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LOCAL ALGEBRAIC APPROXIMATION OF SEMIANALYTIC SETS

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ABSTRACT. Two subanalytic subsets of $\mathbb{R}^n$ are called s-equivalent at a common point P if the Hausdorff distance between their intersections with the sphere centered at P of radius r vanishes to order $> s$ when r tends to 0. In this paper we prove that every s-equivalence class of a closed semianalytic set contains a semialgebraic representative of the same dimension. In other words any semianalytic set can be locally approximated to any order s by means of a semialgebraic set and hence, by previous results, also by means of an algebraic one.

1. INTRODUCTION

In [FFW1] we introduced a notion of local metric proximity between two sets that we called s-equivalence: for a real $s \geq 1$, two subanalytic subsets of $\mathbb{R}^n$ are s-equivalent at a common point P if the Hausdorff distance between their intersections with the sphere centered at P of radius r vanishes to order $> s$ when r tends to 0.

Given a subanalytic set $A \subset \mathbb{R}^n$ and a point $P \in A$, a natural question concerns the existence of an algebraic representative X in the class of s-equivalence of A at P; in that case we also say that X approximates A of order s at P.

The answer to the previous question is in general negative for subanalytic sets which are not semianalytic, even for $s = 1$ (see [FFW3]). Furthermore, in [FFW2] we defined s-equivalence of two subanalytic sets along a common submanifold, and studied 1-equivalence of a pair of strata to the normal cone of the pair. By example we showed that a semianalytic normal cone to a linear X may be not 1-equivalent to any semialgebraic set along X. It is still an open problem whether a semialgebraic normal cone along a linear X is s-equivalent to an algebraic variety along X, for all s.

On the other hand some partial positive answers were given in [FFW1] and [FFW3]; in particular we proved that a subanalytic set $A \subset \mathbb{R}^n$ can be approximated of any order by an algebraic one in each of the following cases:
- A is a closed semialgebraic set of positive codimension,
- A is the zero-set $V(f)$ of a real analytic map f whose regular points are dense in $V(f)$,
- A is the image of a real analytic map f having a finite fiber at P.

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Using the previous results we also obtained that one-dimensional subanalytic sets, analytic surfaces in $\mathbb{R}^3$ and real analytic sets having a Puiseux-type parametrization admit an algebraic approximation of any order.

In the present paper we prove that any closed semianalytic set can be locally approximated of any order by a semialgebraic one having the same dimension. Using the main result of [FFW1], it follows that any closed semianalytic set of positive codimension admits an algebraic approximation of any order. Thus we obtain a complete positive answer to our question for the class of semianalytic sets.

The algebraic approximation, elaborating the methods introduced in [FFW3], is obtained by taking sufficiently high order truncations of the analytic functions appearing in a presentation of the semianalytic set.

Finally, let us mention some possible future developments of these notions and ideas. Since we can prove that two subanalytic sets A, B are 1-equivalent if and only if their tangent cones coincide (see also [FFW1]), it would be interesting to extend the notion of tangent cone associating to A a sort of "tangent cone of order s", say $C_s(A)$, in such a way that A and B are s-equivalent if and only if $C_s(A) = C_s(B)$.

There is currently much interest in bi-Lipschitz equivalence of varieties. Most of the work has been in the complex case. One recent such example is [BFGO]. The theory is closely tied up with the notion of the tangent cone, exceptional subcones, and limits of tangent spaces. The real case has been little studied. A good place to start is in the case of surfaces in $\mathbb{R}^3$, which is the only real case in which the tangent cone, exceptional lines, and limits of tangent planes have been deeply analyzed (see [OW]). The s-equivalence classes are Lipschitz invariants, so they should be a useful tool in this analysis.

2. Basic notions and preliminary results

If A and B are non-empty compact subsets of $\mathbb{R}^n$, we denote by $D(A, B)$ the classical Hausdorff distance, i. e.

$$D(A, B) = \inf \{\epsilon \mid A \subseteq N_\epsilon(B),\ B \subseteq N_\epsilon(A)\},$$

where $N_\epsilon(A) = \{x \in \mathbb{R}^n \mid d(x, A) < \epsilon\}$ and $d(x, A) = \inf_{y \in A} \|x - y\|$.

If we let $\delta(A, B) = \sup_{x \in B} d(x, A)$, then $D(A, B) = \max\{\delta(A, B),\ \delta(B, A)\}$.

We will denote by O the origin of $\mathbb{R}^n$ for any n.

We are going to introduce the notion of s-equivalence at a point; without loss of generality we can assume that this point is O.

Definition 2.1. Let A and B be closed subanalytic subsets of $\mathbb{R}^n$ with $O \in A \cap B$. Let s be a real number ≥ 1. Denote by S_r the sphere of radius r centered at the origin.

(1) We say that $A \leq_s B$ if either O is isolated in A or if O is non-isolated both in A and in B and
$$\lim_{r \to 0} \frac{\delta(B \cap S_r, A \cap S_r)}{r^s} = 0.$$

(2) We say that A and B are s–equivalent (and we will write $A \sim_s B$) if $A \leq_s B$ and $B \leq_s A$.

Observe that if O is non-isolated both in A and in B, then

$$A \sim_s B \qquad \text{if and only if} \qquad \lim_{r \to 0} \frac{D(A \cap S_r, B \cap S_r)}{r^s} = 0.$$

Moreover, if $A \subseteq B$, then $A \leq_s B$ for any $s \geq 1$. It is easy to check that $\leq_s$ is transitive and that $\sim_s$ is an equivalence relation. The following result shows that s-equivalence behaves well with respect to the union of sets:

Proposition 2.2 ([FFW3]). *Let A, A', B and B' be closed subanalytic subsets of $\mathbb{R}^n$.*

1. *If $A \leq_s B$ and $A' \leq_s B'$, then $A \cup A' \leq_s B \cup B'$.*
2. *If $A \sim_s B$ and $A' \sim_s B'$, then $A \cup A' \sim_s B \cup B'$.*

Given a closed subanalytic set A and $s \geq 1$, the problem we are interested in is whether there exists an algebraic subset Y which is s-equivalent to A; in this case we also say that Y approximates A to order s. Evidently the question is trivially true when O is an isolated point in A.

Among the partial answers to the previous question that have been already achieved, we recall only the following one which will be used later on:

Theorem 2.3 ([FFW1]). *For any real number $s \geq 1$ and for any closed semialgebraic set $A \subset \mathbb{R}^n$ of codimension ≥ 1, there exists an algebraic subset Y of $\mathbb{R}^n$ such that $A \sim_s Y$.*

The following definition introduces a geometric tool which is very useful to test the s-equivalence of two subanalytic sets:

Definition 2.4. Let A be a closed subanalytic subset of $\mathbb{R}^n$, $O \in A$. For any real $\sigma > 1$, we will refer to the set

$$\mathcal{H}(A, \sigma) = \{x \in \mathbb{R}^n \mid d(x, A) < \|x\|^\sigma\}$$

as the *horn-neighbourhood* with center A and exponent σ.

Note that, if O is isolated in A, then $\mathcal{H}(A, \sigma) = \emptyset$ near O.

Proposition 2.5 ([FFW3]). *Let A, B be closed subanalytic subsets of $\mathbb{R}^n$ with $O \in A \cap B$ and let $s \geq 1$. Then $A \leq_s B$ if and only if there exist $\sigma > s$ and an open neighbourhood Ω of O such that $(A \setminus \{O\}) \cap \Omega \subseteq \mathcal{H}(B, \sigma) \cap \Omega$.*

The following technical result suggests that horn-neighbourhoods can be used to modify a subanalytic set producing subanalytic sets s-equivalent to the original one:

Lemma 2.6. *Let $X \subset Y \subset \mathbb{R}^n$ be closed subanalytic sets such that $O \in X$ and let $s \geq 1$. Then:*

1. *for any $\sigma > s$ we have $Y \sim_s Y \cup \mathcal{H}(X, \sigma)$;*
2. *if $\overline{Y \setminus X} = Y$, there exists $\sigma > s$ such that $Y \setminus \mathcal{H}(X, \sigma) \sim_s Y$.*

Proof. (a) Since $Y \cup \mathcal{H}(X, \sigma) \subseteq \mathcal{H}(Y, \sigma)$, by Proposition 2.5 for any $\sigma > s$ we have that $Y \cup \mathcal{H}(X, \sigma) \leq_s Y$ and hence $Y \cup \mathcal{H}(X, \sigma) \sim_s Y$.

(b) Let $\mathcal{U}(X, q) = \{x \in \mathbb{R}^n \mid \exists\, y \in X, \|x\| = \|y\|, \|x - y\| < \|x\|^q\}$.

Arguing as in [FFW1, Corollary 2.6], there exists q such that $Y \setminus \mathcal{U}(X, q) \sim_s Y$. Since X and $Y \setminus \mathcal{U}(X, q)$ are subanalytic sets and meet only in O, they are regularly situated, i.e. there exists β such that $d(x, X) + d(x, Y \setminus \mathcal{U}(X, q)) > \|x\|^\beta$ for all x near O. Then $\mathcal{H}(X, \beta) \subseteq \mathcal{U}(X, q)$ and hence taking $\sigma > \max\{\beta, s\}$ we have that $Y \setminus \mathcal{H}(X, \sigma) \sim_s Y$. □

Another essential tool will be Łojasiewicz' inequality, which we will use in the following slightly modified version.

Proposition 2.7. *Let A be a compact subanalytic subset of $\mathbb{R}^n$. Assume f and g are subanalytic functions defined on A such that f is continuous, $V(f) \subseteq V(g)$, g is continuous at the points of $V(g)$ and such that $|g| < 1$ on A. Then there exists a positive constant α such that $|g|^\alpha \leq |f|$ on A and $|g|^\alpha < |f|$ on $A \setminus V(f)$.*

Proof. The result will be obtained by adapting the proof given by Łojasiewicz under the stronger hypothesis that g is continuous on A (see [Ł, Théorème 1]); in that paper he used the following lemma ([Ł, Lemma 4]):

If $E \subset [0,\infty)\times\mathbb{R}$ is a compact semianalytic subset of $\mathbb{R}^2$ such that $E\cap(\{0\}\times\mathbb{R}) \subseteq \{(0,0)\}$, then there exist positive constants c, α such that $E \subseteq \{(x,y) \in \mathbb{R}^2 \mid |y|^\alpha \leq c|x|\}$.

The map $\Phi = (|f|, g)\colon A \to \mathbb{R}^2$ is subanalytic and bounded; hence $\Phi(A)$ is a subanalytic subset of $\mathbb{R}^2$ and therefore semianalytic ([Ł, Proposition 2]). Then $E = \overline{\Phi(A)}$ is a compact semianalytic subset of $[0,\infty) \times \mathbb{R}$.

We have that $E \cap (\{0\} \times \mathbb{R}) \subseteq \{(0,0)\}$: namely, if $(0, y_0) \in E$, then there exists a sequence $\{a_i\} \subset A$ such that $\lim_{i\to\infty} \Phi(a_i) = (0, y_0)$ with a_i converging to $a_0 \in A$. By continuity $f(a_0) = 0$ and hence $g(a_0) = 0$. By the continuity of g at a_0, we have that $y_0 = g(a_0) = 0$.

So E fulfills the hypotheses of the lemma recalled above and therefore there exist positive constants c, α such that $|g|^\alpha \leq c|f|$ on A.

Since $|g| < 1$, increasing α if necessary we can obtain the thesis. □

3. Main results

This section is devoted to the proof of the local approximation theorem for semianalytic sets.

Since s-equivalence depends only on the set-germs at O, all the sets we will work with will be considered as subsets of a suitable open ball Ω in $\mathbb{R}^n$ centered at O; we will shrink such a ball whenever necessary without mention.

Definition 3.1. Let A be a closed semianalytic subset of Ω. We will say that A admits a *good presentation* if the minimal analytic variety V_A containing A is irreducible and there exist analytic functions $f_1, \ldots, f_p$ which generate the ideal $I(V_A)$ and $g_1, \ldots, g_l$ analytic functions on Ω such that

$$A = \{x \in \Omega \mid f_i(x) = 0, g_j(x) \geq 0, i = 1, \ldots, p, j = 1, \ldots, l\}.$$

We start with a preliminary result concerning a way to decompose and present semianalytic sets:

Lemma 3.2. *Let A be a closed semianalytic subset of Ω with $\dim_O A = d > 0$. Then there exist closed semianalytic sets $\Gamma_1, \ldots, \Gamma_r, \Gamma'$ such that*

(1) $A = (\bigcup_{i=1}^r \Gamma_i) \cup \Gamma'$,
(2) *for each i, $\dim_O \Gamma_i = d$ and Γ_i admits a good presentation,*
(3) $\dim \Gamma' < d$.

Proof. Let V_A be the minimal analytic variety containing A (in particular $\dim_O V_A = d$). Let $V_1 \cup \ldots \cup V_m$ be the decomposition of V_A into irreducible components. Then $A = W_1 \cup \ldots \cup W_m$ where $W_i = A \cap V_i$. Then V_i is the minimal analytic variety containing W_i and $\dim_O V_i = \dim_O W_i$.

Each W_i is a finite union of sets of the form $\Gamma = \{h_1 = 0, \ldots, h_q = 0, g_1 \geq 0, \ldots, g_l \geq 0\}$.

Let Γ' be the union, over $i = 1, \dots, m$, of the Γs having dimension less than d.

For any $\Gamma \subseteq V_i$ of dimension d, V_i is the minimal analytic variety containing Γ. It follows that $\Gamma = \{f_1 = 0, \dots, f_p = 0, g_1 \geq 0, \dots, g_l \geq 0\}$ where $f_1, \dots, f_p$ are generators of the ideal $I(V_i)$. Thus we can take as $\Gamma_1, \dots, \Gamma_r$ these latter Γs (over $i = 1, \dots, m$) suitably indexed. □

Notation 3.3. Let $g_1, \dots, g_l$ be analytic functions on Ω and let $f = (f_1, \dots, f_p)\colon \Omega \to \mathbb{R}^p$ be an analytic map. If $A = \{x \in \Omega \mid f(x) = O, g_i(x) \geq 0, i = 1, \dots, l\}$, we will use the following notation:

(1) $A_i = \{x \in \Omega \mid f(x) = O, g_i(x) \geq 0\}$ for $i = 1, \dots, l$ (so that $A = \bigcap A_i$),
(2) $b(A) = \bigcup_{i=1}^{l} (V(g_i) \cap A)$.

Lemma 3.4. *Consider the closed semianalytic set*

$$A = \{x \in \Omega \mid f(x) = O, g_i(x) \geq 0, i = 1, \dots, l\},$$

where $f\colon \Omega \to \mathbb{R}^p$ is an analytic map and $g_1, \dots, g_l$ are analytic functions on Ω. Assume that $O \in A$. Let σ be a real number > 1 and let $H \subseteq \mathbb{R}^n$ be an open subanalytic set such that $H \supseteq \mathcal{H}(b(A), \sigma)$. Then there exists η such that, for each $x \in V(f) \setminus (A \cup H)$, there exists i so that $x \notin \mathcal{H}(A_i, \eta)$.

Proof. Since the functions $\sum_i d(x, A_i)$ and $d(x, A)$ are subanalytic and vanish exactly on A, by Proposition 2.7 there exists $\alpha > 0$ such that, for any x,

$$\sum_i d(x, A_i) \geq d(x, A)^\alpha.$$

Let d_g denote the geodesic distance on $V(f)$.

If $x \in V(f) \setminus A$, we have $d_g(x, A) = d_g(x, b(A))$. In a suitable closed ball centered at O we can assume that $V(f)$ is connected; hence, by a result of Kurdyka and Orro ([KO]) for any $\epsilon > 0$ there exists a subanalytic distance $\Delta(x, y)$ on $V(f)$ such that

$$\forall x, y \in V(f) \qquad 0 \leq \Delta(x, y) \leq d_g(x, y) \leq (1 + \epsilon)\Delta(x, y).$$

Then, if we take for instance $\epsilon = 1$,

$$\forall x \in V(f) \qquad 0 \leq \Delta(x, A) \leq d_g(x, A) \leq 2\Delta(x, A)$$

and so the subanalytic function $\Delta(x, A)$ is continuous at each point of A. Hence by Proposition 2.7 there exists $\mu > 0$ such that, for any x in $V(f)$,

$$d(x, A) \geq \Delta(x, A)^\mu$$

and so

$$\sum_i d(x, A_i) \geq \Delta(x, A)^{\mu\alpha} \geq \left(\frac{d_g(x, A)}{2}\right)^{\mu\alpha}.$$

Moreover for any $x \in V(f) \setminus (A \cup H)$ we have that

$$d_g(x, A) = d_g(x, b(A)) \geq d(x, b(A)) \geq \|x\|^\sigma.$$

Let us show that the thesis holds choosing $\eta > \sigma\mu\alpha$.

If, for a contradiction, any neighbourhood of O contains a point $x \in \bigcap_i \mathcal{H}(A_i, \eta) \cap (V(f) \setminus (A \cup H)$, then we have that

$$\frac{1}{2^{\mu\gamma}} \|x\|^{\sigma\mu\alpha} \leq \sum_{i=1}^{l} d(x, A_i) \leq l\|x\|^\eta,$$

which is impossible when x tends to O. □

For any analytic map ψ defined in a neighbourhood of O, we will denote by $T^k\psi(x)$ the polynomial map whose components are the Taylor polynomials of order k at O of the components of ψ.

Lemma 3.5. *Let φ be an analytic function on Ω such that $\varphi(O) = 0$. Let X be a closed semianalytic subset of Ω, $O \in X$. Then for any real positive θ there exists $\alpha > 0$ such that, for all integers $k > \alpha$, the function $T^k\varphi$ has the same sign as φ on $X \setminus (\mathcal{H}(X \cap V(\varphi), \theta) \cup \{O\})$.*

Proof. Denote $Z = X \setminus \mathcal{H}(X \cap V(\varphi), \theta)$. Since $V(\varphi) \cap Z = \{O\}$, by Proposition 2.7 there exists $\alpha > 0$ such that $\|x\|^\alpha < |\varphi(x)|$ for all $x \in Z \setminus \{O\}$.

For all integers $k > \alpha$

$$\lim_{x \to O} \frac{\varphi(x) - T^k\varphi(x)}{\|x\|^\alpha} = 0.$$

If O is isolated in Z, there is nothing to prove. Otherwise assume, for a contradiction, that any neighbourhood of O contains a point $x \in Z$ such that $\varphi(x)$ and $T^k\varphi(x)$ have different signs (for instance $\varphi(x) > 0$ and $T^k\varphi(x) \leq 0$). Then

$$|\varphi(x) - T^k\varphi(x)| \geq |\varphi(x)| > \|x\|^\alpha$$

and hence

$$\frac{|\varphi(x) - T^k\varphi(x)|}{\|x\|^\alpha} > 1$$

arbitrarily near to O, which is impossible. □

Notation 3.6. Let $g_1, \dots, g_l$ be analytic functions on Ω and let $f \colon \Omega \to \mathbb{R}^p$ be an analytic map. If $A = \{x \in \Omega \mid f(x) = O, g_i(x) \geq 0, i = 1, \dots, l\}$, for any $h, k \in \mathbb{N}$ let

(1) $T^h(A) = \{x \in \Omega \mid T^h f(x) = O, g_i(x) \geq 0 \; i = 1, \dots, l\}$,
(2) $T_k(A) = \{x \in \Omega \mid f(x) = O, T^k g_1(x) \geq 0, \dots, T^k g_l(x) \geq 0\}$,
(3) $T^h_k(A) = T^h(T_k(A)) = \{x \in \Omega \mid T^h f(x) = O, T^k g_1(x) \geq 0, \dots, T^k g_l(x) \geq 0\}$.

Moreover, for any analytic map $\varphi \colon \Omega \to \mathbb{R}^p$, denote $\Sigma_r(\varphi) = \{x \in \Omega \mid \text{rk } d_x\varphi < r\}$, and $\Sigma(\varphi) = \Sigma_p(\varphi)$.

Lemma 3.7. *Let A be a closed semianalytic subset of Ω, with $\dim_O A = d > 0$. Assume that $A = \{f(x) = O, g_i(x) \geq 0, i = 1, \dots, l\}$, with $g_1, \dots, g_l$ analytic functions on Ω and $f \colon \Omega \to \mathbb{R}^{n-d}$ an analytic map. Assume also that*

$$\dim_O(A \setminus (\Sigma(f) \cup b(A))) > 0.$$

Then for any $s \geq 1$ there exist $h_0 > 0, k_0 > 0$ such that, for all integers h, k with $h \geq h_0$ and $k \geq k_0$, we have

(1) $T^h_k(A) \leq_s A$,
(2) $\overline{A \setminus (\Sigma(f) \cup b(A))} \leq_s T^h_k(A)$,
(3) $\dim_O T^h_k(A) = d$.

Proof. Let $s \geq 1$ and let $\sigma > s$. Denote $X = (\Sigma(f) \cap A) \cup b(A)$.

(1) Let $H = \mathcal{H}(X, \sigma)$. By Lemma 3.4 there exists η such that, for each $x \in V(f) \setminus (A \cup H)$, there exists i_0 so that $x \notin \mathcal{H}(A_{i_0}, \eta)$.

For all j, applying Lemma 3.5 to $V(f)$, g_j and η, we find $\alpha_1 > 0$ such that, for all integers $k > \alpha_1$, the functions g_j and $T^k g_j$ have the same sign on $V(f) \setminus (\mathcal{H}(V(f) \cap V(g_j), \eta) \cup \{O\})$.

Let $x \in V(f) \setminus (A \cup H)$. Then $x \notin \mathcal{H}(A_{i_0}, \eta)$ for some i_0 and hence $g_{i_0}(x) < 0$; moreover, since $V(f) \cap V(g_{i_0}) \subseteq A_{i_0}$, we have that $x \in V(f) \setminus (\mathcal{H}(V(f) \cap V(g_{i_0}), \eta) \cup \{O\})$ and hence, for all integers $k > \alpha_1$, $T^k g_{i_0}(x) < 0$. This implies that $T_k(A) \subseteq A \cup H$.

Applying Lemma 2.6 (1) to the sets X and A, we have $A \sim_s A \cup H$, and so $T_k(A) \leq_s A$.

Let $B_k = \{x \in \Omega \mid T^k g_i \geq 0, \; i = 1, \dots, l\}$.

Since $T_k(A) = B_k \cap V(f)$, by Proposition 2.7 there exists $\rho > 0$ such that $\|f(x)\| \geq d(x, T_k(A))^\rho$ for all $x \in B_k$; then for $x \in B_k \setminus \mathcal{H}(T_k(A), \sigma)$ we have that $\|f(x)\| \geq \|x\|^{\rho\sigma}$.

Let h be an integer such that $h \geq \rho\sigma$. Then

$$\lim_{x \to O} \frac{\|f(x) - T^h f(x)\|}{\|x\|^{\rho\sigma}} = 0.$$

We have that $T^h(T_k(A)) \setminus \{O\} \subseteq \mathcal{H}(T_k(A), \sigma)$; otherwise there would exist a sequence of points $y_i \neq O$ converging to O such that $y_i \in T^h(T_k(A)) \setminus \mathcal{H}(T_k(A), \sigma)$ and hence

$$\lim_{i \to \infty} \frac{\|f(y_i) - T^h f(y_i)\|}{\|y_i\|^{\rho\sigma}} = \lim_{i \to \infty} \frac{\|f(y_i)\|}{\|y_i\|^{\rho\sigma}} \geq 1,$$

which is a contradiction.

Thus by Proposition 2.5 we get that $T_k^h(A) \leq_s T_k(A) \leq_s A$.

(2) Let $Y = \overline{A \setminus X}$. By our hypotheses O is not isolated in Y.

Since $\overline{Y \setminus X} = Y$, applying Lemma 2.6 (2) to the sets $X \cap Y$ and Y and increasing σ if needed, we have that $Y \setminus \mathcal{H}(X \cap Y, \sigma) \sim_s Y$. Denote

$$Y' = Y \setminus \mathcal{H}(X \cap Y, \sigma) \quad \text{and} \quad H_i = \mathcal{H}(V(g_i) \cap Y, \sigma).$$

If for each i we apply Lemma 3.5 to Y, g_i and σ, we can find $\alpha_2 > 0$ such that, for all integers $k > \alpha_2$, the functions g_i and $T^k g_i$ have the same sign on $Y \setminus (H_i \cup \{O\})$.

Since $V(g_i) \cap Y \subseteq X \cap Y$ for each i, then $\bigcup H_i \subseteq \mathcal{H}(X \cap Y, \sigma)$, and therefore $Y' \setminus \{O\} \subseteq \bigcap_i (Y \setminus (H_i \cup \{O\}))$. In particular

$$Y' \setminus \{O\} \subseteq \{T^k g_1 > 0, \dots, T^k g_l > 0\}.$$

From now on, assume that $k > \alpha_2$. We will get the result by replacing f with a suitable truncation of it in the presentation of $T_k(A)$. We will denote by $B(x, r)$ the open ball centered at x of radius r.

By the last inclusion, the distance $d(x, b(B_k))$ is subanalytic and positive on $Y' \setminus \{O\}$ so, by Proposition 2.7, there exists $\nu > 0$ (and we can assume $\nu > s$) such that $d(x, b(B_k)) > \|x\|^\nu$ for all x in $Y' \setminus \{O\}$. As a consequence

$$B(x, \|x\|^\nu) \subseteq \{T^k g_1 > 0, \dots, T^k g_l > 0\}.$$

Following [FFW3] consider the real-valued function

$$\Lambda f(x) = \begin{cases} 0 & \text{if rk } d_x f < n - d \\ \inf_{v \perp \ker d_x f, \|v\| = 1} \|d_x f(v)\| & \text{if rk } d_x f = n - d \end{cases}.$$

Observe that $\Lambda f(x)$ is subanalytic, continuous and positive where f is submersive, in particular on $Y' \setminus \{O\}$. Hence, again by Proposition 2.7, there exists $\beta > 0$ such that $\Lambda f(x) > \|x\|^\beta$ for all x in $Y' \setminus \{O\}$.

Consider the subanalytic set $W = \{(x,y) \in Y' \times \Omega \mid \Lambda f(y) \geq \|x\|^\beta\}$ and let $W_0 = \{(x,y) \in Y' \times \Omega \mid \Lambda f(y) = \|x\|^\beta\}$; then the set $\{(x,x) \mid x \in Y' \setminus \{O\}\}$ is contained in the open subanalytic set $W \setminus W_0$.

The function $\varphi \colon Y' \setminus \{O\} \to \mathbb{R}$ defined by $\varphi(x) = d((x,x), W_0)$ is subanalytic and positive. Then again by Proposition 2.7 there exists $\tau > 0$ (and we can assume $\tau > \nu$) such that $\varphi(x) > \|x\|^\tau$ on $Y' \setminus \{O\}$. Then for all $x \in Y' \setminus \{O\}$ and for all $y \in B(x, \|x\|^\tau)$ we have

$$\|(x,y) - (x,x)\| = \|y - x\| < \|x\|^\tau < \varphi(x).$$

Hence $(x,y) \in W \setminus W_0$, i.e. for all x in $Y' \setminus \{O\}$ and for all $y \in B(x, \|x\|^\tau)$ we have $\Lambda f(y) > \|x\|^\beta$. In particular $\Lambda f(y) > 0$ and hence $d_y f$ is surjective for all $y \in B(x, \|x\|^\tau)$.

Let h be an integer such that $h > \beta + 1$ and let $\widetilde{f}(x) = T^h f(x)$.

Then $T^{h-1} d_y f = d_y \widetilde{f}$; thus we have that $\|d_y f - d_y \widetilde{f}\| \leq \|y\|^{h-1}$ for all y near to O, where we consider $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n-d})$ endowed with the standard norm

$$\|L\| = \max_{u \neq 0} \frac{\|L(u)\|}{\|u\|}.$$

Thus by [FFW3, Proposition 3.3] we have

$$|\Lambda f(y) - \Lambda \widetilde{f}(y)| \leq \|y\|^{h-1}.$$

Claim. For $x \in Y' \setminus \{O\}$ and for $y \in B(x, \|x\|^\tau)$, we have

$$\Lambda \widetilde{f}(y) \geq \|x\|^{\beta+1}.$$

To see this, assume for a contradiction that there exist a sequence $x_i \in Y' \setminus \{O\}$ converging to O and a sequence $y_i \in B(x_i, \|x_i\|^\tau)$ such that $\Lambda \widetilde{f}(y_i) < \|x_i\|^{\beta+1}$. Thus we have

$$\frac{\Lambda f(y_i) - \Lambda \widetilde{f}(y_i)}{\|x_i\|^\beta} > \frac{\|x_i\|^\beta - \|x_i\|^{\beta+1}}{\|x_i\|^\beta} = 1 - \|x_i\|.$$

On the other hand

$$\frac{\Lambda f(y_i) - \Lambda \widetilde{f}(y_i)}{\|x_i\|^\beta} \leq \frac{\|y_i\|^{h-1}}{\|x_i\|^\beta} \leq \frac{(\|y_i - x_i\| + \|x_i\|)^{h-1}}{\|x_i\|^\beta}$$
$$= \left(\frac{\|y_i - x_i\|}{\|x_i\|^q} + \|x_i\|^{1-q} \right)^{h-1} \leq \left(\|x_i\|^{\tau - q} + \|x_i\|^{1-q} \right)^{h-1}$$

where $q = \frac{\beta}{h-1}$. Since $\tau > 1$ and $q < 1$, we have that

$$\frac{\Lambda f(y_i) - \Lambda \widetilde{f}(y_i)}{\|x_i\|^\beta}$$

converges to 0, which is a contradiction. So the Claim is proved.

Consequently, for all $x \in Y' \setminus \{O\}$ the map $\widetilde{f}$ is a submersion on $B(x, \|x\|^\tau)$. Hence, using [FFW3, Lemma 3.5], we get $\widetilde{f}(B(x, \|x\|^\tau)) \supseteq B(\widetilde{f}(x), \|x\|^\lambda)$ with $\lambda = \beta + 1 + \tau$.

Observe that if $x \in Y' \setminus \{O\}$, we have that

$$\lim_{x \to O} \frac{\|\widetilde{f}(x)\|}{\|x\|^h} = \lim_{x \to O} \frac{\|\widetilde{f}(x) - f(x)\|}{\|x\|^h} = 0.$$

So, for any $h \geq \lambda$ and $x \in Y'$, the point O belongs to $B(\widetilde{f}(x), \|x\|^{\lambda})$ and hence there exists $y \in B(x, \|x\|^{\tau})$ such that $\widetilde{f}(y) = O$.

Since $\tau > \nu > s$, then $y \in B(x, \|x\|^{\nu})$ so that $T^k g_i(y) > 0$ for all i, i.e. $y \in T_k^h(A)$; hence $Y' \setminus \{O\} \subseteq \mathcal{H}(T_k^h(A), \lambda)$. Then by Proposition 2.5 we have $Y' \leq_s T_k^h(A)$ and hence, since $Y' \sim_s Y$, we have that

$$\overline{A \setminus (\Sigma(f) \cup b(A))} = Y \leq_s T_k^h(A).$$

Therefore, taking $h_0 = \max\{\rho\sigma, \lambda\}$ and $k_0 = \max\{\alpha_1, \alpha_2\}$, we have the thesis.

(3) The previous argument shows that, for all $h \geq h_0$ and $k \geq k_0$, there exist points $y \in V(T^h f)$ arbitrarily near to O where $T^h f$ is submersive and such that $T^k g_i(y) > 0$ for all i. Hence $\dim_O T_k^h(A) = d$. □

Theorem 3.8. *Let A be a closed semianalytic subset of Ω with $O \in A$. Then for any $s \geq 1$ there exists a closed semialgebraic set $S \subseteq \Omega$ such that $A \sim_s S$ and $\dim_O S = \dim_O A$.*

Proof. We will prove the thesis by induction on $d = \dim_O A$.

If $d = 0$ the result holds trivially. So let $d \geq 1$ and assume that the result holds for all semianalytic germs of dimension less that d.

By Lemma 3.2, by Proposition 2.2 and by the inductive hypothesis, we can assume that

$$A = \{x \in \Omega \mid f(x) = O, g_i(x) \geq 0, i = 1, \dots, l\}$$

with $f = (f_1, \dots, f_p)$ such that $V(f)$ is irreducible, $V(f)$ is the minimal analytic variety containing A and $f_1, \dots, f_p$ generate the ideal $I(V(f))$. In particular $\dim_O(\Sigma_{n-d}(f) \cap A) < d$. Moreover, removing from the previous presentation of A the inequalities $g_i(x) \geq 0$ where g_i vanishes identically on A (if any), we can assume that $\dim_O b(A) < d$.

If $p = n - d$, the thesis follows easily by using Lemma 3.7. In general p can be larger than $n-d$; in this case we introduce a semianalytic set $\widetilde{A}$ of dimension d which is s-equivalent to A and which satisfies the hypotheses of Lemma 3.7. In order to prove the thesis it will be sufficient to approximate $\widetilde{A}$ by means of a semialgebraic set having the same dimension.

Denote by Π the set of surjective linear maps from $\mathbb{R}^p$ to $\mathbb{R}^{n-d}$ and consider the smooth map $\Phi\colon (\mathbb{R}^n - V(f)) \times \Pi \to \mathbb{R}^{n-d}$ defined by $\Phi(x, \pi) = (\pi \circ f)(x)$ for all $x \in \mathbb{R}^n - V(f)$ and $\pi \in \Pi$.

The map Φ is transverse to $\{O\}$: namely the partial Jacobian matrix of Φ with respect to the variables in Π (considered as an open subset of $\mathbb{R}^{p(n-d)}$) is the $(n-d) \times p(n-d)$ matrix

$$\begin{bmatrix} f(x) & 0 & 0 & \dots & 0 \\ 0 & f(x) & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & f(x) \end{bmatrix};$$

thus, for all $x \in \mathbb{R}^n - V(f)$ and for all $\pi \in \Pi$ the Jacobian matrix of Φ has rank $n - d$.

As a consequence, by a well-known result of singularity theory (see for instance [BK, Lemma 3.2]), we have that the map $\Phi_\pi\colon \mathbb{R}^n - V(f) \to \mathbb{R}^{n-d}$ defined by $\Phi_\pi(x) = \Phi(x, \pi) = (\pi \circ f)(x)$ is transverse to $\{O\}$ for all π outside a set $\Gamma \subset \Pi$ of measure zero and hence $\pi \circ f$ is a submersion on $V(\pi \circ f) \setminus V(f)$ for all such π.

Let $x \in V(f)$ be a point at which f has rank $n-d$. Then there is an open dense set $U \subset \Pi$ such that for all $\pi \in U$ the map $\pi \circ f$ is a submersion at x, and hence off some subvariety of $V(f)$ of dimension less than d.

Thus, if we choose $\pi_0 \in (\Pi \setminus \Gamma) \cap U$, the map $F = \pi_0 \circ f$ has $n-d$ components, $\Sigma(F) \cap V(F) \subseteq V(f) \subseteq V(F)$, $\dim_O V(F) = d$ and $\dim_O(\Sigma(F) \cap V(F)) < d$. In particular $V(f)$ is an irreducible component of $V(F)$.

For each $m \in \mathbb{N}$ denote $\widetilde{A}_m = \{F(x) = 0, \|x\|^{2m} - \|f(x)\|^2 \geq 0, g_i(x) \geq 0, i = 1, \ldots, l\}$.

Since $A \subseteq \widetilde{A}_m \subseteq V(F)$, we have that $A \leq_s \widetilde{A}_m$ and $\dim_O \widetilde{A}_m = d$.

We claim that there exists m such that $\widetilde{A}_m \sim_s A$; to show that it is sufficient to prove that there exists m such that $\widetilde{A}_m \leq_s A$. Namely, let $B = \{g_i(x) \geq 0, i = 1, \ldots, l\}$. Since $V(\|f\|) \cap B = V(d(x,A)) \cap B$, by Proposition 2.7 there exists q such that $d(x,A)^q \leq \|f(x)\|$ for all $x \in B$. Let $m > sq$. Then $d(x,A) \leq \|f(x)\|^{\frac{1}{q}} \leq \|x\|^{\frac{m}{q}}$ for all $x \in \widetilde{A}_m$, i.e. $\widetilde{A}_m \subseteq \mathcal{H}(A, \frac{m}{q})$ and hence $\widetilde{A}_m \leq_s A$.

Fix m as above and let $\widetilde{A} = \widetilde{A}_m$. Also let $\widetilde{X} = (\Sigma(F) \cap \widetilde{A}) \cup b(\widetilde{A})$.

Observe that $b(\widetilde{A}) \cap A = b(A)$ and so $\widetilde{X} \cap A = (\Sigma(F) \cap A) \cup b(A)$.

Denote $K = \widetilde{X} \cap (\widetilde{A} \setminus A)$ so that $\widetilde{X} = (\widetilde{X} \cap A) \cup K$.

By Lemma 3.7 there exist positive integers h, k such that

$$\overline{\widetilde{A} \setminus \widetilde{X}} \leq_s T_k^h(\widetilde{A}) \leq_s \widetilde{A} \qquad \text{and} \qquad \dim_O T_k^h(\widetilde{A}) = d.$$

Since $\dim_O(\widetilde{X} \cap A) < d$, by induction there exists a semialgebraic set S_0 such that $S_0 \sim_s \widetilde{X} \cap A$ and $\dim_O S_0 < d$. Moreover, since $A \subseteq \overline{\widetilde{A} \setminus K} \subseteq \widetilde{A}$, we have that $\overline{\widetilde{A} \setminus K} \sim_s \widetilde{A}$.

Then

$$\widetilde{A} \sim_s \overline{\widetilde{A} \setminus K} = \overline{\widetilde{A} \setminus \widetilde{X}} \cup (\widetilde{X} \cap A) \leq_s T_k^h(\widetilde{A}) \cup S_0 \leq_s \widetilde{A} \cup (\widetilde{X} \cap A) = \widetilde{A}$$

so we can choose $S = T_k^h(\widetilde{A}) \cup S_0$. □

From Theorem 3.8 and from Theorem 2.3 we immediately obtain:

Theorem 3.9. *Let A be a closed semianalytic subset of Ω of codimension ≥ 1 with $O \in A$. Then for any $s \geq 1$ there exists an algebraic set $Y \subset \mathbb{R}^n$ such that $A \sim_s Y$.*

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