A LIOUVILLE-TYPE THEOREM ON HALF-SPACES FOR SUB-LAPLACIANS

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Abstract. Let $L$ be a sub-Laplacian on $\mathcal{L}^N$ and let $G = (\mathcal{L}^N, \circ, \delta_\lambda)$ be its related homogeneous Lie group. Let $E$ be a Euclidean subgroup of $\mathcal{L}^N$ such that the orthonormal projection $\pi : G \rightarrow E$ is a homomorphism of homogeneous groups, and let $\langle \ , \ \rangle$ be an inner product in $E$. Given $\alpha \in E$, $\alpha \neq 0$, define $\Omega(\alpha) := \{ x \in G : \langle \alpha, \pi(x) \rangle > 0 \}$. We prove the following Liouville-type theorem.

If $u$ is a nonnegative $L$-superharmonic function in $\Omega(\alpha)$ such that $u \in L^1(\Omega(\alpha))$, then $u \equiv 0$ in $\Omega(\alpha)$.

1. Introduction

In [14] F. Uguzzoni proved the following Liouville-type theorem.

Theorem A. Let $\Delta_{H_n}$ be a sub-Laplacian on the Heisenberg group $H_n$ and let $\Omega$ be a half-space of $H_n$ whose boundary is parallel to the center of $H_n$. If $u$ is a nonnegative $\Delta_{H_n}$-superharmonic function such that $u \in L^1(\Omega)$, then $u \equiv 0$.

The aim of this note is to show that an analogous result holds in the general setting of the sub-Laplacians on $\mathbb{R}^N$.

Let $L$ be a sub-Laplacian in $\mathbb{R}^N$ whose related homogeneous Lie group is $(G, \circ, \delta_\lambda)$. Let $E$ be an Euclidean subgroup of $\mathbb{R}^N$ such that the orthonormal projection $\pi : G \rightarrow E$ is a homomorphism of homogeneous Lie groups, i.e.,

$$\pi(x \circ y^{-1}) = \pi(x) - \pi(y), \quad \pi(\delta_\lambda(x)) = \lambda \pi(x),$$

for every $x, y \in G$ and every $\lambda > 0$.

Let $\langle \ , \ \rangle$ be an inner product in $E$ and, for every $\alpha \in E$, $\alpha \neq 0$, define $\Omega(\alpha) := \{ x \in G : \langle \alpha, \pi(x) \rangle > 0 \}$.

The main result of this paper is the following Liouville-type theorem.

Theorem 1.1. Let $u : \Omega(\alpha) \rightarrow [0, \infty]$ be a $L$-superharmonic function in $\Omega(\alpha)$. If $u \geq 0$ and $u \in L^1(\Omega(\alpha))$, then

$$u \equiv 0 \text{ in } \Omega(\alpha).$$
Liouville-type theorems in half-spaces for sub-Laplacian play a crucial role in looking for solutions to semilinear boundary value problems; see, e.g., [2], [1], [3], [7]. Liouville-type theorems in the whole space in a sub-Riemannian setting have received increasing attention in recent years; see, e.g., [4] (Section 5.8), [10], [11], [12], [13], the references therein, and the recent deep papers by D’Ambrosio and Mitidieri both for Riemannian and sub-Riemannian results ([8], [9]).

We would like to stress that to prove Theorem 1.1 we exploit a technique which is different with respect to the one used in the previous quoted papers. We follow the approach of Uguzzoni in [14] based on suitable mean value operators on the level set of the fundamental solution of

\[ L \]

and, moreover, a kind of invariance of \( \Omega(\alpha) \) with respect to suitable left translations of \( G \). For this last reason our method cannot work for half-spaces without this invariance property.

We would also like to stress that our result, in the case of the Heisenberg group \( H_n \), gives back the result of Uguzzoni. As already noticed in [14], the assumption \( u \in L^1(\Omega(\alpha)) \) cannot be improved in the following sense.

**Proposition 1.2.** Let \( p \in ]1, +\infty[ \) be fixed, and let \( G \) be a Lie group whose homogeneous dimension \( Q \) satisfies

\[
\frac{Q}{2} > \frac{p}{p-1}.
\]

Then for every \( \alpha \in \mathbb{E} \) there exists a strictly positive \( \Delta_G \)-harmonic function \( u \) in \( \Omega(\alpha) \) such that

\[
\int_{\Omega(\alpha)} u^p \, dx < +\infty.
\]

In particular this statement holds for the classical Laplacian \( \Delta \) in \( \mathbb{R}^N \) if \( N > \frac{p}{p-1} \).

In Remark 3.1 we will recognize also that the assumption \( u \geq 0 \) cannot be removed from Theorem 1.1.

We close this introduction by showing some explicit examples of applications of our Theorem 1.1.

**Example 1.3.** In \( \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n \), whose point is denoted by \((x, t), x \in \mathbb{R}^m, t \in \mathbb{R}^n\), consider the linear second order partial differential operator (PDO)

\[
L = \Delta_x + \frac{1}{4} |x|^2 \Delta_t + \sum_{k=1}^{n} \langle B^{(k)}x, \nabla x \rangle \partial_t^k,
\]

where \( \Delta_x = \sum_{j=1}^{m} \partial^2_{x_j} \) and \( \Delta_t = \sum_{j=1}^{n} \partial^2_{t_j} \) are the usual Laplace operator in \( \mathbb{R}^m \) and in \( \mathbb{R}^n \), respectively. \( \nabla x = (\partial_{x_1}, \ldots, \partial_{x_m}) \) and \( B^{(1)}, \ldots, B^{(m)} \) are \( m \times m \) matrices having the following properties:

(i) \( B^{(k)} \) is skew-symmetric and orthogonal, \( k = 1, \ldots, m; \)

(ii) \( B^{(i)}B^{(j)} = -B^{(j)}B^{(i)} \) for every \( i, j \in \{j = 1, \ldots, m\}, i \neq j. \)

Then \( L \) in (1.1) is a sub-Laplacian on a group of Heisenberg type \( \mathbb{H} \), and the map \( \pi : \mathbb{H} \rightarrow \mathbb{R}^m, \pi(x, t) = x \) is a homomorphism of homogeneous groups (see [6, Section 3.6]).

For every fixed \( \alpha \in \mathbb{R}^m, \alpha \neq 0, \)

\[
\Omega(\alpha) := \{ x \in \mathbb{G} : \langle \alpha, \pi(x) \rangle > 0 \},
\]

is a half-space to which our Liouville-type Theorem 1.1 applies.
Example 1.4. In $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, whose point is denoted by $(x, y, t)$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$, consider the linear second order PDO
\begin{equation}
L = \Delta_x + (x \cdot \nabla_y - \partial_t)^2.
\end{equation}
This operator is a sub-Laplacian on a group $\mathbb{K}$ named in [6] of Kolmogorov-type. Taking into account the composition law and the dilations on $\mathbb{K}$ defined in [6, Section 4.3.4], one immediately recognizes that the half-spaces to which our Liouville-type Theorem 1.1 applies are of the kind
\[ \{(x, y, t) \in \mathbb{R}^N : \langle \alpha, x \rangle + \beta t > 0\}, \]
where $|\alpha|^2 + \beta^2 > 0$.

Our paper is organized as follows.

The next section is devoted to the notation, definitions, and results needed in the note.

In section 3 we will prove Theorem 1.1, Proposition 1.2, and Remark 3.1.

2. Sub-Laplacians and related sub-harmonic functions

We call a sub-Laplacian on $\mathbb{R}^N$ any linear second order partial differential operator $L$ of the kind
\[ L = \sum_{j=1}^{m} X_j^2 \]
where the $X_j$'s are smooth vector fields (i.e., linear partial differential operator of order one and smooth coefficients) satisfying the following conditions:

(H1) the Lie algebra
\[ a := \text{Lie}\{X_1, \ldots, X_m\} \]
is a vector space of dimension $N$; moreover,
\[ \text{rank } a(x) = N \text{ at any point } x \in \mathbb{R}^N; \]

(H2) there exists a group of dilations $(\delta_{\lambda})_{\lambda>0}$ in $\mathbb{R}^N$ such that every $X_j$ is $\delta_{\lambda}$-homogeneous of degree one.

A group of dilations in $\mathbb{R}^N$ is a family of diagonal linear functions $(\delta_{\lambda})_{\lambda>0}$ of the kind
\[ \delta_{\lambda}(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N \]
where $\sigma_1 = 1 \leq \sigma_2 \leq \cdots \leq \sigma_N$, $\sigma_j \in \mathbb{N}$.

Condition (H1) implies the hypoellipticity of $L$: in particular, the $L$-harmonic functions, i.e., the solution to $Lu = 0$, are smooth. Moreover, conditions (H1) and (H2) imply the existence of a group law $\circ$ in $\mathbb{R}^N$ such that $G = (\mathbb{R}^N, \circ, \delta_{\lambda})$ is a homogeneous Lie group on which the vector fields $X_j$'s are left translation invariant and $\delta_{\lambda}$-homogeneous of degree one (see [4]). The natural number
\[ Q = \sigma_1 + \ldots + \sigma_N \]
is called the homogeneous dimension of $G$. Throughout the paper we always assume $Q \geq 3$ (if $Q = 2$, then $G$ is the Euclidean group). Then there exists a
continuous function \( d : \mathbb{G} \rightarrow \mathbb{R} \), smooth and strictly positive outside the origin, \( \delta_\lambda \)-homogeneous of degree one and such that
\[
\gamma(x) := \left( \frac{1}{d(x)} \right)^{Q-2}
\]
is \( \mathcal{L} \)-harmonic in \( \mathbb{R}^N \setminus \{0\} \) (see [6, Section 5.4]). This function \( d \) is called an \( \mathcal{L} \)-gauge and for \( \mathcal{L} \) plays a role analogous to the one played by the Euclidean norm with respect to the classical Laplacian. In particular, the \( d \)-balls
\[
B_d(x, r) := \{ y \in \mathbb{G} : d(x^{-1} \circ y) < r \}
\]
support averaging operators characterizing the \( \mathcal{L} \)-harmonicity. To be precise, define
\[
\psi := |\nabla \mathcal{L}d|^2, \quad \nabla \mathcal{L} = (X_1, \ldots, X_m),
\]

\[
M_r u(x) := \frac{1}{c_d r^Q} \int_{B_d(x, r)} \psi(x^{-1} \circ y) u(y) \, dy
\]
and
\[
N_r (\mathcal{L} u)(x) = \frac{1}{(Q-2)c_d r^Q} \int_0^r \rho^{Q-1} \left( \int_{B_d(x, \rho)} \mathcal{L} u(y) \left( d(x^{-1} \circ y)^{2-Q} - \rho^{2-Q} \right) dy \right) \, d\rho
\]
where \( c_d = \int_{B_d(0,1)} \psi \, dy \).

Then, if \( \Omega \) is an open subset of \( \mathbb{G} \), \( u \in C^2(\Omega) \) and \( \overline{B_d(x, r)} \subseteq \Omega \),
\[
(2.1) \quad u(x) = M_r u(x) - N_r (\mathcal{L} u)(x)
\]
(see [6, Theorem 5.6.1]).

We stress that \( \psi \) is smooth outside the origin, \( \delta_\lambda \)-homogeneous of degree zero, and nonconstant unless \( \mathbb{G} \) is the Euclidean group (see [5]; see also [6, Proposition 9.8.9]). In some particular important cases, such as, e.g., the group of Heisenberg type, explicit expressions of \( \psi \) are known (see [6, Example 5.5.3]). In any case it is known that \( \psi > 0 \) in a dense open subset of \( \mathbb{R}^N \) (see [6, page 262]).

With these mean value operators, one can prove a version of the Gauss-Koebe Theorem in our setting (see [6, Section 5.6]):

**Theorem 2.1** (Gauss-Koebe-type Theorem). If \( \Omega \subseteq \mathbb{R}^N \) is open and \( u : \Omega \rightarrow \mathbb{R} \) is \( \mathcal{L} \)-harmonic, then
\[
(2.2) \quad u(x) = M_r u(x)
\]
for every \( x \in \Omega \) and \( r > 0 \) such that \( \overline{B_d(x, r)} \subseteq \Omega \).

Vice versa, if \( u \) is merely continuous in \( \Omega \) and satisfies (2.2), then \( u \) is \( C^\infty \) and \( \mathcal{L} \)-harmonic in \( \Omega \).

The average operator \( M_r \) can also be used to fix the notion of \( \mathcal{L} \)-superharmonic function.

A lower semicontinuous function \( u : \Omega \rightarrow ]-\infty, \infty[ \) is called \( \mathcal{L} \)-superharmonic if \( u \) is finite in a dense subset of \( \Omega \) and
\[
u \geq M_r u(x)
\]
for every \( x \in \Omega \) and \( r > 0 \) such that \( \overline{B_d(x, r)} \subseteq \Omega \).

A quite exhaustive theory of \( \mathcal{L} \)-subharmonic functions is presented in the monograph [6, Chapter 8]. In particular, there it is proved that every \( \mathcal{L} \)-subharmonic
function is $L^1_{\text{loc}}$ and that if $u$ is of class $C^2$, then $u$ is $L$-subharmonic if and only if $L u \geq 0$.

3. Proof of Theorem 1.1, Proposition 1.2 and Remark 3.1

The most important part of this section is the

Proof of Theorem 1.1. Let $\alpha \in \mathbb{E}$, $\alpha \neq 0$, be fixed and let

$$\Omega(\alpha) := \{ x \in \mathbb{G} : \langle \alpha, \pi(x) \rangle > 0 \}.$$ 

For every $x \in \Omega(\alpha)$ we define

$$r(x) := \varepsilon \langle \alpha, \pi(x) \rangle,$$

where $\varepsilon > 0$ is fixed in such a way that

$$B(x, r(x)) \subseteq \Omega(\alpha) \quad \forall x \in \Omega(\alpha). \quad (3.1)$$

We will show in a moment the existence of a suitable $\varepsilon > 0$ satisfying (3.1).

For a function $u \in L^1_{\text{loc}}(\Omega(\alpha))$ we let

$$T(u) : \Omega(\alpha) \rightarrow \mathbb{R}, \quad T(u)(x) := M_{r(x)}(u)(x).$$

Hence,

$$T(u)(x) = \int_{\Omega(\alpha)} K(x, y) u(y) \, dy, \quad x \in \Omega(\alpha),$$

where

$$K(x, y) = \frac{1}{c_d(r(x))^q} \psi(x^{-1} \circ y) \chi_{B_x}(y). \quad (3.2)$$

In what follows we also use the following notation:

$$A_x := \{ y \in \Omega(\alpha) \mid d(y^{-1} \circ x) < r(y) \}.$$ 

With this notation, we have

$$\chi_{B_x}(y) = \chi_{A_y}(x).$$

Indeed

$$y \in B_x \iff d(x^{-1} \circ y) < r(x) \iff x \in A_y.$$ 

Let us now show (3.1). We first remark that $\mathbb{E} \ni e \mapsto d(e) \in \mathbb{R}$ is homogeneous of degree one with respect to the Euclidean dilation $e \mapsto \lambda e$. As a consequence, by a suitable constant $c > 0$, we have

$$d(e) \geq c |e| \quad \forall e \in \mathbb{E}, \quad |\cdot| = \text{Euclidean norm}.$$ 

Moreover, we can also assume that

$$d(x) \geq c|\pi(x)| \quad \forall x \in \mathbb{G}.$$ 

Then, if $x \in \Omega(\alpha)$, for every $z \in B_d(x, r(x))$, we have $r(x) > d(z, x) \geq c|\pi(z) - \pi(x)|$.

Hence

$$\langle \alpha, \pi(z) \rangle = \langle \alpha, \pi(x) \rangle + \langle \alpha, \pi(z) - \pi(x) \rangle \geq \langle \alpha, \pi(x) \rangle - |\alpha||\pi(z) - \pi(x)|$$

$$\geq \langle \alpha, \pi(x) \rangle - \frac{|\alpha|}{c} r(x) = \langle \alpha, \pi(x) \rangle \left( 1 - \frac{|\alpha|}{c} \varepsilon \right).$$

Thus, if $0 < \varepsilon < \frac{c}{|\alpha|}$, we get $\langle \alpha, \pi(z) \rangle > 0$; i.e., $z \in \Omega(\alpha)$ and (3.1) is proved.

The proof of Theorem 1.1 will immediately follow from the next lemma.
Main Lemma.

(i) $K(x,y) \geq 0$ for every $x, y \in \Omega(\alpha)$;
(ii) $\int_{\Omega(\alpha)} K(x,y) \, dy = 1$ for every $x \in \Omega(\alpha)$;
(iii) $\int_{\Omega(\alpha)} K(x,y) \, dx = \int_{\Omega(\alpha)} K(x,\alpha) \, dx$ for every $y \in \Omega(\alpha)$;
(iv) $c^* := \int_{\Omega(\alpha)} K(x,\alpha) \, dx > 1$.

Proof of the Main Lemma.

(i) It straightforwardly follows from (3.2).
(ii) By the Gauss-Koebe-type Theorem 2.1 for $L$-harmonic functions, if $u$ is $L$-harmonic in $\Omega(\alpha)$, then $T(u) = u$. In particular $T(1) = 1$, that is, $1 = \int_{\Omega(\alpha)} K(x,y) \, dy$ for every $x \in \Omega(\alpha)$.

(iii) This is the crucial part of the Main Lemma. We start by proving the following property of $\Omega(\alpha)$:

\[ \delta_\lambda(\alpha) \circ y^{-1} \circ x \in \Omega(\alpha) \text{ and } r(\delta_\lambda(\alpha) \circ y^{-1} \circ x) = r(x) \]

for every $x \in \Omega(\alpha)$.

Indeed, let $y,x \in \Omega(\alpha)$ and consider

\[
\langle \alpha, \pi(\delta_\lambda(x) \circ y^{-1} \circ x) \rangle = \langle \alpha, \pi(\delta_\lambda(x)) \rangle + \langle \alpha, \pi(y^{-1}) \rangle + \langle \alpha, \pi(x) \rangle = \langle \alpha, \pi(x) \rangle + \lambda \langle \alpha, \alpha \rangle - \langle \alpha, \pi(y) \rangle.
\]

Then, if we choose $\lambda = \frac{\langle \alpha, \pi(y) \rangle}{|\alpha|^2}$ we have $\lambda > 0$ and

\[
\langle \alpha, \pi(\delta_\lambda(x) \circ y^{-1} \circ x) \rangle > 0, \quad r(\delta_\lambda(x) \circ y^{-1} \circ x) = r(x).
\]

This completes the proof of the stated property of $\Omega(\alpha)$.

In what follows we also use a homogeneity property of $x \mapsto r(x)$, precisely

\[
r(\delta_\lambda(x)) = \lambda r(x) \text{ for every } x \in \Omega(\alpha) \text{ and } \lambda > 0.
\]

Indeed

\[
r(\delta_\lambda(x)) = \varepsilon \langle \alpha, \pi(\delta_\lambda(x)) \rangle = \varepsilon \langle \alpha, \lambda \pi(x) \rangle = \lambda r(x).
\]
Let us now fix \( y \in \Omega(\alpha) \) and compute

\[
\int_{\Omega(\alpha)} K(x, \alpha) \, dx = \frac{1}{c_d} \int_{\Omega(\alpha)} \left( \frac{1}{r(x)} \right)^Q \psi \left( x^{-1} \circ \alpha \right) \chi_{B(x)}(\alpha) \, dx
\]

(letting \( \hat{\psi}(z) = \psi(z^{-1}) \))

\[
= \frac{1}{c_d} \int_{A_{\alpha}} \left( \frac{1}{r(x)} \right)^Q \hat{\psi}(\alpha^{-1} \circ x) \chi_{A_{\alpha}}(x) \, dx
\]

(\( Q = 1 \) and noticing that \( r (\delta_{\frac{1}{x}}(\xi)) = \frac{1}{\lambda} r(\xi) \) and that \( dx = \theta^{-Q} d\xi \))

\[
= \frac{1}{c_d} \int_{\delta_{\lambda}(A_{\alpha})} \left( \frac{1}{r(x)} \right)^Q \psi^{-1} (\delta_{\lambda}(\alpha^{-1}) \circ \xi) \, d\xi.
\]

Using the change of variables \( x, y \) and noticing that \( \psi \) is \( \delta_{\lambda} \)-homogeneous of degree zero,

We now choose \( \lambda = \lambda(y) > 0 \) such that \( r(\delta_{\lambda}(\alpha) \circ y^{-1} \circ x) = r(x) \) for every \( x \in \Omega(\alpha) \) and use the change of variable

\[
\xi = \delta_{\lambda}(\alpha) \circ y^{-1} \circ x.
\]

We obtain

\[
\int_{\Omega(\alpha)} K(x, \alpha) \, dx = \frac{1}{c_d} \int_{y \circ \delta_{\lambda}(\alpha^{-1}) \circ \delta_{\lambda}(A_{\alpha})} \left( \frac{1}{r(x)} \right)^Q \psi^{-1}(y^{-1} \circ x) \, dx.
\]

On the other hand, as we will recognize in a moment,

\[
(3.3) \quad y \circ \delta_{\lambda}(\alpha^{-1}) \circ \delta_{\lambda}(A_{\alpha}) = A_{y}.
\]

Then

\[
\int_{\Omega(\alpha)} K(x, \alpha) \, dx = \frac{1}{c_d} \int_{A_{y}} \left( \frac{1}{r(x)} \right)^Q \psi^{-1}(y^{-1} \circ x) \, dx = \int_{\Omega(\alpha)} K(x, y) \, dx,
\]

and (iii) is proved.

We are left to prove (3.3). One has

\[
x \in y \circ \delta_{\lambda}(\alpha^{-1}) \circ \delta_{\lambda}(A_{\alpha}) \iff z := \alpha \circ \delta_{\frac{1}{x}}(y^{-1} \circ x) \in A_{\alpha} \iff d(z, \alpha) < r(z).
\]

We know that \( r(z) = \frac{1}{\lambda} r(\delta_{\lambda}(\alpha) \circ y^{-1} \circ x) = \frac{1}{\lambda} r(x) \), while

\[
d(z, \alpha) = d(\alpha^{-1} \circ z) = d(\delta_{\frac{1}{x}}(y^{-1} \circ x)) = \frac{1}{\lambda} d(y^{-1}, x).
\]

We have thus proved that

\[
x \in y \circ \delta_{\lambda}(\alpha^{-1}) \circ \delta_{\lambda}(A_{\alpha}) \iff d(y^{-1} \circ x) < r(x) \iff x \in A_{y}.
\]

This completes the proof of (iii).

(iv) Let \( x_0 \in \mathbb{G} \setminus \Omega(\alpha) \) and consider the function

\[
v : \Omega(\alpha) \to \mathbb{R}, \quad v(x) = (d(x^{-1}_0 \circ x))^{-1-Q}.
\]
Obviously the function $v$ is smooth in $\Omega(\alpha)$. Moreover, $v > 0$ and $v \in L^1(\Omega(\alpha))$. By using the left invariance of $\mathcal{L}$ on $G$ and the form of $\mathcal{L}$ for radial functions \footnote{If $w = f(d)$, then $\mathcal{L}(w) = \psi(f''(d) + \frac{Q-1}{d}f'(d))$ (see [6, Proposition 5.4.3]).}, we also have

\[
(\mathcal{L}v)(x) = (\mathcal{L}d^{-Q-1})(x_0^{-1} \circ x) = \psi(x_0^{-1} \circ x)((Q+1)(Q+2) - (Q+1)(Q-1))d^{-Q-3}(x_0^{-1} \circ x) = 3(Q+1)(\psi d^{-Q-3})(x_0^{-1} \circ x).
\]

Then $\mathcal{L}v > 0$ in a dense open set of $\Omega(\alpha)$. As a consequence, using the representation formula (2.1), we get

\[
T(v)(x) - v(x) > 0 \quad \forall x \in \Omega(\alpha),
\]

that is, $T(v) > v$ in $\Omega(\alpha)$. It follows that

\[
\int_{\Omega(\alpha)} v \, dx < \int_{\Omega(\alpha)} T(v) \, dx = \int_{\Omega(\alpha)} \left( \int_{\Omega(\alpha)} K(x,y)v(y) \, dy \right) \, dx = \int_{\Omega(\alpha)} v(y) \left( \int_{\Omega(\alpha)} K(x,y) \, dx \right) \, dy = c^* \int_{\Omega(\alpha)} v(y) \, dy.
\]

Then

\[
\int_{\Omega(\alpha)} v \, dx < c^* \int_{\Omega(\alpha)} v \, dy,
\]

which implies $c^* > 1$, since $\int_{\Omega(\alpha)} v \, dx > 0$. This completes the proof of the Main Lemma. \hfill \Box

We can now conclude the proof of Theorem 1.1. Since $u$ is $\mathcal{L}$-superharmonic, we have $T(u) \leq u$ in $\Omega(\alpha)$. Therefore,

\[
\int_{\Omega(\alpha)} u \, dx \geq \int_{\Omega(\alpha)} T(u) \, dx \quad \text{(as in the proof of the Main Lemma (iv))} = c^* \int_{\Omega(\alpha)} u \, dx.
\]

Then, since $c^* > 1$,

\[
\int_{\Omega(\alpha)} u \, dx \leq 0,
\]

which implies $u \equiv 0$ since $u \geq 0$ and lower semicontinuous. \hfill \Box

Proof of Proposition 1.2 Let $d$ be a gauge function for $\mathcal{L}$ and define

\[
u(x) = (d(x_0^{-1} \circ x))^{-Q+2}, \quad x \in \Omega(\alpha),
\]

where, as before, $x_0 \notin \overline{\Omega(\alpha)}$. The function $\nu$ is smooth in $\Omega(\alpha)$ and

\[
\mathcal{L}(\nu)(x) = (\mathcal{L}d^{2-Q})(x_0^{-1} \circ x) = 0, \quad x \in \Omega(\alpha).
\]

Moreover, $\nu > 0$ and $\nu \in L^p(\Omega(\alpha))$ since, from the assumption $\frac{Q}{2} > \frac{p}{p-1}$, it follows that

\[
p(Q-2) > Q.
\]

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Remark 3.1. The assumption $u \geq 0$ in Theorem 1.1 cannot be removed.

Indeed, if $x_0 \notin \Omega(\alpha)$, the function 

$$u_k(x) := \partial_{x_N}^k \left( d(x_0^{-1} \circ x) \right)^2 - Q$$

is $\mathcal{L}$-harmonic in $\Omega(\alpha)$ for every $k \in \mathbb{N}$, and $\delta_\lambda$-homogeneous of degree $2 - Q - k\sigma_N$.

Then, if $k > \frac{2}{\sigma_N}$, $u_k \in L^1(\Omega(\alpha))$. Thus, with this choice of $k$, $u_k$ is a summable $\mathcal{L}$-harmonic function in $\Omega(\alpha)$ and $u_k \neq 0$.

We would like to stress that in the previous argument we used the following properties:

(i) the differential operator $\partial_{x_N}$ is $\delta_\lambda$-homogeneous of degree $\sigma_N$ and commutes with $\mathcal{L}$;

(ii) $\mathcal{L}$ is left translation invariant with respect to the composition law $\circ$;

(iii) $d^2 - Q$ is $\mathcal{L}$-harmonic out of the origin.

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