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ON THE EISENBUD-GREEN-HARRIS CONJECTURE

ABED ABEDELFATAH

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ABSTRACT. It has been conjectured by Eisenbud, Green and Harris that if I is a homogeneous ideal in $k[x_1,\dots,x_n]$ containing a regular sequence $f_1,\dots,f_n$ of degrees $\deg(f_i)=a_i$, where $2\le a_1\le\cdots\le a_n$, then there is a homogeneous ideal J containing $x_1^{a_1},\dots,x_n^{a_n}$ with the same Hilbert function. In this paper we prove the Eisenbud-Green-Harris Conjecture when f_i splits into linear factors for all i.

1. Introduction

Let $S=k[x_1,\dots,x_n]$ be a polynomial ring over a field k. The ring $S=\bigoplus_{d\ge0}S_d$ is graded by $\deg(x_i)=1$ for all i. In 1927, F. Macaulay proved that if $I=\bigoplus_{d\ge0}I_d$ is a graded ideal in S, then there exists a lex ideal L such that L has the same Hilbert function as I [13]; i.e., every Hilbert function in S is attained by a lex ideal. Let M be a monomial ideal in S. It is natural to ask if we have the same result in S/M. In [5], Clements and Lindström proved that every Hilbert function in $S/\langle x_1^{a_1},\dots,x_n^{a_n}\rangle$ is attained by a lex ideal, where $2\le a_1\le\cdots\le a_n$. In the case $a_1=\cdots=a_n=2$, the result was obtained earlier by Katona [11] and Kruskal [12]. Another generalization of Macaulay's theorem can be found in [17], [15] and [1].

Let $f_1,\dots,f_n$ be a regular sequence in S such that $2\le a_1=\deg(f_1)\le\cdots\le a_n=\deg(f_n)$. A well-known result says that $\langle f_1,\dots,f_n\rangle$ has the same Hilbert function as $\langle x_1^{a_1},\dots,x_n^{a_n}\rangle$ (see Exercise 6.2. of [9]). It is natural to ask what happens if $I\subseteq S$ is a homogeneous ideal containing a regular sequence in fixed degrees. This question bring us to the Eisenbud-Green-Harris Conjecture, denoted by EGH.

Conjecture 1.1 (EGH [8])**.** *If I is a homogeneous ideal in S containing a regular sequence $f_1,\dots,f_n$ of degrees $\deg(f_i)=a_i$, where $2\le a_1\le\cdots\le a_n$, then I has the same Hilbert function as an ideal containing $x_1^{a_1},\dots,x_n^{a_n}$.*

The original conjecture is equivalent to Conjecture 1.1 in the case $a_i=2$ for all i. The EGH Conjecture is known to be true in few cases. The conjecture has been proven in the case $n=2$ [16]. Caviglia and Maclagan [3] have proven that the EGH Conjecture is true if $a_j>\sum_{i=1}^{j-1}(a_i-1)$ for all $j>1$. Richert [16] says that the EGH Conjecture in degree 2 ($a_i=2$ for all i) holds for $n\le5$, but this result was not published. Herzog and Popescu [10] proved that if k is a field of characteristic zero and I is minimally generated by generic quadratic forms, then the EGH Conjecture in degree 2 holds. Cooper [6, 7] has done some work in a geometric direction. She studied the EGH Conjecture for some cases with $n=3$.

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Let $f_1, \ldots, f_n$ be a regular sequence in S such that f_i splits into linear factors for all i. For all $1 \le i \le n$, let $p_i \in S_1$ such that $p_i | f_i$. Since $p_1, \ldots, p_n$ must be a k-linear independent, it follows that the k-algebra map $\alpha : S \to S$ defined by $\alpha(x_i) = p_i$ for all $1 \le i \le n$, is a graded isomorphism. So the Hilbert function is preserved under this map and we may assume that $p_i = x_i$ for all i.

In section 2, we study the dimension growth of some ideals containing a regular sequence $x_1 l_1, \ldots, x_n l_n$, where $l_i \in S_1$ for all i. In section 3, we prove the EGH Conjecture when f_i splits into linear factors for all i. This answers a question of R. X. Chen, who asked if the EGH Conjecture holds when $f_i = x_i l_i$, where $l_i \in S_1$ for all $1 \le i \le n$ (see Example 3.8 of [4]).

2. The dimension growth of some ideals containing a reducible regular sequence

Let $f_1 = x_1 l_1, \ldots, f_n = x_n l_n$ be a regular sequence in S, where $l_i \in S_1$ for all i. Set $P = \langle f_1, \ldots, f_n \rangle$ and $M = \langle x_1^2, \ldots, x_n^2 \rangle$. Let V_d be a vector space spanned by P_d and square-free monomials $w_1, \ldots, w_t$ in S_d, and W_d be the vector space spanned by M_d and $w_1, \ldots, w_t$. In this section, we prove that $\dim(S_1 V_d) = \dim(S_1 W_d)$. We also compute $\dim(S_1 K_d)$, where K_d is the space generated by P_d and the biggest (in lex order) square-free monomials $v_1, \ldots, v_t$ in S_d.

For a matrix $A \in M_{n \times n}(k)$, we denote by $A[i_1, \ldots, i_r]$ the submatrix of A formed by rows $i_1, \ldots, i_r$ and columns $i_1, \ldots, i_r$, where $1 \le r \le n$ and $1 \le i_1 < \cdots < i_r \le n$. We begin with the following lemma, which characterize the structure of $f_1, \ldots, f_n$.

Lemma 2.1 (Example 3.8 of [4]). *Let $f_1 = x_1 l_1, \ldots, f_n = x_n l_n$ be a sequence of homogeneous polynomials in S, where $l_i = \sum_{j=1}^n a_{ij} x_j$ with $a_{ij} \in k$ and A be the $n \times n$ matrix (a_{ij}). Then $f_1, \ldots, f_n$ is a regular sequence if and only if $\det A[i_1, \ldots, i_r] \neq 0$ for all $1 \le r \le n$ and $1 \le i_1 < \cdots < i_r \le n$.*

Proof. Assume that $f_1, \ldots, f_n$ is regular. We prove that $\det A[i_1, \ldots, i_r] \neq 0$ for all $1 \le r \le n$ and $1 \le i_1 < \cdots < i_r \le n$, by induction on n, starting with $n = 1$. Let $n > 1$. Assume that $1 \le i_1 < \cdots < i_r \le n$, where $1 \le r \le n - 1$. Let $j \notin \{i_1, \ldots, i_r\}$. Note that $x_j l_j$ is regular modulo an ideal I if and only if both x_j and l_j are regular modulo I. Since $\overline{f_1}, \ldots, \overline{f_{j-1}}, \overline{f_{j+1}}, \ldots, \overline{f_n}$ is a regular sequence in $S/\langle x_j \rangle$, by the inductive step, it follows that $\det A[i_1, \ldots, i_r] \neq 0$. It remains to show that $\det(A) \neq 0$. From the permutability property of regular sequences of homogeneous polynomials, we obtain that $l_1, \ldots, l_n$ is a regular sequence. So $l_1, \ldots, l_n$ is k-linearly independent.

Assume now $\det A[i_1, \ldots, i_r] \neq 0$ for all $1 \le r \le n$ and $1 \le i_1 < \cdots < i_r \le n$. We prove that $f_1, \ldots, f_n$ is a regular sequence by induction on n, starting with $n = 1$. Let $n > 1$. By the inductive step, the sequence $\overline{f_1}, \ldots, \overline{f_{n-1}}$ is regular in $S/\langle x_n \rangle$. So $f_1, \ldots, f_{n-1}, x_n$ is a regular sequence in S. It remains to show that $f_1, \ldots, f_{n-1}, l_n$ is a regular sequence. Since $\det(A) \neq 0$, it follows that the k-algebra map $\alpha : S \to S$ defined by $\alpha(x_i) = l_i$, for all i, is an isomorphism. By the inductive step, $\alpha^{-1}(f_1), \ldots, \alpha^{-1}(f_{n-1}), \alpha^{-1}(l_n) = x_n$ is a regular sequence. So $f_1, \ldots, f_{n-1}, l_n$ is a regular sequence, as desired. ☐

The special structure of the regular sequence in Lemma 2.1 implies the following lemma.

Lemma 2.2. *Let $f_1 = x_1l_1, \dots, f_n = x_nl_n$ be a regular sequence of homogeneous polynomials in S, where $l_i = \sum_{j=1}^n a_{ij}x_j$ with $a_{ij} \in k$, and $P = \langle f_1, \dots, f_n\rangle$. If $g \notin P$ is a homogeneous polynomial in S, then*

$$g \equiv h \pmod P$$

where $\deg(h) = \deg(g)$ *and h is a k-linear combination of square-free monomials.*

Proof. Since $g \notin P$, we have $\deg(g) \le n$. It is sufficient to prove the lemma when $g \notin P$ is a monomial in $\langle x_1^2, \dots, x_n^2\rangle$ of degree $\le n$. We prove by induction on $\deg(g)$. The lemma is true when $\deg(g) = 2$, since $a_{ii} \neq 0$ for all i. Let g be a monomial in $\langle x_1^2, \dots, x_n^2\rangle$ of degree $d > 2$ and A be the $n \times n$ matrix (a_{ij}). By the inductive step, we may assume that $\frac{g}{x_i}$ is a square-free monomial for some i. By Lemma 2.1, we have $\det A[j : j \in A_g] \neq 0$. So there exist scalars $(c_j)_{j\in A_g}$, such that $\sum_{j\in A_g} c_jl_j \equiv x_i \pmod{\langle x_j : j \notin A_g\rangle}$. It follows that $x_i = \sum_{j\in A_g} c_jl_j + \sum_{j\notin A_g} c_jx_j$, where $c_j \in k$ for all $j \notin A_g$. Then $g = \sum_{j\in A_g} c_jl_j\frac{g}{x_i} + \sum_{j\notin A_g} c_jx_j\frac{g}{x_i}$. Let $h = \sum_{j\notin A_g} c_jx_j\frac{g}{x_i}$. Note that $h \neq 0$ is a k-linear combination of square-free monomials of degree d. Since $\sum_{j\in A_g} c_jl_j\frac{g}{x_i} \in P$, we obtain that $g \equiv h \pmod P$. □

By the proof of Lemma 2.2, we obtain the following.

Remark 2.3. Let P be as in Lemma 2.2 and $0 \le d \le n$. If w is a square-free monomial in S_d and $q \in S_1$, then

$$qw = \widetilde{q}w + \widehat{q}w$$

where $\widetilde{q}, \widehat{q} \in S_1$, $\widehat{q}w \in P$ and $\widetilde{q}w$ is a k-linear combination of square-free monomials.

Example 2.4. Assume that $S = \mathbb{C}[x_1, x_2, x_3]$ and

$$\begin{aligned} f_1 &= x_1^2 + x_1x_2 + x_1x_3 = x_1(x_1 + x_2 + x_3),\\ f_2 &= -x_1x_2 + x_2^2 + x_2x_3 = x_2(-x_1 + x_2 + x_3),\\ f_3 &= -x_1x_3 - x_2x_3 + x_3^2 = x_3(-x_1 - x_2 + x_3). \end{aligned}$$

In this case, $A = \begin{pmatrix} 1 & 1 & 1\\ -1 & 1 & 1\\ -1 & -1 & 1\end{pmatrix}$ is the matrix defined in Lemma 2.1. Since $\det A[i_1, \dots, i_r] \neq 0$ for all $1 \le r \le 3$ and $1 \le i_1 < \dots < i_r \le 3$, we have that f_1, f_2, f_3 is a regular sequence in S. Set $P = \langle f_1, f_2, f_3\rangle$ and let $g = x_1^3 + x_1^2x_2$. Since $x_1^2 \equiv -x_1x_2 - x_1x_3 \pmod P$, we have $x_1^3 \equiv -x_1^2x_2 - x_1^2x_3 \pmod P$. So $g \equiv -x_1^2x_3 \pmod P$. Also, we see that $x_3f_1 - x_1f_3 = 2x_1^2x_3 + 2x_1x_2x_3 \in P$. So $g \equiv x_1x_2x_3 \pmod P$ and $x_1x_2x_3 \notin \langle x_1^2, x_2^2, x_3^2\rangle$.

Remark 2.5. Lemma 2.2 is not true if $f_1, \dots, f_n$ is an arbitrary regular sequence. For example, consider the sequence

$$f_1 = x_1^2 + x_1x_2 + x_2^2,\ f_2 = x_1x_2 \text{ in } \mathbb{C}[x_1, x_2].$$

Since f_1 and f_2 have no common factor, it follows that f_1, f_2 is a regular sequence. Let $g = x_2^2$. It is easy to show that $g \notin \langle f_1, f_2\rangle$. If $g \equiv ax_1x_2 \pmod{\langle f_1, f_2\rangle}$, for some $a \in \mathbb{C}$, then there exist $c_1, c_2, c_3 \in \mathbb{C}$, not all zero, such that $c_1f_1 + c_2f_2 + c_3(g - ax_1x_2) = 0$. But the equation

$$c_1x_1^2 + (c_1 + c_2 - ac_3)x_1x_2 + (c_1 + c_3)x_2^2 = 0,$$

implies that $c_1 = c_2 = c_3 = 0$, a contradiction.

As a result of Lemma 2.2, we obtain the following.

Lemma 2.6. *If P is as in Lemma 2.2, then the set of all square-free monomials form a k-basis of S/P.*

Proof. Denote by $\mathcal{A}$ the set of all square-free monomials in S. Lemma 2.2 shows that S/P generated by $\mathcal{A}$. Let $w = x_1 \cdots x_n$. Assume that $w \in P$. Since $H(S/P) = H(S/\langle x_1^2, \ldots, x_n^2 \rangle)$, it follows that there is a polynomial $f \in S_n$ such that $f \notin P$. By Lemma 2.2, $f \equiv bx_1 \cdots x_n \pmod{P}$, where $0 \neq b \in k$. Since $w \in P$, it follows that $f \in P$, a contradiction. So $w \notin P$. Suppose that $\sum_{w \in \mathcal{A}} a_w w \in P$, where $a_w \in k$ and $a_w = 0$ for almost all $w \in \mathcal{A}$. Assume that $a_w \neq 0$ for some w. Let $v \in \mathcal{A}$ be a monomial with minimal degree such that $a_v \neq 0$. So $\overline{v} \in \langle \overline{f_i} : i \in A_v \rangle$ in the ring $S/\langle x_i : i \notin A_v \rangle$, a contradiction. □

Lemma 2.7. *Let P be as in Lemma 2.2. If w is a square-free monomial in S_d, where $0 \leq d \leq n$, then*

(a) $|S_1(w) \cap P_{d+1}| = d$.

(b) $|S_1(w) \cap (P_{d+1} + S_1(w_1, \ldots, w_t))| = |S_1(w) \cap P_{d+1}| + |S_1(w) \cap S_1(w_1, \ldots, w_t)|$ *for every square-free monomial $w_1, \ldots, w_t$ of degrees d such that $w_i \neq w$ for all $1 \leq i \leq t$.*

Proof. (a) Let $q = \sum_{i=1}^n c_i l_i \in S_1$, where $c_i \in k$ for all i, such that $qw \in P_{d+1}$. Assume that $c_j \neq 0$ for some $j \notin A_w$. Since $qw \prod_{j \neq k \notin A_w} x_k \in P$, it follows that $c_j l_j w \prod_{j \neq k \notin A_w} x_k \in P$. Thus, $c_j l_j w \prod_{j \neq k \notin A_w} x_k = h_1 f_1 + \cdots + h_n f_n$, where $h_i \in S$ for all $1 \leq i \leq n$. So

$$h_1 f_1 + \cdots + h_{j-1} f_{j-1} + (x_j h_j - c_j w \prod_{j \neq k \notin A_w} x_k) l_j + h_{j+1} f_{j+1} + \cdots + h_n f_n = 0,$$

which implies that

$$x_j h_j - c_j w \prod_{j \neq k \notin A_w} x_k \in \langle f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n \rangle.$$

So $\overline{w \prod_{j \neq k \notin A_w} x_k} \in \langle \overline{f_1}, \ldots, \overline{f_{j-1}}, \overline{f_{j+1}}, \ldots, \overline{f_n} \rangle$ in the ring $S/\langle x_j \rangle$, a contradiction to Lemma 2.6. It follows that q belongs to the k-vector space $(l_i : i \in A_w)$. On the other hand, $l_i w \in P$, for all $i \in A_w$. So $|S_1(w) \cap P_{d+1}| = \dim(l_i w : i \in A_w) = d$.

(b) First, we show that

$$S_1(w) \cap (P_{d+1} + S_1(w_1, \ldots, w_t)) = S_1(w) \cap P_{d+1} + S_1(w) \cap S_1(w_1, \ldots, w_t).$$

Assume $qw \in P_{d+1} + S_1(w_1, \ldots, w_t)$, where $q \in S_1$. There exist $f \in S_1(w_1, \ldots, w_t)$ and $g \in P_{d+1}$ such that $qw = g + f$. If $f \in P$, then $qw \in S_1(w) \cap P_{d+1}$. So assume that $f \notin P$. By Remark 2.3, we may assume that f is a k-linear combination of square-free monomials. Also, we obtain $qw = \widetilde{q}w + \widehat{q}w$, where $\widetilde{q}, \widehat{q} \in S_1$, $\widehat{q}w \in P$ and $\widetilde{q}w$ is a k-linear combination of square-free monomials. So $\widetilde{q}w - f \in P$, which implies that $\widetilde{q}w = f \in S_1(w_1, \ldots, w_t)$. Hence $qw \in S_1(w) \cap P_{d+1} + S_1(w) \cap S_1(w_1, \ldots, w_t)$ and we obtain the desired equality.

It remains to show that

$$S_1(w) \cap S_1(w_1, \ldots, w_t) \cap P_{d+1} = (0).$$

Let $qw \in S_1(w_1, \ldots, w_t) \cap P_{d+1}$, where $q \in S_1$. By (a), we have $q = \sum_{j \in A_w} c_j l_j$, where $c_j \in k$ for all $j \in A_w$. For every $1 \leq j \leq t$, let $i_j \in A_{w_j} \setminus A_w$ and let $B = \{i_j : 1 \leq j \leq t\}$. By the hypothesis, we obtain that $qw = \sum_{i=1}^t q_i w_i$, where $q_i \in S_1$ for all $1 \leq i \leq t$. So $\overline{qw} = \overline{0}$ in the ring $S/\langle x_j : j \in B \rangle$, which implies that $\sum_{j\in A_w} \overline{c_j l_j} = \overline{0}$. By Lemma 2.1, we obtain that $c_j = 0$, for all $j \in A_w$. Thus, $qw = 0$. □

Remark 2.8. Part (b) of Lemma 2.7 is not true if we replace $w, w_1, \ldots, w_t$ by homogeneous polynomials which are a k-linear combination of square-free monomials in S_d. For example, let $S = k[x_1, x_2, x_3, x_4]$ and $P = \langle x_1^2, x_2^2, x_3^2, x_4^2\rangle$. Suppose that $h = x_1x_2+x_2x_4+x_3x_4$ and $h_1 = x_1x_2+x_1x_3$. Computation with Macaulay2 shows that

$$|S_1(h) \cap (P_3 + S_1(h_1))| = 2 \text{ and } |S_1(h) \cap P_3| = |S_1(h) \cap S_1(h_1))| = 0.$$

Now, we prove the main results of this section.

Theorem 2.9. *Let P be as in Lemma 2.2 and $M = \langle x_1^2, \ldots, x_n^2\rangle$. Assume that $V = P_d + (w_1, \ldots, w_t)$ and $W = M_d + (w_1, \ldots, w_t)$, where w_i is a square-free monomial of degree d, for all i. Then*

$$\dim S_1W = \dim S_1V.$$

Proof. We may assume that $d \geq 2$ and prove by induction on t. If $t = 1$, then

$$\begin{aligned}\dim S_1W &= \dim M_{d+1} + \dim S_1(w_1) - \dim S_1(w_1) \cap M_{d+1}\\ &= \dim P_{d+1} + \dim S_1(w_1) - \dim S_1(w_1) \cap P_{d+1} = \dim S_1V.\end{aligned}$$

Let $t > 1$, and set $W_1 = M_d + (w_1, \ldots, w_{t-1})$, $V_1 = P_d + (w_1, \ldots, w_{t-1})$ and $Z = S_1(w_t) \cap S_1(w_1, \ldots, w_{t-1})$. By Lemma 2.7 and the inductive step, we have

$$\begin{aligned}\dim S_1W &= \dim S_1W_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap S_1W_1\\ &= \dim S_1V_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap S_1W_1\\ &= \dim S_1V_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap M_{d+1} - \dim Z\\ &= \dim S_1V_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap P_{d+1} - \dim Z\\ &= \dim S_1V_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap S_1V_1\\ &= \dim S_1V.\end{aligned}$$

□

Proposition 2.10. *Let P be as in Lemma 2.2 and $V = P_d + (w_1, \ldots, w_t)$ be the k-vector space spanned by P_d and the t biggest (in lex order) square-free monomials in S_d. Then*

$$\dim S_1V = \binom{d+n}{d+1} - \binom{n}{d+1} + \sum_{i=1}^{t}(n - m(w_i)),$$

where $m(w_i) = \max\{j : \; x_j | w_i\}$, $1 \leq i \leq t$.

Proof. We claim that

$$|S_1V| = |P_{d+1}| + \sum_{i=1}^{t} |S_1(w_i)| - \sum_{i=1}^{t} |S_1(w_i) \cap P_{d+1}| - \sum_{i=2}^{t} |S_1(w_i) \cap S_1(w_1, \ldots, w_{i-1})|.$$

We prove the claim by induction on t. If $t = 1$, then

$$|S_1V| = |P_{d+1}| + |S_1(w_1)| - |S_1(w_1) \cap P_{d+1}|.$$

Let $t > 1$ and $V_1 = P_d + (w_1, \ldots, w_{t-1})$. By the inductive step we obtain that $|S_1V|$ is equal to

$$|P_{d+1}| + \sum_{i=1}^{t} |S_1(w_i)| - \sum_{i=1}^{t-1} |S_1(w_i) \cap P_{d+1}| \\ - \sum_{i=2}^{t-1} |S_1(w_i) \cap S_1(w_1, \ldots, w_{i-1})| - |S_1(w_t) \cap S_1\overline{V}|.$$

By Lemma 2.7 we have $|S_1(w_t) \cap S_1V_1| = |S_1(w_t) \cap P_{d+1}| + |S_1(w_t) \cap S_1(w_1, \ldots, w_{t-1})|$. We proved the claim.

Let $2 \leq j \leq t$. If $i < m(w_j)$ such that $x_i \nmid w_j$, then $x_i w_j \in S_1(w_1, \ldots, w_{j-1})$. So $|S_1(w_i) \cap S_1(w_1, \ldots, w_{i-1})| = m(w_j) - d$. Therefore

$$\begin{aligned} |S_1V| &= |S_{d+1}| - \binom{n}{d+1} + tn - td - \sum_{i=2}^{t}(m(w_i) - d) \\ &= \binom{d+n}{d+1} - \binom{n}{d+1} + tn - td - \sum_{i=2}^{t}(m(w_i) - d) \\ &= \binom{d+n}{d+1} - \binom{n}{d+1} + tn - td - \sum_{i=1}^{t}(m(w_i) - d) \\ &= \binom{d+n}{d+1} - \binom{n}{d+1} + \sum_{i=1}^{t}(n - m(w_i)). \end{aligned}$$ □

3. The main result

In this section we prove that the EGH Conjecture is true if f_i splits into linear factors for all i. We begin with the following lemma.

Lemma 3.1. *Let $P = \langle f_1, \ldots, f_n \rangle$ be an ideal of S generated by a regular sequence with $\deg(f_i) = a_i$ and $n \geq 2$. Assume that $f_n = q_1 \cdots q_s$, where $q_1, \ldots, q_s \in S_1$. Then*

(a) *$H(S/P + \langle q_m \rangle) = H(S/P + \langle q_k \rangle)$ for all $1 \leq m, k \leq s$.*
(b) *$H(S/(P : q_1 \cdots q_j) + \langle q_m \rangle) = H(S/(P : q_1 \cdots q_j) + \langle q_k \rangle)$ for all $1 \leq j \leq s-1$ and $j < m, k \leq s$.*

Proof. First, we will prove (a). Let $1 \leq m, k \leq s$. Note that $P + \langle q_m \rangle / \langle q_m \rangle$ and $\underline{P + \langle q_k \rangle} / \langle q_k \rangle$ are ideals in $S/\langle q_m \rangle$ and $S/\langle q_k \rangle$, respectively, generated by $\overline{f_1}, \ldots, \overline{f_{n-1}}$. Note also that $f_1, \ldots, f_{n-1}, q_m$ and $f_1, \ldots, f_{n-1}, q_k$ are regular sequences. We obtain that $\overline{f_1}, \ldots, \overline{f_{n-1}}$ is a regular sequence in $S/\langle q_m \rangle$ and $S/\langle q_k \rangle$. So

$$H(S/P + \langle q_m \rangle) = H(S/P + \langle q_k \rangle).$$

Now, we prove (b). Let $1 \leq j \leq s - 1$ and $j < m, k \leq s$. Assume that

$$h = h_1 + h_2 \in (P : q_1 \cdots q_j) + \langle q_m \rangle,$$

where $h_1 \in (P : q_1 \cdots q_j)$ and $h_2 \in \langle q_m \rangle$. Since $h_1 q_1 \cdots q_j \in P$, it follows that $h_1 q_1 \cdots q_j = g_1 f_1 + \cdots + g_n f_n$, where $g_1, \ldots, g_n \in S$; i.e.,

$$g_1 f_1 + \cdots + g_{n-1} f_{n-1} + q_1 \cdots q_j (g_n q_{j+1} \cdots q_s - h_1) = 0.$$

Since $f_1, \ldots, f_{n-1}, q_1 \cdots q_j$ is a regular sequence, it follows that $g_n q_{j+1} \cdots q_s - h_1 \in \langle f_1, \ldots, f_{n-1} \rangle$. So $\overline{h_1} \in \langle \overline{f_1}, \ldots, \overline{f_{n-1}} \rangle$ in the ring $S/\langle q_m \rangle$, which implies that $\overline{h} \in \langle \overline{f_1}, \ldots, \overline{f_{n-1}} \rangle$. Conversely, $\overline{f_i} \in (P : q_1 \cdots q_j) + \langle q_m \rangle / \langle q_m \rangle$ for all $1 \leq i \leq n-1$. So $(P : q_1 \cdots q_j) + \langle q_m \rangle / \langle q_m \rangle$ is an ideal in $S/\langle q_m \rangle$ generated by $\overline{f_1}, \ldots, \overline{f_{n-1}}$. Similarly, $(P : q_1 \cdots q_j) + \langle q_k \rangle / \langle q_k \rangle$ is an ideal in $S/\langle q_k \rangle$ generated by $\overline{f_1}, \ldots, \overline{f_{n-1}}$. So

$$H(S/(P : q_1 \cdots q_j) + \langle q_k \rangle) = H(S/(P : q_1 \cdots q_j) + \langle q_m \rangle). \qquad \square$$

Theorem 3.2. *Let I be a graded ideal in $S = k[x_1, \ldots, x_n]$ containing a regular sequence $f_1, \ldots, f_{n-1}, f_n = q_1 \cdots q_s$ of degrees $\deg(f_i) = a_i$ such that $q_i \in S_1$ for all $1 \leq i \leq s$. If the image of the sequence $f_1, \ldots, f_{n-1}$ in $S/\langle q_i \rangle$ satisfies the EGH Conjecture for all i, then I has the same Hilbert function as a graded ideal in S containing $x_1^{a_1}, \ldots, x_n^{a_n}$.*

Proof. We show that for every $d \geq 0$, there exists a graded ideal K in S containing $x_1^{a_1}, \ldots, x_n^{a_n}$ such that $H(S/I, d) = H(S/K, d)$ and $H(S/I, d+1) \leq H(S/K, d+1)$. Let $d \geq 0$ and J be the ideal generated by $f_1, \ldots, f_n$ and I_d. By renaming the linear polynomials $q_1, \ldots, q_s$, we may assume without loss of generality that

$$\begin{gathered} |J_d \cap \langle q_1 \rangle_d| \geq |J_d \cap \langle q_i \rangle_d| \text{ for all } 2 \leq i \leq s, \\ |(J : q_1)_{d-1} \cap \langle q_2 \rangle_{d-1}| \geq |(J : q_1)_{d-1} \cap \langle q_i \rangle_{d-1}| \text{ for all } 3 \leq i \leq s, \\ |(J : q_1 q_2)_{d-2} \cap \langle q_3 \rangle_{d-2}| \geq |(J : q_1 q_2)_{d-2} \cap \langle q_i \rangle_{d-2}| \text{ for all } 4 \leq i \leq s, \\ \vdots \\ |(J : q_1 \cdots q_{s-2})_{d-(s-2)} \cap \langle q_{s-1} \rangle_{d-(s-2)}| \geq |(J : q_1 \cdots q_{s-2})_{d-(s-2)} \cap \langle q_s \rangle_{d-(s-2)}|. \end{gathered}$$

By considering the short exact sequences

$$\begin{gathered} 0 \to S/(J : q_1) \longrightarrow S/J \longrightarrow S/J + \langle q_1 \rangle \to 0, \\ 0 \to S/(J : q_1 q_2) \longrightarrow S/(J : q_1) \longrightarrow S/(J : q_1) + \langle q_2 \rangle \to 0, \\ 0 \to S/(J : q_1 q_2 q_3) \longrightarrow S/(J : q_1 q_2) \longrightarrow S/(J : q_1 q_2) + \langle q_3 \rangle \to 0, \\ \vdots \\ 0 \to S/(J : q_1 \cdots q_{s-1}) \longrightarrow S/(J : q_1 \cdots q_{s-2}) \longrightarrow S/(J : q_1 \cdots q_{s-2}) + \langle q_{s-1} \rangle \to 0, \end{gathered}$$

we see that $H(S/J, t)$ is equal to

$$H(S/J + \langle q_1 \rangle, t) + \sum_{i=1}^{s-2} H(S/(J : q_1 \cdots q_i) + \langle q_{i+1} \rangle, t - i) + H(S/(J : q_1 \cdots q_{s-1}), t - (s-1))$$

for all $t \geq 0$. Let $J_0 = J + \langle q_1 \rangle$, $J_{s-1} = (J : q_1 \cdots q_{s-1})$, and for $1 \leq i \leq s-2$ let $J_i = (J : q_1 \cdots q_i) + \langle q_{i+1} \rangle$. Note that $q_{i+1} \in J_i$ and $H(\frac{S/\langle q_{i+1} \rangle}{J_i/\langle q_{i+1} \rangle}) = H(S/J_i)$ for all $0 \leq i \leq s-1$. Set $\overline{S} = k[x_1, \ldots, x_{n-1}]$. For all $0 \leq i \leq s-1$, $S/\langle q_{i+1} \rangle$ is isomorphic to $\overline{S}$, so by the hypothesis there is an ideal in $\overline{S}$ containing $x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}}$ with the same Hilbert function as J_i. For all $0 \leq i \leq s-1$, let L_i be the lex-plus-powers ideal in $\overline{S}$ containing $x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}}$ such that $H(\overline{S}/L_i) = H(S/J_i)$.

Claim. $L_{i,j} \subseteq L_{i+1,j}$ for all $0 \leq i \leq s-2$ and $j \leq d-i$, where $L_{i,j}$ is the j-th component of the ideal L_i.

Proof of the Claim. Assume that $i = 0$. If $j < d$, then by part (a) of Lemma 3.1 we obtain

$$|J_{0,j}| = |J_j + \langle q_1 \rangle_j| = |P_j + \langle q_1 \rangle_j| = |P_j + \langle q_2 \rangle_j| \leq |J_{1,j}|.$$

If $j = d$, then by our assumption we obtain

$$\begin{aligned}|J_{0,d}| &= |J_d| + |\langle q_1\rangle_d| - |J_d \cap \langle q_1\rangle_d| \\ &\le |J_d| + |\langle q_1\rangle_d| - |J_d \cap \langle q_2\rangle_d| \\ &= |J_d| + |\langle q_2\rangle_d| - |J_d \cap \langle q_2\rangle_d| \\ &= |J_d + \langle q_2\rangle_d| \\ &\le |J_{1,d}|.\end{aligned}$$

This means that $H(S/J_0, j) \ge H(S/J_1, j)$ for all $j \le d$. So $H(\overline{S}/L_0, j) \ge H(\overline{S}/L_1, j)$ for all $j \le d$. Since L_0 and L_1 are lex-plus-powers ideals, it follows that $L_{0,j} \subseteq L_{1,j}$ for all $j \le d$.

Let $0 < i \le s-2$. If $j < d-i$, then by part (b) of Lemma 3.1 we obtain

$$\begin{aligned}|J_{i,j}| &= |(J : q_1 \cdots q_i)_j + \langle q_{i+1}\rangle_j| = |(P : q_1 \cdots q_i)_j + \langle q_{i+1}\rangle_j| \\ &= |(P : q_1 \cdots q_i)_j + \langle q_{i+2}\rangle_j| \le |J_{i+1,j}|.\end{aligned}$$

If $j = d-i$, then by our assumption we obtain

$$\begin{aligned}|J_{i,d-i}| &= |(J : q_1 \cdots q_i)_{d-i}| + |\langle q_{i+1}\rangle_{d-i}| - |(J : q_1 \cdots q_i)_{d-i} \cap \langle q_{i+1}\rangle_{d-i}| \\ &\le |(J : q_1 \cdots q_i)_{d-i}| + |\langle q_{i+1}\rangle_{d-i}| - |(J : q_1 \cdots q_i)_{d-i} \cap \langle q_{i+2}\rangle_{d-i}| \\ &= |(J : q_1 \cdots q_i)_{d-i}| + |\langle q_{i+2}\rangle_{d-i}| - |(J : q_1 \cdots q_i)_{d-i} \cap \langle q_{i+2}\rangle_{d-i}| \\ &= |(J : q_1 \cdots q_i)_{d-i} + \langle q_{i+2}\rangle_{d-i}| \\ &\le |J_{i+1,d-i}|.\end{aligned}$$

Similarly, we conclude that $L_{i,j} \subseteq L_{i+1,j}$ for all $j \le d-i$, and prove the claim. □

Let $K_s = \{zx_n^{s+j} : z \in \mathrm{Mon}(\overline{S}) \wedge j \ge 0\}$ and $K_i = \{zx_n^i : z \in \mathrm{Mon}(L_i)\}$ for all $0 \le i \le s-1$. Define K to be the ideal generated by $\bigcup_{0 \le i \le s} K_i$. Since $x_n^s \in K_s$ and $x_i^{a_i} \in K_0$ for all $1 \le i \le n-1$, it follows that $x_1^{a_1}, \ldots, x_n^{a_n} \in K$.

Claim. If w is a monomial in K of degree t, where $0 \le t \le d+1$, then $w \in \bigcup_{0 \le i \le s} K_i$.

Proof of the Claim. There exists a monomial u in $\bigcup_{0 \le i \le s} K_i$ such that $u|w$; i.e., $w = vu$ for some monomial $v \in S$. If $u \in K_s$, then $w \in K_s$. Assume that $u = zx_n^i \in K_i$, where $z \in L_i$ for some $0 \le i \le s-1$. If $x_n \nmid v$, then $w \in \bigcup_{0 \le i \le s} K_i$. Assume that $x_n|v$. Let $r = \max\{j : x_n^j|v\}$. If $i + r \ge s$, then $w \in K_s$. So we may assume that $i + r < s$. By the previous claim, we obtain that $L_{i,j} \subseteq L_{i+r,j}$ for all $j \le d - (i + r - 1)$. Since $\deg(z) \le d + 1 - (i + r)$, it follows that $z \in L_{i+r}$. So $\frac{v}{x_n^r} z \in L_{i+r}$, and then $\frac{v}{x_n^r} z x_n^{r+i} = w \in K_{i+r}$. Hence, we have proved the claim. □

We conclude that the number of monomials in K of degree t, where $0 \le t \le d+1$, is equal to $\sum_{i=0}^{s-1} |L_{i,t-i}| + \sum_{i=0}^{t-s} |\overline{S}_i|$. Since $|S_t| = \sum_{0 \le i \le t} |\overline{S}_i|$, it follows that

$$|S_t| - |K_t| = \sum_{i=t-(s-1)}^{t} |\overline{S}_i| - \sum_{i=0}^{s-1} |L_{i,t-i}| = \sum_{i=0}^{s-1} |\overline{S}_{t-i}| - \sum_{i=0}^{s-1} |L_{i,t-i}|.$$

So $H(S/K, t) = \sum_{i=0}^{s-1} H(\overline{S}/L_i, t-i) = \sum_{i=0}^{s-1} H(S/J_i, t-i) = H(S/J, t)$. In particular,

$$H(S/K, d) = H(S/J, d) = H(S/I, d)$$

and

$$H(S/K, d+1) = H(S/J, d+1) \ge H(S/I, d+1). \qquad \square$$

Corollary 3.3. *If I is a graded ideal in S containing a regular sequence $f_1, \dots, f_n$ with $\deg(f_i) = a_i$ such that f_i splits into linear factors for all i, then I has the same Hilbert function as a graded ideal in S containing $x_1^{a_1}, \dots, x_n^{a_n}$.*

Since the EGH Conjecture holds when $n = 2$, we obtain the following.

Corollary 3.4. *Let $n \geq 3$. If I is a graded ideal in S containing a regular sequence $f_1, \dots, f_n$ with $\deg(f_i) = a_i$ such that f_i splits into linear factors for all $3 \leq i \leq n$, then I has the same Hilbert function as a graded ideal in S containing $x_1^{a_1}, \dots, x_n^{a_n}$.*

Example 3.5. Let $S = \mathbb{C}[x_1, \dots, x_5]$, $f_i = x_i(\sum_{j=1}^{i-1} -x_j) + x_i(\sum_{j=i}^{5} x_j)$ for all $1 \leq i \leq 5$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

Since $\det A[i_1, \dots, i_r] \neq 0$ for all $1 \leq r \leq 5$ and $1 \leq i_1 < \cdots < i_r \leq 5$, it follows that $f_1, \dots, f_5$ is a regular sequence in S. Assume that $I = \langle f_1, \dots, f_5, x_1x_2 + x_1x_3, x_1^2 + x_4x_5 \rangle$. In this example, we construct an ideal in S with the same Hilbert function as I, using the Hilbert functions of $J_0 = I + \langle x_5 \rangle$ and $J_1 = (I : x_5)$. Computation with Macaulay2 shows that

$$H_{S/I} = (1, 5, 8, 3, 0, 0, \dots), H_{S/J_0} = (1, 4, 4, 1, 0, 0, \dots) \text{ and } H_{S/J_1} = (1, 4, 2, 0, 0, \dots)$$

are the Hilbert sequence of I, J_0 and J_1, respectively. Denote by R the polynomial ring $\mathbb{C}[x_1, \dots, x_4]$. Let

$$L_0 = \langle x_1^2, \dots, x_4^2, x_1x_2, x_1x_3 \rangle \subset R \text{ and}$$
$$L_1 = \langle x_1^2, \dots, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3 \rangle \subset R.$$

Note that L_0 and L_1 are lex-plus-powers ideals in R. We can see that $L_{0,0} = L_{0,1} = (0)$ and

$$L_{0,2} = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_3^2, x_4^2),$$
$$L_{0,3} = (w : \ w \in \text{Mon}(R_3) \text{ and } w \neq x_2x_3x_4),$$
$$L_{0,j} = R_j \text{ for all } j \geq 4.$$

So we have $H_{R/L_0} = H_{S/J_0}$. Also, we have $L_{1,0} = L_{1,1} = (0)$ and

$$L_{1,2} = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_3^2, x_4^2),$$
$$L_{1,j} = R_j \text{ for all } j \geq 3.$$

So we have $H_{R/L_1} = H_{S/J_1}$. Let K be the ideal in S generated by

$$\text{Mon}(L_0) \cup \{wx_5 : \ w \in \text{Mon}(L_1)\} \cup \{x_5^2\}.$$

Then $K = \langle x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2, x_1x_3, x_1x_4x_5, x_2x_3x_5 \rangle$. It is clear that $|S_0/K_0| = 1$ and $|S_1/K_1| = 5$. Since $S_2/K_2 = (\overline{x_1x_4}, \overline{x_1x_5}, \overline{x_2x_3}, \overline{x_2x_4}, \overline{x_2x_5}, \overline{x_3x_4}, \overline{x_3x_5}, \overline{x_4x_5})$, it follows that $|S_3/K_3| = 8$. Also we have $S_3/K_3 = (\overline{x_2x_3x_4}, \overline{x_2x_4x_5}, \overline{x_3x_4x_5})$ and $K_j = S_j$ for all $j \geq 4$. Thus $H_{S/K} = (1, 5, 8, 3, 0, 0, \dots) = H_{S/I}$.

Example 3.6. Let $S = \mathbb{C}[x_1, \dots, x_6]$, $f_i = x_i(\sum_{j=1}^{i-1} -x_j) + x_i(\sum_{j=i}^{6} x_j)$ for all $1 \leq i \leq 5$ and $f_6 = x_6^2(-x_1 - x_2 - x_3 - x_4 - x_5 + x_6)$. Since $f_1, \dots, f_5, \frac{f_6}{x_6}$ is a regular sequence, it follows that $f_1, \dots, f_6$ is a regular sequence in S. Assume that

$$I = \langle f_1, \dots, f_6, x_1x_2 + x_3x_4, x_1x_6 + x_5^2, x_2^2x_3 \rangle.$$

Computation with Macaulay2 shows that

$$\begin{aligned} H_{S/I} &=(1,6,14,13,2,0,\dots),\\ H_{S/I+\langle x_6\rangle} &=(1,5,8,2,0,\dots),\\ H_{S/(I:x_6)+\langle x_6\rangle} &=(1,5,6,0,\dots),\\ H_{S/(I:x_6^2)} &=(1,5,2,0,\dots). \end{aligned}$$

Also we have

$$|I_2 \cap \langle x_6\rangle_2| = |I_2 \cap \langle -x_1-x_2-x_3-x_4-x_5+x_6\rangle_2|$$

and

$$|(I:x_6)_1 \cap \langle x_6\rangle_1| = |(I:x_6)_1 \cap \langle -x_1-x_2-x_3-x_4-x_5+x_6\rangle_1|.$$

We construct an ideal in S with the same Hilbert function as I, using the Hilbert functions of $I+\langle x_6\rangle$, $(I:x_6)+\langle x_6\rangle$ and $(I:x_6^2)$. Denote by J_0, J_1 and J_2 the ideals $I+\langle x_6\rangle$, $(I:x_6)+\langle x_6\rangle$ and $(I:x_6^2)$, respectively. Let $R=\mathbb{C}[x_1,\dots,x_5]$ and $L_0 = \langle x_1^2,\dots,x_5^2, x_1x_2, x_1x_3, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5\rangle \subset R$. An easy calculation shows that L_0 is a lex-plus-powers ideal in R and $H_{R/L_0} = (1,5,8,0,\dots) = H_{S/I+\langle x_6\rangle}$. Let $L_1 = \langle x_1^2,\dots,x_5^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5\rangle \subset R$. We can see that L_1 is a lex-plus-powers ideal and $H_{R/L_1} = (1,5,6,0,\dots) = H_{S/J_1}$. Let $L_2 = \langle x_1^2,\dots,x_5^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4\rangle \subset R$. Also we have that L_2 is a lex-plus-powers ideal in R and $H_{R/L_2} = (1,5,3,0,\dots) = H_{S/J_2}$. Let K be the ideal in S generated by $\mathrm{Mon}(L_0)\cup\{wx_6 :\ w\in \mathrm{Mon}(L_1)\}\cup\{wx_6^2 :\ w\in \mathrm{Mon}(L_2)\}\cup\{x_6^3\}$. The ideal K is generated by

$$\{x_1^2,\dots,x_5^2,x_6^3,x_1x_2,x_1x_3,x_1x_4x_5,x_2x_3x_4,x_2x_3x_5,x_1x_4x_6\}$$

and

$$\{x_1x_5x_6, x_2x_4x_5x_6, x_3x_4x_5x_6, x_2x_3x_6^2, x_2x_4x_6^2, x_2x_5x_6^2, x_3x_4x_6^2\}.$$

Computation with Macaulay2 shows that $H_{S/K} = (1,6,14,13,2,0,\dots) = H_{S/I}$.

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Department of Mathematics, University of Haifa, Mount Carmel, Haifa 31905, Israel
E-mail address: abed@math.haifa.ac.il