{m_n} \subset X^*$ (we take $m_0 = 0$) and the numbers $\delta_n = \frac{1}{2^n(\lambda+\varepsilon)(\lambda+\varepsilon+1)}$.

$T(B_Y)$ is relatively compact, so we can find a finite set $(e_i^{**})_{i=1}^p$ such that for every $y \in B_Y$, there is some i such that $\|Ty - e_i^{**}\| < \delta_1$. Define a finite dimensional space

$$E_1 = \operatorname{span}((e_i^{**})_{i=1}^p, (Ty_k)_{k=1}^{m_1}) \subset X^{**}.$$

We can now find an operator $S_1 \in \mathcal{L}(X^{**}, X^{**})$ such that

$$S_1(E_1) \subset X, \tag{2.4}$$

$$\|S_1\| < \lambda + \varepsilon, \tag{2.5}$$

and

$$\langle x^{**}, x^*\rangle = \langle S_1 x^{**}, x^*\rangle, \quad \forall\, x^* \in F_1, \forall\, x^{**} \in E_1. \tag{2.6}$$

Define $T_1 = S_1 T \in \mathcal{K}(Y, X^{**})$. By the choice of m_1, (2.6), and (2.5), we get

$$\begin{aligned} |\langle u, T\rangle - \langle u, T_1\rangle| &\leq |\sum_{k \leq m_1} x_k^*(Ty_k - S_1 Ty_k)| + |\sum_{k > m_1} x_k^*(Ty_k - S_1 Ty_k)| \\ &\leq 0 + \sum_{k > m_1} \|x_k^*\| \|T\|(1 + \lambda + \varepsilon) < \varepsilon/2. \end{aligned} \tag{2.7}$$

Let $y \in B_Y$ and choose $e^{**} \in E_1$ such that $\|Ty - e^{**}\| < \delta_1$. Then $\|T_1 y - S_1 e^{**}\| < \delta_1(\lambda + \varepsilon) < 1/2$, and by (2.4) we get

$$T_1(B_Y) \subset X + B_{X^{**}}(0, 1/2). \tag{2.8}$$

Assume that operators $(S_k)_{k=1}^n \subset \mathcal{L}(X^{**}, X^{**})$ have been found and that $T_k = S_k T_{k-1}$ for $1 \leq k \leq n$ (take $T_0 = T$). Choose $(e_i^{**})_{i=1}^p \subset T_n(B_Y)$ such that for every $y \in B_Y$ there exists i such that $\|T_n y - e_i^{**}\| < \delta_{n+1}$. Define

$$E_{n+1} = \operatorname{span}((e_i^{**})_{i=1}^p, (T_n y_k)_{k=m_n+1}^{m_{n+1}}, S_n(E_n)) \subset X^{**}.$$

Now we can find an operator $S_{n+1} \in \mathcal{L}(X^{**}, X^{**})$ such that

$$S_{n+1}(E_{n+1}) \subset X, \tag{2.9}$$

$$\|S_{n+1}\| < \lambda + \varepsilon, \tag{2.10}$$

and

$$\langle x^{**}, x^* \rangle = \langle S_{n+1} x^{**}, x^* \rangle, \quad \forall\, x^* \in F_{n+1}, \forall\, x^{**} \in E_{n+1}. \tag{2.11}$$

Moreover, by Proposition 3.12 in [9], we can assume that

$$S_{n+1} x = x, \quad \forall\, x \in E_{n+1} \cap X. \tag{2.12}$$

Define $T_{n+1} = S_{n+1} T_n \in \mathcal{K}(Y, X^{**})$.

By construction we have $(Ty_k)_{k=1}^{m_1} \subset E_1$ and $(T_i y_k)_{k=m_i+1}^{m_{i+1}} \subset E_{i+1}$ for $1 \le i \le n$. Since $S_i(E_i) \subset E_{i+1}$ we get from (2.12) that

$$\sum_{k=m_i+1}^{m_{i+1}} x_k^*(T_{n+1} y_k) = \sum_{k=m_i+1}^{m_{i+1}} x_k^*(S_{n+1} S_n \dots S_{i+1} T_i y_k) = \sum_{k=m_i+1}^{m_{i+1}} x_k^*(S_{i+1} T_i y_k).$$

By using (2.11) we get

$$\sum_{k=m_i+1}^{m_{i+1}} x_k^*(S_{i+1} T_i y_k) = \sum_{k=m_i+1}^{m_{i+1}} x_k^*(T_i y_k).$$

It now follows that

$$\sum_{k=1}^{m_1} x_k^*(Ty_k - T_{n+1} y_k) = \sum_{k=1}^{m_1} x_k^*(Ty_k - S_1 T y_k) = \sum_{k=1}^{m_1} x_k^*(Ty_k - Ty_k) = 0,$$

and for $1 \le i \le n$,

$$\sum_{k=m_i+1}^{m_{i+1}} x_k^*(Ty_k - T_{n+1} y_k) = \sum_{k=m_i+1}^{m_{i+1}} x_k^*(Ty_k - T_i y_k).$$

But then we get

$$\begin{aligned} |\langle u, T \rangle - \langle u, T_{n+1} \rangle| &\le \sum_{i=1}^{n} \Big| \sum_{k=m_i+1}^{m_{i+1}} x_k^*(Ty_k - T_i y_k) \Big| + \Big| \sum_{k > m_{n+1}} x_k^*(Ty_k - T_{n+1} y_k) \Big| \\ &\le \sum_{i=1}^{n} \sum_{k=m_i+1}^{m_{i+1}} \|x_k^*\| \|T\| (1 + (\lambda + \varepsilon)^i) \\ &\quad + \sum_{k > m_{n+1}} \|x_k^*\| \|T\| (1 + (\lambda + \varepsilon)^{n+1}) \\ &\le \sum_{i=1}^{n+1} \varepsilon / 2^i. \end{aligned}$$

Let $y \in B_Y$ and choose $e^{**} \in E_{n+1}$ such that $\|T_n y - e^{**}\| < \delta_{n+1}$. Then we get by using (2.9) that

$$\|T_{n+1} y - S_{n+1} e^{**}\| \le \|S_{n+1}\| \|T_n y - e^{**}\| < 1/2^{n+1}.$$

Thus we get

$$T_{n+1}(B_Y) \subset X + B_{X^{**}}(0, 1/2^{n+1}).$$

If $y \in B_Y$, then there exists $e^{**} \in E_n$ such that $\|T_{n-1}y - e^{**}\| < \delta_n$. We get by using $S_{n+1}S_n e^{**} = S_n e^{**}$ and (2.10) that

$$\begin{aligned}\|T_{n+1}y - T_n y\| &\leq \|T_{n+1}y - S_{n+1}S_n e^{**}\| + \|S_{n+1}S_n e^{**} - T_n y\| \\ &\leq (1 + \|S_{n+1}\|)\|S_n e^{**} - T_n y\| \leq (1+\lambda+\varepsilon)\|S_n\|\|T_{n-1}y - e^{**}\| \\ &< (1+\lambda+\varepsilon)(\lambda+\varepsilon)\delta_n = 1/2^n.\end{aligned}$$

Hence, $\|T_{n+1} - T_n\| \leq 1/2^n$. □

References

[1] J. Diestel and J. J. Uhl Jr., *Vector measures.* With a foreword by B. J. Pettis, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977. MR0453964 (56 #12216)

[2] William B. Johnson and Timur Oikhberg, *Separable lifting property and extensions of local reflexivity*, Illinois J. Math. **45** (2001), no. 1, 123–137. MR1849989 (2002j:46014)

[3] Åsvald Lima and Vegard Lima, *Geometry of spaces of compact operators*, Ark. Mat. **46** (2008), no. 1, 113–142, DOI 10.1007/s11512-007-0060-y. MR2379687 (2009a:46038)

[4] Åsvald Lima, Vegard Lima, and Eve Oja, *Bounded approximation properties in terms of* $C[0,1]$, Math. Scand. **110** (2012), no. 1, 45–58. MR2900069

[5] Åsvald Lima and Eve Oja, *The weak metric approximation property*, Math. Ann. **333** (2005), no. 3, 471–484, DOI 10.1007/s00208-005-0656-0. MR2198796 (2006i:46025)

[6] Eve Oja, *The impact of the Radon-Nikodým property on the weak bounded approximation property* (English, with English and Spanish summaries), RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **100** (2006), no. 1-2, 325–331. MR2267414 (2007h:46027)

[7] Eve Oja, *Lifting of the approximation property from Banach spaces to their dual spaces*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3581–3587 (electronic), DOI 10.1090/S0002-9939-07-08996-4. MR2336573 (2008f:46025)

[8] Eve Oja, *Inner and outer inequalities with applications to approximation properties*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5827–5846, DOI 10.1090/S0002-9947-2011-05241-4. MR2817411 (2012d:46042)

[9] Timur Oikhberg and Haskell P. Rosenthal, *Extension properties for the space of compact operators*, J. Funct. Anal. **179** (2001), no. 2, 251–308, DOI 10.1006/jfan.2000.3674. MR1809112 (2002b:47138)

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