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EIGENVALUES FOR A FOURTH ORDER ELLIPTIC PROBLEM

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ABSTRACT. We study the fourth order nonlinear eigenvalue problem with a $p(x)$-biharmonic operator

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda w(x) f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p \in C(\overline{\Omega})$ with $p(x) > 1$ on $\overline{\Omega}$, $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator, and $\lambda > 0$ is a parameter. Under some appropriate conditions on the functions p, a, w, f, we prove that there exists $\overline{\lambda} > 0$ such that any $\lambda \in (0, \overline{\lambda})$ is an eigenvalue of the above problem. Our analysis mainly relies on variational arguments based on Ekeland's variational principle and some recent theory on the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$.

1. INTRODUCTION AND PRELIMINARY RESULTS

Differential equations and variational problems with nonstandard $p(x)$-growth conditions have many applications in mathematical physics such as in the modelling of electrorheological fluids and of other phenomena related to image processing, elasticity, and the flow in porous media ([11, 20, 23]). Such problems have been studied by many authors in the literature. The reader is referred to [1, 2, 5–8, 12, 13, 18, 19] for some recent work on this subject. It is well known that problems with $p(x)$-growth conditions possess more complicated nonlinearities than the constant case. For instance, it is not homogeneous, and thus many techniques which can be applied when $p(x)$ is a positive constant fail to work in this new setting.

In this paper, we are concerned with the existence of weak solutions of the following fourth order nonlinear elliptic equation with a $p(x)$-biharmonic operator:

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda w(x) f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a bounded domain with smooth boundary, $p \in C(\overline{\Omega})$ with $p(x) > 1$ on $\overline{\Omega}$, $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the so-called $p(x)$-biharmonic operator, $\lambda > 0$ is a parameter, $a \in C(\overline{\Omega})$ is nonnegative, $f \in C(\mathbb{R})$, and $w \in L^{r(x)}(\Omega)$ for some $r \in C(\overline{\Omega})$.

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Several variations of problem (1.1) have been studied in the literature. For instance, Ayoujil and El Amrouss [1, 2] studied the problem

$$\begin{cases} \Delta^2_{p(x)} u = \lambda |u|^{q(x)-2} u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In [1], the case $p(x) = q(x)$ was considered. By the Ljusternik-Schnirelmann principle on C^1-manifolds, the existence of a sequence of eigenvalues was proved. Let Λ be the set of all nonnegative eigenvalues. It was shown that $\sup \Lambda = \infty$. Sufficient conditions were also found to guarantee that $\inf \Lambda = 0$. We comment that when $p(x) = p > 1$ (a positive constant), we always have $\inf \Lambda > 0$. In [2], using the Mountain Pass Theorem and Ekeland's variational principle, several existence criteria for eigenvalues were established for problem (1.2) when $p(x) \neq q(x)$. Recently, the existence of three weak solutions of the problem

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = f(x,u) + \lambda g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

was investigated in [5] by using Ricceri's variational principle ([6, 21]). In a recent paper [13], the present author studied the existence of weak solutions to the problem

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda \left(b(x)u^{\gamma(x)-1} - c(x)u^{\beta(x)-1} \right) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

by applying variational arguments.

We also want to point out that when $p(x)$ is a positive constant, a number of variations of problem (1.2) have been investigated in the literature. See, for example, [3, 9, 10, 15, 16] and the references therein.

In this paper, by using simple variational arguments based on Ekeland's variational principle and the theory of the generalized Lebesgue–Sobolev spaces, we study the existence of a continuous family of eigenvalues for problem (1.1) in a neighborhood of the origin. More precisely, under some appropriate conditions, we show that there exists $\overline{\lambda} > 0$ such that any $\lambda \in (0, \overline{\lambda})$ is an eigenvalue of problem (1.1). For more applications of Ekeland's variational principle to other problems, see, for example, [2, 12, 19]. Our result is partly motivated by these nice papers.

In the remainder of this section, we recall some definitions and basic properties of variable spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, where Ω is given as in problem (1.1). The presentation here can be found in, for example, [1, 4, 5, 7, 8, 14, 22].

Let

$$C_+(\overline{\Omega}) = \left\{ h \ : \ h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ on } \overline{\Omega} \right\}.$$

Throughout this paper, for any $h \in C(\overline{\Omega})$, we use the notation

$$h^+ := \max_{x \in \overline{\Omega}} h(x) \quad \text{and} \quad h^- := \min_{x \in \overline{\Omega}} h(x).$$

Let $p \in C_+(\overline{\Omega})$ be fixed. We define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u \ : \ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

Then, equipped with the so-called Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(x)}(\Omega)$ is a separable and reflexive Banach space. It is clear that when $p(x) = p > 1$ (a positive constant), the space $L^{p(x)}(\Omega)$ becomes the well-known Lebesgue space $L^p(\Omega)$ and the norm $|u|_{p(x)}$ reduces to the stand norm $|u|_p = \left(\int_\Omega |u|^p\right)^{1/p}$ in $L^p(\Omega)$.

As in the constant exponent case, for any positive integer k, let

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) \ : \ D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \le k\right\},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$, and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_N} x_N}$. Then, equipped with the norm

$$||u||_{k,p(x)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)},$$

$W^{k,p(x)}(\Omega)$ is also a separable and reflexive Banach space. We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Throughout this paper, we let

$$X = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega).$$

Define a norm $||\cdot||_X$ of X by

$$||u||_X = ||u||_{1,p(x)} + ||u||_{2,p(x)}.$$

Then, endowed with $||\cdot||_X$, X is a separable and reflexive Banach space. Moreover, by [22, Theorem 4.4], $||u||$ and $|\Delta u|_{p(x)}$ are two equivalent norms of X.

Let

$$||u||_a = \inf\left\{\lambda > 0 \ : \ \int_\Omega \left(\left|\frac{\Delta u}{\lambda}\right|^{p(x)} + a(x)\left|\frac{u}{\lambda}\right|^{p(x)}\right) dx \le 1\right\} \quad \text{for } u \in X.$$

In view of $a^- \ge 0$, it is easy to see that $||u||_a$ is equivalent to the norms $||u||$ and $|\Delta u|_{p(x)}$ in X. In this paper, for the convenience of discussion, we use the norm $||u||_a$ for X.

Proposition 1.1 ([5, Proposition 2.3]). *Let* $\rho_a(u) = \int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right) dx$ *for* $u \in X$. *Then, we have*

(a) *if* $||u||_a \ge 1$, *then* $||u||_a^{p^-} \le \rho_a(u) \le ||u||_a^{p^+}$;
(b) *if* $||u||_a \le 1$, *then* $||u||_a^{p^+} \le \rho_a(u) \le ||u||_a^{p^-}$.

Proposition 1.2 ([7, Propositions 2.4 and 2.5] or [14, Theorem 2.1 and Corollary 2.7]). *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $1/p(x) + 1/q(x) = 1$. *Moreover, for* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have the following inequality of Hölder type:*

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

Moreover, if $h_i \in C_+(\overline{\Omega})$ *with* $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$*, then, for* $u \in L^{h_1(x)}(\Omega)$*,* $v \in L^{h_2(x)}(\Omega)$*, and* $w \in L^{h_3(x)}(\Omega)$*, we have*

$$\left|\int_\Omega uvw dx\right| \le \left(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-}\right) |u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)} \le 3|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}.$$

Proposition 1.3 ([4, Lemma 2.1]). *Let* $q \in L^\infty(\Omega)$ *be such that* $1 \le p(x)q(x) \le \infty$ *for a.e.* $x \in \Omega$*. Let* $u \in L^{p(x)}(\Omega)$*,* $u \ne 0$*. Then, we have*

(a) *if* $|u|_{p(x)q(x)} \le 1$*, then* $|u|^{q^+}_{p(x)q(x)} \le ||u|^{q(x)}|_{p(x)} \le |u|^{q^-}_{p(x)q(x)}$*;*
(b) *if* $|u|_{p(x)q(x)} \ge 1$*, then* $|u|^{q^-}_{p(x)q(x)} \le ||u|^{q(x)}|_{p(x)} \le |u|^{q^+}_{p(x)q(x)}$*.*

For any $x \in \overline{\Omega}$, let

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2}, \\ \infty & \text{if } p(x) \ge \frac{N}{2}. \end{cases}$$

Proposition 1.4 ([1, Theorem 3.2]). *Assume that* $q \in C_+(\overline{\Omega})$ *satisfy* $q(x) < p^*(x)$ *on* $\overline{\Omega}$*. Then, there exists a continuous and compact embedding* $X \hookrightarrow L^{q(x)}(\Omega)$*.*

In the next section, we state our main result and give its proof.

2. Main result

We need the following assumptions.

(H1) There exist $k_1 \ge k_2 > 0$, $0 < \delta < 1$ and $q_1, q_2, r \in C_+(\overline{\Omega})$ such that

$$1 < q_1(x) \le q_2(x) < p(x) \le \frac{N}{2} < r(x) \quad \text{on } \overline{\Omega} \tag{2.1}$$

$$0 \le tf(t) \le k_1|t|^{q_1(x)} \quad \text{for } t \in \mathbb{R}, \tag{2.2}$$

and

$$tf(t) \ge k_2|t|^{q_2(x)} \quad \text{for } t \in [-\delta, \delta]. \tag{2.3}$$

(H2) $w \in L^{r(x)}(\Omega)$ and there exists a subset $\Omega_1 \subset \Omega$ with meas$(\Omega_1) > 0$ such that $w(x) > 0$ for $x \in \Omega_1$, where meas$(\cdot)$ denotes the Lebesgue measure of a set.

Remark 2.1. Regarding the condition (H1), we make the following comments.

(a) Let

$$r'(x) = \frac{r(x)}{r(x)-1} \quad \text{and} \quad s_i(x) = \frac{r(x)q_i(x)}{r(x)-q_i(x)}, \ i = 1, 2.$$

Then, $1/r(x) + 1/r'(x) = 1$ and it is easy to check that (2.1) implies

$$r'(x)q_i(x) < p^*(x) \quad \text{and} \quad s_i(x) < p^*(x) \quad \text{for } i = 1, 2 \text{ and } x \in \overline{\Omega}.$$

Thus, by Proposition 1.4, the embeddings $X \hookrightarrow L^{r'(x)q_i(x)}(\Omega)$ and $X \hookrightarrow L^{s_i(x)}(\Omega)$, $i = 1, 2$, are continuous and compact.

(b) There are many functions f satisfying both (2.2) and (2.3). For instance, it is easy to check that the following are several simple examples of such functions $f(t)$:

$$f(t) = |t|^{q(x)-2}t, \quad t \in \mathbb{R},$$

or

$$f(t) = t|\sin t^{q(x)-2}|, \quad t \in \mathbb{R},$$

or

$$f(t) = |t|^{\gamma(x)-2}t - |t|^{\beta(x)-2}t, \quad t \in \mathbb{R},$$

where $q, \beta, \gamma \in C_+(\overline{\Omega})$ satisfy $q(x) < p(x)$ and $\gamma(x) < \beta(x) < p(x)$ on $\overline{\Omega}$.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there exists $u \in X \setminus \{0\}$ such that

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \int_\Omega a(x)|u|^{p(x)-2}uv dx - \lambda \int_\Omega w(x)f(u)v dx = 0$$

for all $v \in X$. When λ is an eigenvalue of problem (1.1), the corresponding function $u \in X \setminus \{0\}$ is a weak solution of problem (1.1).

We now state our main theorem.

Theorem 2.1. *Assume that* (H1) *and* (H2) *hold. Then, there exists* $\overline{\lambda} > 0$ *such that any* $\lambda \in (0, \overline{\lambda})$ *is an eigenvalue of problem* (1.1).

Remark 2.2. By Remark 2.1 (b), we see that Theorem 2.1 can be applied to problems (1.2) and (1.3) with $b(x) = c(x)$ in Ω as well as to some other problems.

In the rest of this section, we assume that (H1) and (H2) hold and we will prove Theorem 2.1. To this end, define functionals $\Phi, \Psi, I_\lambda : X \to R$ by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx,$$

$$\Psi(u) = \int_\Omega w(x)F(u)dx,$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where $F(t) = \int_0^t f(s)ds$.

Lemma 2.1. *We have the following:*

(a) Φ *is weakly lower semicontinuous,* $\Phi \in C^1(X, \mathbb{R})$*, and*

$$\langle \Phi'(u), v \rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \int_\Omega a(x)|u|^{p(x)-2}uv dx$$

for all $u, v \in X$.

(b) $\Phi'(u) : X \to X^*$ *is of type* (S_+)*; i.e., if* $u_n \rightharpoonup u$ *and* $\liminf_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle \le 0$*, then* $u_n \to u$*, where* X^* *is the dual space of* X*,* $\to$ *and* $\rightharpoonup$ *denote the strong and weak convergence, respectively.*

(c) Ψ *is weakly lower semicontinuous,* $\Psi \in C^1(X, \mathbb{R})$*, and*

$$\langle \Psi'(u), v \rangle = \int_\Omega w(x)f(u)v dx$$

for all $u, v \in X$.

Proof. Parts (a) and (b) follow from [5, Proposition 2.5]. Part (c) can be proved by a standard argument, and hence the details are omitted. ☐

Remark 2.3. By Lemma 2.1 (a) and (c), I_λ is (weakly) lower semicontinuous, $I_\lambda \in C^1(X,\mathbb{R})$, and

$$\langle I'_\lambda(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \int_\Omega a(x)|u|^{p(x)-2}uvdx - \lambda \int_\Omega w(x)f(u)vdx$$

for all $v \in X$. Thus, u is a critical point of I_λ if and only if u is a weak solution of problem (1.1).

Note from Remark 2.1 (a) that the embedding $X \hookrightarrow L^{r'(x)q_1(x)}(\Omega)$ is continuous. Then, there exists a constant $C > 1$ such that

$$|u|_{r'(x)q_1(x)} \leq C||u||_a \quad \text{for any } u \in X. \tag{2.4}$$

Lemma 2.2. *For any $\varrho \in (0, 1/C)$, there exist $\overline{\lambda} > 0$ and $\kappa > 0$ such that $I_\lambda(u) \geq \kappa$ for any $\lambda \in (0, \overline{\lambda})$ and $u \in X$ with $||u||_a = \varrho$,*

Proof. Let $\varrho \in (0, 1/C)$ be fixed. Then, $\varrho < 1$, and from (2.4), it is clear that

$$|u|_{r'(x)q_1(x)} < 1 \quad \text{for any } u \in X \text{ and } ||u||_a = \varrho.$$

By (2.2), we see that

$$0 \leq F(t) \leq \frac{1}{q_1(x)} k_1 |t|^{q_1(x)} \quad \text{for } t \in \mathbb{R}.$$

Thus, for $u \in X$ with $||u||_a = \varrho$, from Propositions 1.1–1.3 and (2.4), it follows that

$$\begin{aligned}
I_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right) dx - \lambda \int_\Omega |w(x)| F(u) dx \\
&\geq \frac{1}{p^+}||u||_a^{p^+} - \frac{1}{q_1^-}\lambda k_1 \int_\Omega |w(x)|\ |u|^{q_1(x)} dx \\
&\geq \frac{1}{p^+}||u||_a^{p^+} - \frac{2}{q_1^-}\lambda k_1 |w|_{r(x)} \left||u|^{q_1(x)}\right|_{r'(x)} \\
&\geq \frac{1}{p^+}||u||_a^{p^+} - \frac{2}{q_1^-}\lambda k_1 |w|_{r(x)} |u|^{q_1^-}_{r'(x)q(x)} \\
&\geq \frac{1}{p^+}||u||_a^{p^+} - \frac{2}{q_1^-}\lambda k_1 C^{q_1^-} |w|_{r(x)} ||u||_a^{q_1^-} \\
&= \frac{1}{p^+}\varrho^{p^+} - \frac{2}{q_1^-}\lambda k_1 C^{q_1^-} |w|_{r(x)} \varrho^{q_1^-} \\
&= \varrho^{q_1^-}\left(\frac{1}{p^+}\varrho^{p^+ - q_1^-} - \frac{2}{q_1^-}\lambda k_1 C^{q_1^-} |w|_{r(x)}\right).
\end{aligned} \tag{2.5}$$

Hence, if we let

$$\overline{\lambda} = \frac{\varrho^{p^+ - q_1^-}}{2\varrho^+} \frac{q_1^-}{2k_1 C^{q_1^-} |w|_{r(x)}}, \tag{2.6}$$

then, for any $\lambda \in (0, \overline{\lambda})$ and $u \in X$ with $||u||_a = \varrho$, there exists $\kappa = \frac{\varrho^{p^+}}{2p^+} > 0$ such that $I_\lambda(u) \geq \kappa$. This completes the proof of the lemma. ☐

Lemma 2.3. *There exists $\phi \in X$ such that $\phi \geq 0$, $\phi \neq 0$, and $I_\lambda(t\phi) < 0$ for $t > 0$ small enough.*

Proof. Let $\Omega_1 \subset \Omega$ be given as in (H2). Then, (2.1) implies that $q_2(x) < p(x)$ on $\overline{\Omega}_1$. If we let $\hat{q} = \min_{x\in\overline{\Omega}_1} q_2(x)$ and $\hat{p} = \min_{x\in\overline{\Omega}_1} p(x)$, then there exists $\epsilon_0 > 0$ such that $\hat{q} + \epsilon_0 < \hat{p}$. Moreover, since $q_2 \in C(\overline{\Omega}_1)$, there exists an open set $\Omega_2 \subset \Omega_1$ such that meas$\,(\Omega_2) > 0$ and $|q_2(x) - \hat{q}| < \epsilon_0$ for $x \in \Omega_2$. Thus, $q_2(x) < \hat{q} + \epsilon_0 < \hat{p}$ in Ω_2.

Let $\phi \in C_0^\infty(\Omega)$ be nontrivial such that supp$\,(\phi) \subset \Omega_2 \subset \Omega_1$, $\phi \geq 0$, and $\phi \neq 0$ in Ω_2. Note from (2.3) that

$$F(t) \geq \frac{1}{q_2(x)} k_2 |t|^{q_2(x)} \quad \text{for } t \in [-\delta, \delta].$$

Then, for $0 < t < \min\left\{1, \frac{\delta}{\max_{x\in\Omega_2} \phi(x)}\right\}$, we have

$$\begin{aligned} I_\lambda(t\phi) &= \int_\Omega \frac{1}{p(x)} \left(|\Delta(t\phi)|^{p(x)} + a(x)|t\phi|^{p(x)}\right) dx - \lambda \int_\Omega w(x) F(t\phi) dx \\ &= \int_{\Omega_2} \frac{t^{p(x)}}{p(x)} \left(|\Delta\phi|^{p(x)} + a(x)|\phi|^{p(x)}\right) dx - \lambda \int_{\Omega_2} w(x) F(t\phi) dx \\ &\leq \frac{t^{\hat{p}}}{p^-} \int_{\Omega_2} \left(|\Delta\phi|^{p(x)} + a(x)|\phi|^{p(x)}\right) dx - \lambda k_2 \int_{\Omega_2} \frac{t^{q_2(x)}}{q_2(x)} w(x)|\phi|^{q_2(x)} dx \\ &\leq \frac{t^{\hat{p}}}{p^-} \int_{\Omega_2} \left(|\Delta\phi|^{p(x)} + a(x)|\phi|^{p(x)}\right) dx - \frac{\lambda k_2 t^{\hat{q}+\epsilon_0}}{q_2^+} \int_{\Omega_2} w(x)|\phi|^{q_2(x)} dx. \end{aligned}$$

Hence, $I_\lambda(t\phi) < 0$ for $0 < t < \eta^{1/(\hat{p}-\hat{q}-\epsilon_0)}$, where

$$0 < \eta < \min\left\{1, \frac{\delta}{\max_{x\in\Omega_2} \phi(x)}, \frac{\lambda k_2 p^- \int_{\Omega_2} w(x)|\phi|^{q_2(x)} dx}{q_2^+ \int_{\Omega_2} \left(|\Delta\phi|^{p(x)} + a(x)|\phi|^{p(x)}\right) dx}\right\}.$$

Here, we point out that

$$\int_{\Omega_2} \left(|\Delta\phi|^{p(x)} + a(x)|\phi|^{p(x)}\right) dx > 0.$$

In fact, if this is not true, then

$$\int_{\Omega} \left(|\Delta\phi|^{p(x)} + a(x)|\phi|^{p(x)}\right) dx = 0.$$

By Proposition 1.1, we have $||\phi||_a = 0$, and so $\phi \equiv 0$ in Ω, which is a contradiction. This completes the proof of the lemma. □

The proof of our main result will utilize Ekeland's variation principle, which is recalled below.

Lemma 2.4 ([17, Theorem 4.1]). *Let M be a complete metric space and let $J : M \to (-\infty, \infty]$ be a lower semicontinuous functional, bounded from below, and not identically equal to ∞. Let $\epsilon > 0$ be given and $z \in M$ be such that*

$$J(z) \leq \inf_M J + \epsilon.$$

Then, there exists $v \in M$ such that

$$J(v) \le J(z) \le \inf_M J + \epsilon,$$

$$d(z, v) \le 1,$$

and for any $u \ne v$ in M,

$$J(v) < J(u) + \epsilon d(v, u),$$

where $d(\cdot, \cdot)$ denotes the distance between two elements in M.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\overline{\lambda}$ be defined by (2.6) and $\lambda \in (0, \overline{\lambda})$. By Lemma 2.2, we have

$$\inf_{\partial B_\varrho(0)} I_\lambda > 0, \tag{2.7}$$

where $B_\varrho(0)$ is the ball in X centered at the origin and of radius ϱ, and $\partial B_\varrho(0)$ is the boundary of $B_\varrho(0)$. For any $u \in B_\varrho(0)$, by an argument similar to those used in obtaining (2.5), we can derive that

$$I_\lambda(u) \ge \frac{1}{p^+}||u||_a^{p^+} - \frac{2}{q_1^-}\lambda k_1 C^{q_1^-}|w|_{r(x)}||u||_a^{q_1^-}. \tag{2.8}$$

Note from Lemma 2.3 that there exists $\phi \in X$ such that $I_\lambda(t\phi) < 0$ for $t > 0$ small enough. Then, from (2.7) and (2.8), it follows that

$$-\infty < \underline{c}_\lambda := \inf_{\overline{B_\varrho(0)}} I_\lambda < 0.$$

Let

$$0 < \epsilon < \inf_{\partial B_\varrho(0)} I_\lambda - \underline{c}_\lambda.$$

Applying Lemma 2.4 to the functional $I_\lambda : \overline{B_\varrho(0)} \to \mathbb{R}$, we see that there exists $u_\epsilon \in \overline{B_\varrho(0)}$ such that

$$\underline{c}_\lambda \le I_\lambda(u_\epsilon) \le \underline{c}_\lambda + \epsilon \tag{2.9}$$

and

$$I_\lambda(u_\epsilon) < I_\lambda(u) + \epsilon||u - u_\epsilon||_a \quad \text{for } u \ne u_\epsilon. \tag{2.10}$$

Since $I_\lambda(u_\epsilon) \le \underline{c}_\lambda + \epsilon < \inf_{\partial B_\varrho(0)} I_\lambda$, we have $u_\epsilon \in B_\varrho(0)$.

Now, define a functional $J_\lambda : \overline{B_\varrho(0)} \to \mathbb{R}$ by

$$J_\lambda(u) = I_\lambda(u) + \epsilon||u - u_\epsilon||_a.$$

Obviously, by (2.10), u_ϵ is a minimum point of J_λ, and so

$$\frac{J_\lambda(u_\epsilon + tv) - J_\lambda(u_\epsilon)}{t} \ge 0$$

for $t > 0$ small enough and all $v \in B_\varrho(0)$. Then,

$$\frac{I_\lambda(u_\epsilon + tv) - I_\lambda(u_\epsilon)}{t} + \epsilon||v|| \ge 0$$

for $t > 0$ small enough and all $v \in B_\varrho(0)$. Letting $t \to 0$, we obtain

$$\langle I'_\lambda(u_\epsilon), v\rangle + \epsilon||v|| \le 0 \quad \text{for all } v \in B_\varrho(0).$$

Hence, $||I'_\lambda(u_\epsilon)||_{X^*} \leq \epsilon$. This, together with (2.9), implies that there exists a sequence $\{u_n\} \subset B_\varrho(0)$ such that

$$I_\lambda(u_n) \to \underline{c}_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0. \tag{2.11}$$

Obviously, $\{u_n\}$ is bounded in X. Then, by the reflexivity of X, there exists $u_0 \in X$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in X. Note that

$$\begin{aligned} |\langle I'_\lambda(u_n), u_n - u_0\rangle| &\leq |\langle I'_\lambda(u_n), u_n\rangle| + |\langle I'_\lambda(u_n), u_0\rangle| \\ &\leq ||\langle I'_\lambda(u_n)||\ ||u_n|| + ||\langle I'_\lambda(u_n)||\ ||u_0||. \end{aligned}$$

Then, from (2.11) and the fact that $\{u_n\}$ is bounded in X, it follows that

$$\lim_{n\to\infty} \langle I'_\lambda(u_n), u_n - u_0\rangle = 0. \tag{2.12}$$

Now, we claim that

$$\lim_{n\to\infty} \langle \Psi'(u_n), u_n - u_0\rangle = 0. \tag{2.13}$$

In fact, from (2.2) and Propositions 1.2 and 1.3, we have

$$\begin{aligned} |\langle \Psi'(u_n), u_n - u_0\rangle| &= \left|\int_\Omega w(x) f(u_n)(u_n - u_0)dx\right| \\ &\leq \int_\Omega |w(x)|\ |f(u_n)|\ |(u_n - u_0)|dx \\ &\leq k_1 \int_\Omega |w(x)|\ |u_n|^{q_1(x)-1}\ |(u_n - u_0)|dx \\ &\leq 3k_1 |w|_{r(x)} \left||u_n|^{q_1(x)-1}\right|_{\frac{q_1(x)}{q_1(x)-1}} |u_n - u_0|_{s_1(x)} \\ &\leq 3k_1 |w|_{r(x)} \max\left\{|u_n|_{q_1(x)}^{q_1^+-1},\ |u_n|_{q_1(x)}^{q_1^--1}\right\} |u_n - u_0|_{s_1(x)}, \end{aligned}$$

where $s_1(x)$ is defined in Remark 2.1 (a). Then, by the continuous and compact embedding of $X \hookrightarrow L^{q_1(x)}(\Omega)$ and $X \hookrightarrow L^{s_1(x)}(\Omega)$, we see that (2.13) holds. Now, from (2.12) and (2.13), we conclude that

$$\lim_{n\to\infty} \langle \Phi'(u_n), u_n - u_0\rangle = 0.$$

Thus, by Lemma 2.1 (b), we have $u_n \to u_0$ in X. Hence, from (2.11), we see that

$$I_\lambda(u_0) = \underline{c}_\lambda < 0 \quad \text{and} \quad I'_\lambda(u_0) = 0.$$

Therefore, u_0 is a nontrivial weak solution of problem (1.1), and so any $\lambda \in (0, \overline{\lambda})$ is an eigenvalue of problem (1.1). This completes the proof of the theorem. □

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