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A LIOUVILLE-TYPE THEOREM ON HALF-SPACES FOR SUB-LAPLACIANS

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Abstract. Let $\mathcal{L}$ be a sub-Laplacian on $\mathcal{L}^N$ and let $\mathbb{G} = (\mathcal{L}^N, \circ, \delta_\lambda)$ be its related homogeneous Lie group. Let $\mathbb{E}$ be a Euclidean subgroup of $\mathcal{L}^N$ such that the orthonormal projection $\pi : \mathbb{G} \longrightarrow \mathbb{E}$ is a homomorphism of homogeneous groups, and let $\langle\ ,\ \rangle$ be an inner product in $\mathbb{E}$. Given $\alpha \in \mathbb{E}$, $\alpha \neq 0$, define $\Omega(\alpha) := \{x \in \mathbb{G}\ :\ \langle \alpha, \pi(x)\rangle > 0\}$. We prove the following Liouville-type theorem.

If u is a nonnegative $\mathcal{L}$-superharmonic function in $\Omega(\alpha)$ such that $u \in L^1(\Omega(\alpha))$, then $u \equiv 0$ in $\Omega(\alpha)$.

1. Introduction

In [14] F. Uguzzoni proved the following Liouville-type theorem.

Theorem A. *Let $\Delta_{\mathbb{H}_n}$ be a sub-Laplacian on the Heisenberg group $\mathbb{H}_n$ and let Ω be a half-space of $\mathbb{H}_n$ whose boundary is parallel to the center of $\mathbb{H}_n$. If u is a nonnegative $\Delta_{\mathbb{H}_n}$-superharmonic function such that $u \in L^1(\Omega)$, then $u \equiv 0$.*

The aim of this note is to show that an analogous result holds in the general setting of the sub-Laplacians on $\mathbb{R}^N$.

Let $\mathcal{L}$ be a sub-Laplacian in $\mathbb{R}^N$ whose related homogeneous Lie group is $(\mathbb{G}, \circ, \delta_\lambda)$. Let $\mathbb{E}$ be an Euclidean subgroup of $\mathbb{R}^N$ such that the orthonormal projection

$$\pi : \mathbb{G} \longrightarrow \mathbb{E}$$

is a homomorphism of homogeneous Lie groups, i.e.,

$$\pi(x \circ y^{-1}) = \pi(x) - \pi(y), \quad \pi(\delta_\lambda(x)) = \lambda \pi(x),$$

for every $x, y \in \mathbb{G}$ and every $\lambda > 0$.

Let $\langle\ ,\ \rangle$ be an inner product in $\mathbb{E}$ and, for every $\alpha \in \mathbb{E}$, $\alpha \neq 0$, define

$$\Omega(\alpha) := \{x \in \mathbb{G}\ :\ \langle \alpha, \pi(x)\rangle > 0\}.$$

The main result of this paper is the following Liouville-type theorem.

Theorem 1.1. *Let $u : \Omega(\alpha) \longrightarrow]-\infty, \infty]$ be a $\mathcal{L}$-superharmonic function in $\Omega(\alpha)$. If $u \geq 0$ and $u \in L^1(\Omega(\alpha))$, then*

$$u \equiv 0 \text{ in } \Omega(\alpha).$$

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Liouville-type theorems in half-spaces for sub-Laplacian play a crucial role in looking for solutions to semilinear boundary value problems; see, e.g., [2], [1], [3], [7]. Liouville-type theorems in the whole space in a sub-Riemannian setting have received increasing attention in recent years; see, e.g., [4] (Section 5.8), [10], [11], [12], [13], the references therein, and the recent deep papers by D'Ambrosio and Mitidieri both for Riemannian and sub-Riemannian results ([8], [9]).

We would like to stress that to prove Theorem 1.1 we exploit a technique which is different with respect to the one used in the previous quoted papers. We follow the approach of Uguzzoni in [14] based on suitable mean value operators on the level set of the fundamental solution of $\mathcal{L}$ and, moreover, a kind of invariance of $\Omega(\alpha)$ with respect to suitable left translations of $\mathbb{G}$. For this last reason our method cannot work for half-spaces without this *invariance property*.

We would also like to stress that our result, in the case of the Heisenberg group $\mathbb{H}_n$, gives back the result of Uguzzoni. As already noticed in [14], the assumption $u \in L^1(\Omega(\alpha))$ cannot be improved in the following sense.

Proposition 1.2. *Let $p \in]1, +\infty[$ be fixed, and let $\mathbb{G}$ be a Lie group whose homogeneous dimension Q satisfies*

$$\frac{Q}{2} > \frac{p}{p-1}.$$

Then for every $\alpha \in \mathbb{E}$ there exists a strictly positive $\Delta_{\mathbb{G}}$-harmonic function u in $\Omega(\alpha)$ such that

$$\int_{\Omega(\alpha)} u^p \ dx < +\infty.$$

In particular this statement holds for the classical Laplacian Δ in $\mathbb{R}^N$ if $\frac{N}{2} > \frac{p}{p-1}$.

In Remark 3.1 we will recognize also that the assumption $u \geq 0$ cannot be removed from Theorem 1.1.

We close this introduction by showing some explicit examples of applications of our Theorem 1.1.

Example 1.3. In $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$, whose point is denoted by $(x, t), x \in \mathbb{R}^m, t \in \mathbb{R}^n$, consider the linear second order partial differential operator (PDO)

$$\mathcal{L} = \Delta_x + \frac{1}{4}|x|^2\Delta_t + \sum_{k=1}^{n} \langle B^{(k)}x, \nabla_x \rangle \partial_{t_k}, \tag{1.1}$$

where $\Delta_x = \sum_{j=1}^m \partial^2_{x_j}$ and $\Delta_t = \sum_{j=1}^n \partial^2_{t_j}$ are the usual Laplace operator in $\mathbb{R}^m$ and in $\mathbb{R}^n$, respectively. $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_m})$ and $B^{(1)}, \ldots, B^{(m)}$ are $m \times m$ matrices having the following properties:

(i) $B^{(k)}$ is skew-symmetric and orthogonal, $k = 1, \ldots, m$;
(ii) $B^{(i)}B^{(j)} = -B^{(j)}B^{(i)}$ for every $i, j \in \{j = 1, \ldots, m\}, i \neq j$.

Then $\mathcal{L}$ in (3.1) is a sub-Laplacian on a *group of Heisenberg type* $\mathbb{H}$, and the map $\pi : \mathbb{H} \longrightarrow \mathbb{R}^m, \pi(x, t) = x$ is a homomorphism of homogeneous groups (see [6, Section 3.6]).

For every fixed $\alpha \in \mathbb{R}^m$, $\alpha \neq 0$,

$$\Omega(\alpha) := \{x \in \mathbb{G} \ : \ \langle \alpha, \pi(x) \rangle > 0\},$$

is a half-space to which our Liouville-type Theorem 1.1 applies.

Example 1.4. In $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, whose point is denoted by (x, y, t), x,$y \in \mathbb{R}^n$, $t \in \mathbb{R}$, consider the linear second order PDO

$$\mathcal{L} = \Delta_x + (x \cdot \nabla_y - \partial_t)^2. \tag{1.2}$$

This operator is a sub-Laplacian on a group $\mathbb{K}$ named in [6] of Kolmogorov-type. Taking into account the composition law and the dilations on $\mathbb{K}$ defined in [6, Section 4.3.4], one immediately recognizes that the half-spaces to which our Liouville-type Theorem 1.1 applies are of the kind

$$\{(x, y, t) \in \mathbb{R}^N \ : \ \langle \alpha, x\rangle + \beta t > 0\},$$

where $|\alpha|^2 + \beta^2 > 0$.

Our paper is organized as follows.

The next section is devoted to the notation, definitions, and results needed in the note.

In section 3 we will prove Theorem 1.1, Proposition 1.2, and Remark 3.1.

2. Sub-Laplacians and related sub-harmonic functions

We call a sub-Laplacian on $\mathbb{R}^N$ any linear second order partial differential operator $\mathcal{L}$ of the kind

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2$$

where the X_j's are smooth vector fields (i.e. linear partial differential operator of order one and smooth coefficients) satisfying the following conditions:

(H1) the Lie algebra

$$a := \text{Lie}\{X_1, \ldots, X_m\}$$

is a vector space of dimension N; moreover,

$$\text{rank } a(x) = N \text{ at any point } x \in \mathbb{R}^N;$$

(H2) there exists a group of dilations $(\delta_\lambda)_{\lambda>0}$ in $\mathbb{R}^N$ such that every X_j is δ_λ-homogeneous of degree one.

A group of dilations in $\mathbb{R}^N$ is a family of diagonal linear functions $(\delta_\lambda)_{\lambda>0}$ of the kind

$$\delta_\lambda(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N), \qquad x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

where $\sigma_1 = 1 \le \sigma_2 \le \cdots \le \sigma_N$, $\sigma_j \in \mathbb{N}$.

Condition (H1) implies the hypoellipticity of $\mathcal{L}$: in particular, the *$\mathcal{L}$-harmonic functions*, i.e., the solution to $\mathcal{L}u = 0$, are smooth. Moreover, conditions (H1) and (H2) imply the existence of a group law $\circ$ in $\mathbb{R}^N$ such that $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a homogeneous Lie group on which the vector fields $X_j's$ are left translation invariant and δ_λ-homogeneous of degree one (see [4]). The natural number

$$Q = \sigma_1 + \ldots + \sigma_N$$

is called the homogeneous dimension of $\mathbb{G}$. Throughout the paper we always assume $Q \ge 3$ (if $Q = 2$, then $\mathbb{G}$ is the Euclidean group). Then there exists a continuous function $d : \mathbb{G} \longrightarrow \mathbb{R}$, smooth and strictly positive outside the origin, δ_λ-homogeneous of degree one and such that

$$\gamma(x) := \left(\frac{1}{d(x)}\right)^{Q-2}$$

is $\mathcal{L}$-harmonic in $\mathbb{R}^N \backslash \{0\}$ (see [6, Section 5.4]). This function d is called an $\mathcal{L}$-*gauge* and for $\mathcal{L}$ plays a role analogous to the one played by the Euclidean norm with respect to the classical Laplacian. In particular, the d-balls

$$B_d(x,r) := \{y \in \mathbb{G} \ : \ d(x^{-1} \circ y) < r\}$$

support averaging operators characterizing the $\mathcal{L}$-harmonicity. To be precise, define

$$\psi := |\nabla_{\mathcal{L}} d|^2, \qquad \nabla_{\mathcal{L}} = (X_1, \ldots, X_m),$$

$$M_r u(x) := \frac{1}{c_d r^Q} \int_{B_d(x,r)} \psi(x^{-1} \circ y) u(y) \, dy$$

and

$$N_r(\mathcal{L}u)(x) = \frac{1}{(Q-2)c_d r^Q} \int_0^r \rho^{Q-1} \left(\int_{B_d(x,\rho)} \mathcal{L}u(y) \left(d(x^{-1} \circ y)^{2-Q} - \rho^{2-Q} \right) dy \right) d\rho$$

where $c_d = \int_{B_d(0,1)} \psi \, dy$.

Then, if Ω is an open subset of $\mathbb{G}$, $u \in C^2(\Omega)$ and $\overline{B_d(x,r)} \subseteq \Omega$,

$$u(x) = M_r u(x) - N_r(\mathcal{L}u)(x) \tag{2.1}$$

(see [6, Theorem 5.6.1]).

We stress that ψ is smooth outside the origin, δ_λ-homogeneous of degree zero, and nonconstant unless $\mathbb{G}$ is the Euclidean group (see [5]; see also [6, Proposition 9.8.9]). In some particular important cases, such as, e.g., the group of Heisenberg type, explicit expressions of ψ are known (see [6, Example 5.5.3]). In any case it is known that $\psi > 0$ in a dense open subset of $\mathbb{R}^N$ (see [6, page 262]).

With these mean value operators, one can prove a version of the Gauss-Koebe Theorem in our setting (see [6, Section 5.6]):

Theorem 2.1 (Gauss-Koebe-type Theorem). *If $\Omega \subseteq \mathbb{R}^N$ is open and $u : \Omega \longrightarrow \mathbb{R}$ is $\mathcal{L}$-harmonic, then*

$$u(x) = M_r u(x) \tag{2.2}$$

for every $x \in \Omega$ and $r > 0$ such that $\overline{B_d(x,r)} \subseteq \Omega$.

Vice versa, if u is merely continuous in Ω and satisfies (2.2), then u is C^∞ and $\mathcal{L}$-harmonic in Ω.

The average operator M_r can also be used to fix the notion of $\mathcal{L}$-*superharmonic function.*

A lower semicontinuous function $u : \Omega \longrightarrow]-\infty, \infty]$ is called $\mathcal{L}$-*superharmonic* if u is finite in a dense subset of Ω and

$$u(x) \geq M_r u(x)$$

for every $x \in \Omega$ and $r > 0$ such that $\overline{B_d(x,r)} \subseteq \Omega$.

A quite exhaustive theory of $\mathcal{L}$-subharmonic functions is presented in the monograph [6, Chapter 8]. In particular, there it is proved that every $\mathcal{L}$-subharmonic function is L^1_{loc} and that if u is of class C^2, then u is $\mathcal{L}$-subharmonic if and only if $\mathcal{L}u \geq 0$.

3. Proof of Theorem 1.1, Proposition 1.2 and Remark 3.1

The most important part of this section is the

Proof of Theorem 1.1. Let $\alpha \in \mathbb{E}$, $\alpha \neq 0$, be fixed and let

$$\Omega(\alpha) := \{x \in \mathbb{G} \ : \ \langle \alpha, \pi(x) \rangle > 0\}.$$

For every $x \in \Omega(\alpha)$ we define

$$r(x) := \varepsilon \langle \alpha, \pi(x) \rangle,$$

where $\varepsilon > 0$ is fixed in such a way that

$$B(x, r(x)) \subseteq \Omega(\alpha) \qquad \forall x \in \Omega(\alpha). \tag{3.1}$$

We will show in a moment the existence of a suitable $\varepsilon > 0$ satisfying (3.1).

For a function $u \in L^1_{\text{loc}}(\Omega(\alpha))$ we let

$$T(u) : \Omega(\alpha) \longrightarrow \mathbb{R}, \quad T(u)(x) := M_{r(x)}(u)(x).$$

Hence,

$$T(u)(x) = \int_{\Omega(\alpha)} K(x, y) u(y) \, dy, \quad x \in \Omega(\alpha),$$

where

$$K(x, y) = \frac{1}{c_d (r(x))^Q} \psi(x^{-1} \circ y) \mathcal{X}_{B_x}(y). \tag{3.2}$$

In what follows we also use the following notation:

$$A_x := \{y \in \Omega(\alpha) \mid d(y^{-1} \circ x) < r(y)\}.$$

With this notation, we have

$$\mathcal{X}_{B_x}(y) = \mathcal{X}_{A_y}(x).$$

Indeed

$$y \in B_x \iff d(x^{-1} \circ y) < r(x) \iff x \in A_y.$$

Let us now show (3.1). We first remark that $\mathbb{E} \ni e \longmapsto d(e) \in \mathbb{R}$ is homogeneous of degree one with respect to the Euclidean dilation $e \longmapsto \lambda e$. As a consequence, by a suitable constant $c > 0$, we have

$$d(e) \geq c|e| \qquad \forall e \in \mathbb{E}, \quad |\cdot| = \text{ Euclidean norm.}$$

Moreover, we can also assume that

$$d(x) \geq c|\pi(x)| \qquad \forall x \in \mathbb{G}.$$

Then, if $x \in \Omega(\alpha)$, for every $z \in B_d(x, r(x))$, we have $r(x) > d(z, x) \geq c|\pi(z) - \pi(x)|$. Hence

$$\begin{aligned} \langle \alpha, \pi(z) \rangle &= \langle \alpha, \pi(x) \rangle + \langle \alpha, \pi(z) - \pi(x) \rangle \geq \langle \alpha, \pi(x) \rangle - |\alpha||\pi(z) - \pi(x)| \\ &\geq \langle \alpha, \pi(x) \rangle - \frac{|\alpha|}{c} r(x) = \langle \alpha, \pi(x) \rangle \left(1 - \frac{|\alpha|}{c} \varepsilon\right). \end{aligned}$$

Thus, if $0 < \varepsilon < \frac{c}{|\alpha|}$, we get $\langle \alpha, \pi(z) \rangle > 0$; i.e., $z \in \Omega(\alpha)$ and (3.1) is proved.

The proof of Theorem 1.1 will immediately follow from the next lemma.

Main Lemma.

(i) $K(x,y) \geq 0$ *for every* $x, y \in \Omega(\alpha)$*;*
(ii) $\int_{\Omega(\alpha)} K(x,y)\ dy = 1$ *for every* $x \in \Omega(\alpha)$*;*
(iii) $\int_{\Omega(\alpha)} K(x,y)\ dx = \int_{\Omega(\alpha)} K(x,\alpha)\ dx$ *for every* $y \in \Omega(\alpha)$*;*
(iv) $c^* := \int_{\Omega(\alpha)} K(x,\alpha)\ dx > 1$.

Proof of the Main Lemma.

(i) It straightforwardly follows from (3.2).
(ii) By the Gauss-Koebe-type Theorem 2.1 for $\mathcal{L}$-harmonic functions, if u is $\mathcal{L}$-harmonic in $\Omega(\alpha)$, then $T(u) = u$. In particular $T(1) = 1$, that is,

$$1 = \int_{\Omega(\alpha)} K(x,y)\ dy \text{ for every } x \in \Omega(\alpha).$$

(iii) This is the crucial part of the Main Lemma. We start by proving the following property of $\Omega(\alpha)$: $\forall y \in \Omega(\alpha)$ there exists $\lambda = \lambda(y) > 0$ such that

$$\delta_\lambda(\alpha) \circ y^{-1} \circ x \in \Omega(\alpha) \text{ and } r(\delta_\lambda(\alpha) \circ y^{-1} \circ x) = r(x)$$

for every $x \in \Omega(\alpha)$.
Indeed, let $y, x \in \Omega(\alpha)$ and consider

$$\begin{aligned} \langle \alpha, \pi(\delta_\lambda(\alpha) \circ y^{-1} \circ x)\rangle &= \langle \alpha, \pi(\delta_\lambda(\alpha))\rangle + \langle \alpha, \pi(y^{-1})\rangle + \langle \alpha, \pi(x)\rangle \\ &= \langle \alpha, \pi(x)\rangle + \lambda\langle \alpha, \alpha\rangle - \langle \alpha, \pi(y)\rangle. \end{aligned}$$

Then, if we choose $\lambda = \frac{\langle \alpha, \pi(y)\rangle}{|\alpha|^2}$ we have $\lambda > 0$ and

$$\langle \alpha, \pi(\delta_\lambda(\alpha)) \circ y^{-1} \circ x\rangle > 0, \quad r((\delta_\lambda(\alpha) \circ y^{-1} \circ x)) = r(x).$$

This completes the proof of the stated property of $\Omega(\alpha)$.
In what follows we also use a homogeneity property of $x \longmapsto r(x)$, precisely

$$r(\delta_\lambda(x)) = \lambda r(x) \text{ for every } x \in \Omega(\alpha) \text{ and } \lambda > 0.$$

Indeed

$$r(\delta_\lambda(x)) = \varepsilon\langle \alpha, \pi(\delta_\lambda(x))\rangle = \varepsilon\langle \alpha, \lambda\pi(x)\rangle = \lambda r(x).$$

Let us now fix $y \in \Omega(\alpha)$ and compute

$$\begin{aligned}
\int_{\Omega(\alpha)} K(x,\alpha)\,dx &= \frac{1}{c_d}\int_{\Omega(\alpha)} \left(\frac{1}{r(x)}\right)^Q \psi\left(x^{-1}\circ\alpha\right)\mathcal{X}_{B_x}(\alpha)\,dx \\
&\quad \text{(letting } \hat{\psi}(z)=\psi(z^{-1})\text{)} \\
&= \frac{1}{c_d}\int_{A_\alpha} \left(\frac{1}{r(x)}\right)^Q \hat{\psi}(\alpha^{-1}\circ x)\mathcal{X}_{A_\alpha}(x)\,dx \\
&\quad \text{(using the change of variables } x=\delta_{\frac{1}{\lambda}}(\xi) \text{ and noticing} \\
&\quad \text{that } r\left(\delta_{\frac{1}{\lambda}}(\xi)\right)=\frac{1}{\lambda}r(\xi) \text{ and that } dx=\lambda^{-Q}d\xi) \\
&= \frac{1}{c_d}\int_{\delta_\lambda(A_\alpha)} \left(\frac{1}{r(\xi)}\right)^{-Q} \hat{\psi}\left(\alpha^{-1}\circ\delta_{\frac{1}{\lambda}}(\xi)\right)\,d\xi \\
&\quad \text{(keeping in mind that } \hat{\psi} \text{ is } \delta_\lambda\text{-homogeneous of degree zero)} \\
&= \frac{1}{c_d}\int_{\delta_\lambda(A_\alpha)} \left(\frac{1}{r(\xi)}\right)^{Q} \psi^{-1}\left(\delta_\lambda(\alpha^{-1})\circ\xi\right)\,d\xi.
\end{aligned}$$

We now choose $\lambda=\lambda(y)>0$ such that $r(\delta_\lambda(\alpha)\circ y^{-1}\circ x)=r(x)$ for every $x\in\Omega(\alpha)$ and use the change of variable

$$\xi=\delta_\lambda(\alpha)\circ y^{-1}\circ x.$$

We obtain

$$\int_{\Omega(\alpha)} K(x,\alpha)\,dx = \frac{1}{c_d}\int_{y\circ\delta_\lambda(\alpha^{-1})\circ\delta_\lambda(A_\alpha)} \left(\frac{1}{r(x)}\right)^Q \hat{\psi}^{-1}(y^{-1}\circ x)\,dx.$$

On the other hand, as we will recognize in a moment,

$$y\circ\delta_\lambda(\alpha^{-1})\circ\delta_\lambda(A_\alpha)=A_y. \tag{3.3}$$

Then

$$\int_{\Omega(\alpha)} K(x,\alpha)\,dx = \frac{1}{c_d}\int_{A_y}\left(\frac{1}{r(x)}\right)^Q \hat{\psi}(y^{-1}\circ x)\,dx = \int_{\Omega(\alpha)} K(x,y)\,dx,$$

and (iii) is proved.

We are left to prove (3.3). One has

$$\begin{aligned}
x\in y\circ\delta_\lambda(\alpha^{-1})\circ\delta_\lambda(A_\alpha) &\iff z:=\alpha\circ\delta_{\frac{1}{\lambda}}(y^{-1}\circ x)\in A_\alpha \\
&\iff d(z,\alpha)<r(z).
\end{aligned}$$

We know that $r(z)=\frac{1}{\lambda}r(\delta_\lambda(\alpha)\circ y^{-1}\circ x)=\frac{1}{\lambda}r(x)$, while

$$d(z,\alpha)=d(\alpha^{-1}\circ z)=d(\delta_{\frac{1}{\lambda}}(y^{-1}\circ x))=\frac{1}{\lambda}d(y^{-1},x).$$

We have thus proved that

$$x\in y\circ\delta_\lambda(\alpha^{-1})\circ\delta_\lambda(A_\alpha) \iff d(y^{-1}\circ x)<r(x) \iff x\in A_y.$$

This completes the proof of (iii).

(iv) Let $x_0\in\mathbb{G}\backslash\overline{\Omega(\alpha)}$ and consider the function

$$v:\Omega(\alpha)\longrightarrow\mathbb{R},\quad v(x)=(d(x_0^{-1}\circ x))^{-1-Q}.$$

Obviously the function v is smooth in $\Omega(\alpha)$. Moreover, $v > 0$ and $v \in L^1(\Omega(\alpha))$. By using the left invariance of $\mathcal{L}$ on $\mathbb{G}$ and the form of $\mathcal{L}$ for radial functions[1] we also have

$$\begin{aligned}(\mathcal{L}v)(x) &= (\mathcal{L}d^{-Q-1})(x_0^{-1} \circ x)\\ &= \psi(x_0^{-1} \circ x)((Q+1)(Q+2) - (Q+1)(Q-1))d^{-Q-3}(x_0^{-1} \circ x)\\ &= 3(Q+1)(\psi d^{-Q-3})(x_0^{-1} \circ x).\end{aligned}$$

Then $\mathcal{L}v > 0$ in a dense open set of $\Omega(\alpha)$. As a consequence, using the representation formula (2.1), we get

$$T(v)(x) - v(x) > 0 \qquad \forall x \in \Omega(\alpha),$$

that is, $T(v) > v$ in $\Omega(\alpha)$. It follows that

$$\begin{aligned}\int_{\Omega(\alpha)} v\ dx &< \int_{\Omega(\alpha)} T(v)\ dx = \int_{\Omega(\alpha)} \left(\int_{\Omega(\alpha)} K(x,y)v(y)\ dy \right)\ dx\\ &= \int_{\Omega(\alpha)} v(y) \left(\int_{\Omega(\alpha)} K(x,y)\ dx \right)\ dy = c^* \int_{\Omega(\alpha)} v(y)\ dy.\end{aligned}$$

Then

$$\int_{\Omega(\alpha)} v\ dx < c^* \int_{\Omega(\alpha)} v\ dy,$$

which implies $c^* > 1$, since $\int_{\Omega(\alpha)} v\ dx > 0$. This completes the proof of the Main Lemma. □

We can now conclude the proof of Theorem 1.1. Since u is $\mathcal{L}$-superharmonic, we have $T(u) \leq u$ in $\Omega(\alpha)$. Therefore,

$$\begin{aligned}\int_{\Omega(\alpha)} u\ dx &\geq \int_{\Omega(\alpha)} T(u)\ dx\\ &\quad \text{(as in the proof of the Main Lemma (iv))}\\ &= c^* \int_{\Omega(\alpha)} u\ dx.\end{aligned}$$

Then, since $c^* > 1$,

$$\int_{\Omega(\alpha)} u\ dx \leq 0,$$

which implies $u \equiv 0$ since $u \geq 0$ and lower semicontinuous. □

Proof of Proposition 1.2. Let d be a gauge function for $\mathcal{L}$ and define

$$u(x) = (d({x_0}^{-1} \circ x))^{-Q+2}, \quad x \in \Omega(\alpha),$$

where, as before, $x_0 \notin \overline{\Omega(\alpha)}$. The function u is smooth in $\Omega(\alpha)$ and

$$\mathcal{L}(u)(x) = (\mathcal{L}d^{2-Q})({x_0}^{-1} \circ x) = 0, \quad x \in \Omega(\alpha).$$

Moreover, $u > 0$ and $u \in L^p(\Omega(\alpha))$ since, from the assumption $\frac{Q}{2} > \frac{p}{p-1}$, it follows that

$$p(Q-2) > Q.$$

□

[1] If $w = f(d)$, then $\mathcal{L}(w) = \psi(f''(d) + \frac{Q-1}{d} f'(d))$ (see [6, Proposition 5.4.3]).

Remark 3.1. The assumption $u \geq 0$ in Theorem 1.1 cannot be removed.

Indeed, if $x_0 \notin \overline{\Omega(\alpha)}$, the function

$$u_k(x) := \partial_{x_N}^k (d(x_0^{-1} \circ x))^{2-Q}$$

is $\mathcal{L}$-harmonic in $\Omega(\alpha)$ for every $k \in \mathbb{N}$, and δ_λ-homogeneous of degree $2 - Q - k\sigma_N$.

Then, if $k > \frac{2}{\sigma_N}$, $u_k \in L^1(\Omega(\alpha))$. Thus, with this choice of k, u_k is a summable $\mathcal{L}$-harmonic function in $\Omega(\alpha)$ and $u_k \not\equiv 0$.

We would like to stress that in the previous argument we used the following properties:

(i) the differential operator ∂_{x_N} is δ_λ-homogeneous of degree σ_N and commutes with $\mathcal{L}$;

(ii) $\mathcal{L}$ is left translation invariant with respect to the composition law $\circ$;

(iii) d^{2-Q} is $\mathcal{L}$-harmonic out of the origin.

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