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NECESSARY CONDITIONS OF SOLVABILITY AND ISOPERIMETRIC ESTIMATES FOR SOME MONGE-AMPÈRE PROBLEMS IN THE PLANE

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Abstract. This article is mainly devoted to the solvability of the Monge-Ampère equation $\det\left(D^2u\right)=1$, in a C^2 bounded strictly convex domain $\Omega\subset\mathbb{R}^2$, subject to a contact angle boundary condition. A necessary condition for the solvability of this problem, involving the maximal value of the curvature $k\left(s\right)$ of $\partial\Omega$ and the contact angle, was derived by X.-N. Ma in 1999, making use of a maximum principle for an appropriate P-function. Our main goal here is to prove a complementary result. More precisely, we will derive a new necessary condition of solvability, involving the minimal value of the curvature $k\left(s\right)$ of $\partial\Omega$ and the contact angle. The main ingredients of our proof are the derivation of a minimum principle for the P-function employed by X.-N. Ma in his proof, respectively, the use of some computations in normal coordinates with respect to the boundary $\partial\Omega$. Finally, a similar minimum principle will be employed to derive some isoperimetric estimates for the classical convex solution of the Monge-Ampère equation, subject to the homogeneous Dirichlet boundary condition.

1. Introduction

Let $\Omega\subset\mathbb{R}^2$ be a C^2 bounded strictly convex domain. This note is concerned with the study of some boundary value problems for the following Monge-Ampère equation

$$\det\left(D^2u\right)=1 \text{ in } \Omega. \tag{1.1}$$

When the Monge-Ampère equation (1.1) is subject to Dirichlet, Neumann or oblique boundary conditions, various existence results are already known (see, for instance, L. Caffarelli - L. Nirenberg - J. Spruck [5], P.-L. Lions - N.S. Trudinger - J. Urbas [9], respectively J. Urbas [19]). However, when we deal with the solvability of the Monge-Ampère equation (1.1), subject to the contact angle or capillary boundary condition

$$u_n=\cos\theta\sqrt{1+\left|\nabla u\right|^2} \text{ on } \partial\Omega, \tag{1.2}$$

where $\theta\in(0,\pi/2)$ is a constant and u_n is the outward normal derivative of $u\left(\mathbf{x}\right)$ on $\partial\Omega$, existence results are still unknown for general domains Ω. A reason for this

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situation is the fact that, when we deal with the solvability of the Monge-Ampère problems (1.1)-(1.2), there is a strong restriction on the curvature of $\partial\Omega$ and contact angle θ. More precisely, we have the following necessary condition of solvability, derived by X.-N. Ma in [10] (see, also, J. Urbas [20], for an extension to higher dimensions): *If there exists a strictly convex solution* $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ *of the Monge-Ampère problems* (1.1)-(1.2), *then*

$$k_{\min} \leq \max\{\cos\theta, \tan\theta\}, \tag{1.3}$$

where $k_{\min}$ *is the minimum value of the curvature* $k(s)$ *of* $\partial\Omega$.

In our first main theorem we state a complementary result. More precisely, we have:

Theorem 1.1. *If* $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ *is a strictly convex solution of problems* (1.1)-(1.2)*, then*

$$k_{\max} \geq \tan\theta, \tag{1.4}$$

where $k_{\max}$ *is the maximum value of the curvature* $k(s)$ *on* $\partial\Omega$.

The key elements in the proof of Theorem 1.1 are a minimum principle derived for the P-function employed by X.-N. Ma in [10] in his proof of (1.3) and the use of some computations in normal coordinates with respect to the boundary $\partial\Omega$. More exactly, let us introduce the following P-function:

$$P(\mathbf{x}, \alpha) := |\nabla u|^2 - 2\alpha u, \text{ with } \alpha \in \mathbb{R}, \tag{1.5}$$

where $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ is assumed to be a strictly convex solution of problems (1.1)-(1.2). According to X.-N. Ma [10]: $P(\mathbf{x}, 1)$ *attains its maximum value on* $\partial\Omega$, *unless it is constant on* $\overline{\Omega}$. Our minimum principle states the following:

Theorem 1.2. *If* $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ *is a strictly convex solution of problems* (1.1)-(1.2)*, then* $P(\mathbf{x}, 1)$ *attains its minimum value on the boundary* $\partial\Omega$*, unless it is constant on* $\overline{\Omega}$.

From the proof of Theorem 1.2 we will easily notice that a similar minimum principle also holds when we replace in its statement the capillary boundary condition (1.2) with the Dirichlet boundary condition

$$u = 0 \text{ on } \partial\Omega. \tag{1.6}$$

Using such a minimum principle and a related maximum principle derived by G.A. Philippin and A. Safoui in [16], we get the following isoperimetric estimates:

Theorem 1.3. *Let* $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ *be the strictly convex solution of problems* (1.1)-(1.6)*. We then have the following estimates:*

$$-\frac{1}{2k_{\min}^2} \leq u_{\min} \leq -\frac{1}{2k_{\max}^2}, \tag{1.7}$$

where $u_{\min}$ *is the minimum value of* $u(\mathbf{x})$ *on* $\overline{\Omega}$*. The equality signs hold in the above inequalities if and only if* Ω *is a disk.*

We note that only a few minimum principles for *P-functions* are known in the literature; namely, for *the problem of torsional rigidity* [13], [11], [15], *the problem of torsional creep* [14], *the problem of capillary free surfaces without gravity* [12], [14], *the soap film problem* [3], *the problem of a fluid surface in a short capillary tube* [2] or for a general class of quasilinear elliptic problems, whose solutions satisfy some convexity property [17]. This situation is mainly due to the fact that, without some additional information regarding convexity properties of solutions or uniqueness of their critical point, it seems difficult to get useful lower bound estimates for some second order derivatives of the solutions. To prove our results, some ideas and techniques developed in the above papers will also be employed.

We finally mention that, here and in the remainder of this note, the summation convention over repeated indices (from 1 to 2) is employed and the following notation is adopted: $u_1 := \partial u/\partial x_1$, $u_2 := \partial u/\partial x_2$, $u_{ij} := \partial^2 u/(\partial x_i \partial x_j)$, for $i, j \in \{1, 2\}$, or $u_n := \partial u/\partial n$ and $u_s := \partial u/\partial s$, to denote the outward normal derivative of $u(\mathbf{x})$, respectively its tangential derivative.

2. Proof of Theorem 1.2

Differentiating successively (1.4), we obtain

$$P_k = 2u_{ik}u_i - 2\alpha u_k, \tag{2.1}$$

$$u^{kl}P_{kl} = 2u^{kl}u_{ikl}u_i + 2u^{kl}u_{ik}u_{il} - 2\alpha u^{kl}u_{kl}, \tag{2.2}$$

where $\left\{u^{ij}\right\} =: A^{-1}$ is the inverse of the Hessian matrix $A := D^2u = \{u_{ij}\}$. We will compute separately each term of (2.2). First, we note that

$$u^{kl}u_{kl} = 2\det\left(D^2u\right), \tag{2.3}$$

while

$$u^{kl}u_{ik}u_{il} = tr\left(A^{-1}A^2\right) = tr\,(A) = \Delta u. \tag{2.4}$$

Next, to compute $u^{kl}u_{ikl}u_i$, we differentiate equation (1.1) with respect to x_i and multiply the result with u_i. We thus obtain

$$0 = \frac{\partial\left(\det(D^2u)\right)}{\partial x_i}u_i = \frac{\partial\left(\det(D^2u)\right)}{\partial u_{kl}}\frac{\partial u_{kl}}{\partial x_i}u_i = u^{kl}u_{kli}u_i. \tag{2.5}$$

On the other hand, making use of identity (2.1) to replace the right hand terms of the identity (which holds only in $\mathbb{R}^2$!; see G.A. Philippin - A Safoui [17])

$$\det\left(D^2u\right)|\nabla u|^2 = \Delta u u_{ik}u_iu_k - u_{ik}u_ku_{il}u_l, \tag{2.6}$$

we obtain

$$\det\left(D^2u\right) = \alpha\Delta u - \alpha^2 + \text{ terms containing } P_k \text{ in } \Omega\setminus\omega, \tag{2.7}$$

where $\omega := \{\mathbf{x}\in\Omega : \nabla u(\mathbf{x}) = 0\}$ is the set of critical points of u. Now replacing (2.7) in (2.3), we get

$$u^{kl}u_{kl} = 2\alpha\Delta u - 2\alpha^2 + \text{ terms containing } P_k \text{ in } \Omega\setminus\omega. \tag{2.8}$$

Next, inserting (2.4), (2.5) and (2.8) into (2.2), we obtain

$$LP\,(\mathbf{x},\alpha) := u^{kl}P_{kl} + \text{ terms containing } P_k = 2\Delta u - 4\alpha^2\Delta u + 4\alpha^3 \text{ in } \Omega\setminus\omega. \tag{2.9}$$

Therefore, using in (2.9) the inequality

$$\frac{\Delta u}{2} \geq \left(\det(D^2u)\right)^{1/2} = 1 \text{ in } \Omega, \tag{2.10}$$

one may derive the differential inequality

$$LP\,(\mathbf{x},\alpha) \leq 4\,(\alpha-1)\left(\alpha^2-\alpha-1\right) \text{ in } \Omega\setminus\omega, \text{ for } \alpha \geq 1/\sqrt{2}, \tag{2.11}$$

so that

$$LP(\mathbf{x},\alpha)\begin{cases} \leq 0 \text{ in } \Omega\setminus\omega, \text{ for } \alpha=1, \\ <0 \text{ in } \Omega\setminus\omega, \text{ for } \alpha\in\left(1,(1+\sqrt{5})/2\right). \end{cases} \tag{2.12}$$

On the other hand, from Hopf's maximum principles [7], [8] and the strict convexity of $u(\mathbf{x})$ we deduce that, in fact, $u(\mathbf{x})$ has a single critical (minimal) point in Ω. Now applying Hopf's first maximum principle [7] in (2.12), we conclude that, for $\alpha\in\left[1,(1+\sqrt{5})/2\right)$, $P(\mathbf{x},\alpha)$ attains its minimum value at the single critical point of $u(\mathbf{x})$ or on the boundary $\partial\Omega$, unless it is constant on $\overline{\Omega}$.

Next, we analyze separately the following two cases:

I. *The case* $\alpha\in\left(1,(1+\sqrt{5})/2\right)$:

In this case, we assume on the contrary that the minimum of $P(\mathbf{x},\alpha)$ occurs at the (single) critical point of $u(\mathbf{x})$. We will show that this assumption leads to a contradiction. To this end, eventually performing a translation and/or rotation if necessary, we first choose the coordinate axes such that the (single) critical point of $u(\mathbf{x})$ is located at the origin $\mathbf{O}$ and $u_{12}(\mathbf{O})=u_{21}(\mathbf{O})=0$. Thereafter, we evaluate

$$P_{11}(\mathbf{O},\alpha)=2u_{11}(\mathbf{O})(u_{11}(\mathbf{O})-\alpha),\qquad P_{22}(\mathbf{O},\alpha)=2u_{22}(\mathbf{O})(u_{22}(\mathbf{O})-\alpha). \tag{2.13}$$

Now, since $u(\mathbf{x})$ attains its minimum value at the single critical point $\mathbf{O}\in\Omega$, we clearly have $u_{11}(\mathbf{O})\geq 0$ and $u_{22}(\mathbf{O})\geq 0$. Moreover, since $u(\mathbf{x})$ satisfies the Monge-Ampère equation (1.1), we should also have $u_{11}(\mathbf{O})\neq 0$ and $u_{22}(\mathbf{O})\neq 0$. We thus conclude that:

$$u_{11}(\mathbf{O})>0,\qquad u_{22}(\mathbf{O})>0. \tag{2.14}$$

On the other hand, since the minimum of $P(\mathbf{x},\alpha)$ occurs at $\mathbf{O}$, we have

$$P_{11}(\mathbf{O},\alpha)\geq 0,\qquad P_{22}(\mathbf{O},\alpha)\geq 0. \tag{2.15}$$

Now using (2.13) and (2.14) in (2.15), we get

$$u_{11}(\mathbf{O})\geq\alpha,\qquad u_{22}(\mathbf{O})\geq\alpha. \tag{2.16}$$

Moreover, by multiplication and making use of the Monge-Ampère equation (1.1), we obtain

$$\alpha^2\leq\det\left(D^2u\right)(\mathbf{O})=1, \tag{2.17}$$

which contradicts the assumption that $\alpha>1$. Therefore, for $\alpha\in\left(1,(1+\sqrt{5})/2\right)$, the auxiliary function $P(\mathbf{x},\alpha)$ attains its minimum value on the boundary $\partial\Omega$, unless it is constant on $\overline{\Omega}$. However, in this case, namely for $\alpha\in\left(1,(1+\sqrt{5})/2\right)$, $P(\mathbf{x},\alpha)$ cannot be identically constant on $\overline{\Omega}$, since no constant P satisfies the strict inequality in (2.12). In conclusion, for $\alpha\in\left(1,(1+\sqrt{5})/2\right)$, the auxiliary function $P(\mathbf{x},\alpha)$ takes its minimum value on the boundary $\partial\Omega$.

II. *The case* $\alpha=1$:

In this limiting case, we use a continuity argument. Thus, from the above analysis we know that, when α decreases continuously from $(1+\sqrt{5})/2$ to 1, the points at which $P(\mathbf{x},\alpha)$ takes its minimum value have to move continuously (or, possibly, to remain fixed!) on $\partial\Omega$. Moreover, we also know that $P(\mathbf{x},1)$ attains its minimum value at the single critical point of $u(\mathbf{x})$ or on the boundary $\partial\Omega$, unless it is constant on $\overline{\Omega}$. Since the critical point of $u(\mathbf{x})$ is away from $\partial\Omega$, we then conclude that the auxiliary function $P(\mathbf{x},1)$ should attain its minimum value on $\partial\Omega$, unless it is constant on $\overline{\Omega}$. The proof of Theorem 1.2 is thus achieved. □

Remark 2.1. If $P(\mathbf{x},1)$ is constant on $\overline{\Omega}$, then (2.10) implies that we should have equality in inequality (2.10), which means that the Hessian matrix D^2u has equal eigenvalues at every point of Ω. This fact, together with equation (1.1), imply that D^2u is the identity matrix and $u(\mathbf{x}) = (|x|^2 - R^2)/2$, where $R > 0$ is a constant.

Remark 2.2. From the proof, one may easily notice that Theorem 1.2 remains true if we replace in its statement the capillary boundary condition (1.2) with the Dirichlet boundary condition (1.6), since the solution of the Monge-Ampère problems (1.1)-(1.6) also has a single critical point in Ω.

Remark 2.3. While maximum principles for P-functions depending on solutions to some fully nonlinear elliptic problems and their gradients are already known (see, for instance, C. Enache [6] or L. Barbu - C. Enache [4]), Theorem 1.2 provides the first afferent minimum principle in such a fully nonlinear context. However, we believe that the idea of the proof presented here might be successfully employed to get a similar minimum principle in higher dimension for other fully nonlinear problems, whose solutions have only one critical point in Ω.

3. Proof of Theorem 1.1

From Theorem 1.2 we know that $P(\mathbf{x},1)$ takes its minimum value on the boundary $\partial\Omega$. More precisely, taking into account the boundary condition (1.2), we notice that in fact the minimum of $P(\mathbf{x},1)$ on $\partial\Omega$ must be a point $\mathbf{Q} \in \partial\Omega$, where $u(\mathbf{x})$ itself attains its maximum. Therefore, we have

$$u_s(\mathbf{Q}) = 0,\ u_{ss}(\mathbf{Q}) \leq 0, \tag{3.1}$$

and

$$\partial P(\mathbf{Q},1)/\partial n \leq 0. \tag{3.2}$$

Using the fact that $u_s(\mathbf{Q}) = 0$ and $u_n > 0$ on $\partial\Omega$, inequality (3.2) implies

$$u_{nn}(\mathbf{Q}) \leq 1, \tag{3.3}$$

where u_{nn} denotes the second order outward normal derivative of $u(\mathbf{x})$.

On the other hand, since the boundary $\partial\Omega$ is smooth, we evaluate inequality (2.10) on $\partial\Omega$, in normal coordinates with respect to $\partial\Omega$, to obtain

$$u_{nn} + ku_n + u_{ss} \geq 2 \text{ at } \mathbf{Q} \in \partial\Omega. \tag{3.4}$$

Therefore, making use of (3.1) and (3.3) in (3.4), we get

$$k(\mathbf{Q})\, u_n(\mathbf{Q}) \geq 1. \tag{3.5}$$

Also, from the boundary condition (1.2), we note that we have

$$\sin^2\theta u_n^2 = \cos^2\theta + \cos^2\theta u_s^2 \text{ on } \partial\Omega, \tag{3.6}$$

so that, making use of (3.1), we get

$$u_n(\mathbf{Q}) = \cot\theta. \tag{3.7}$$

Now, the insertion of (3.7) in (3.5) achieves the proof of Theorem 1.1. □

Remark 3.1. The inequality obtained in Theorem 1.1 is sharp, since when Ω is a disk of radius, centered at the origin, i.e. $\Omega := B(\mathbf{O},R)$, and $u(\mathbf{x}) := (|\mathbf{x}|^2 - R^2)/2$, we have $k(s) = \tan\theta$ on $\partial\Omega$ (see X.-N. Ma [10]).

4. Proof of Theorem 1.3

Let us consider the auxiliary function $P(\mathbf{x},1)$ defined in (1.4), but this time with $u(\mathbf{x})$ being the solution of problems (1.1)-(1.6). As we have already noticed in Remark 2.2, $P(\mathbf{x},1)$ takes its minimum value at some point $\mathbf{Q}\in\partial\Omega$, unless it is constant on $\overline{\Omega}$. This fact leads to the following inequality:

$$|\nabla u|^2 - 2u \geq q_m^2, \tag{4.1}$$

where q_m is the minimum value of $|\nabla u|$ on $\partial\Omega$. Evaluating (4.1) at the unique minimal point of $u(\mathbf{x})$, we obtain

$$-2u_{\min} \geq q_m^2. \tag{4.2}$$

Next, we would like to construct a lower bound for q_m, in terms of the curvature of $\partial\Omega$. To this end, we first note that, since $P(\mathbf{x},1)$ takes its minimum value at the point $\mathbf{Q}\in\partial\Omega$, we have $\partial P(\mathbf{x},1)/\partial n \leq 0$ or

$$2u_n u_{nn} - 2u_n \leq 0 \text{ at } \mathbf{Q}. \tag{4.3}$$

On the other hand, since the boundary $\partial\Omega$ is smooth, equation (1.1) may be evaluated on $\partial\Omega$, in normal coordinates with respect to $\partial\Omega$, as it follows (see Lemma 8 from G.A. Philippin - A. Safoui [16])

$$k u_{nn} u_n = 1 \text{ on } \partial\Omega, \tag{4.4}$$

where $k(s)$ is the curvature of $\partial\Omega$. Therefore, inserting (4.4) in (4.3) and taking into account that $u_n > 0$ on $\partial\Omega$ (due to Hopf's second maximum principle [8]), we get

$$q_m = |\nabla u|(\mathbf{Q}) \geq \frac{1}{k(\mathbf{Q})} \geq \frac{1}{k_{\max}}. \tag{4.5}$$

Now using (4.5) in (4.2), we obtain

$$u_{\min} \leq -\frac{1}{2k_{\max}^2}. \tag{4.6}$$

In a similar way, using the fact that $P(\mathbf{x},1)$ takes its maximum value on $\partial\Omega$, unless it is constant on $\overline{\Omega}$ (see G.A. Philippin - A. Safoui [16]), one may also derive the first inequality in (1.7). Obviously, both inequalities in (1.7) are sharp, since the equality signs hold when $P(\mathbf{x},1)$ is constant on $\overline{\Omega}$, namely, when Ω is a ball (see Remark 2.1 and use the fact that $u(\mathbf{x})$ satisfies the Dirichlet boundary condition (1.6)). The proof of Theorem 1.3 is thus achieved. □

Remark 4.1. We note that similar isoperimetric inequalities have been obtained for *the stress function*, i.e. the solution $u(x)$ of *the torsional rigidity problem*, by C. Bandle in [1] and X.-N. Ma in [11]. While C. Bandle's proof relies on a purely geometric result and the domain monotonicity of the stress function, X.-N. Ma's proof is based on some similar maximum and minimum principles for P-functions.

Remark 4.2. The isoperimetric estimates (1.7) can also be derived from the fact that, for $R := 1/k_{\min}$ and $r := 1/k_{\max}$, there are some points $\xi_1,\, \xi_2 \in \mathbb{R}^2$ such that $B_r(\xi_1) \subseteq \Omega \subseteq B_R(\xi_2)$. Therefore, the functions

$$\frac{1}{2}\left(|x-\xi_1|^2 - r^2\right), \qquad \frac{1}{2}\left(|x-\xi_2|^2 - R^2\right) \tag{4.7}$$

are upper and lower barriers for $u(\mathbf{x})$ and (1.7) follows immediately using the comparison principle.

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