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A SYMPLECTIC FUNCTIONAL ANALYTIC PROOF OF THE CONFORMAL WELDING THEOREM

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ABSTRACT. We give a new functional-analytic/symplectic geometric proof of the conformal welding theorem. This is accomplished by representing composition by a quasisymmetric map ϕ as an operator on a suitable Hilbert space and algebraically solving the conformal welding equation for the unknown maps f and g satisfying $g \circ \phi = f$. The univalence and quasiconformal extendibility of f and g is demonstrated through the use of the Grunsky matrix.

1. Introduction

In this paper we give a new proof of the conformal welding theorem. The conformal welding theorem states that any quasisymmetric function $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ has a factorization $\phi = g^{-1} \circ f$ where f and g are normalized conformal maps of $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$ with quasiconformal extensions to the sphere. The standard proof uses the existence and uniqueness of solutions (i.e., well-posedness) to the Beltrami equation, which is in turn based on the theory of singular integral operators, and was first proven by A. Pfluger [16]. Following an idea suggested by G. F. Mandéavidze and B. V. Hvedelidze in [6], F. D. Gakhov gave a proof for the case of diffeomorphisms (i.e., smooth quasisymmetries) based on the Hilbert transform on the circle which is presented in his monograph [3]. Later, a proof was given by O. Lehto and K. I. Virtanen [13, II.7.4-II.7.5] using an approximation method. References can be found in the standard text [12]. Around the same time that [12] was published, A. A. Kirillov [9] independently gave Pfluger's proof using the Beltrami equation. However, he only stated the theorem for the case of diffeomorphisms of the circle (on Kirillov's work, see below). After the completion of our manuscript, another proof by P. Ebenfelt, D. Khavinson and H. S. Shapiro [2] came to our attention. They prove the conformal welding theorem (referring to it as Kirillov's theorem) in the case of diffeomorphisms of the circle, using an approximation argument distinct from that of Lehto-Virtanen.

In this paper, we give a proof via an entirely new approach, which uses a representation of quasisymmetries of the circle as bounded operators on the Sobolev space of L^2 functions on the circle with L^2 half-order derivatives and mean value zero, due to Nag and Sullivan. This allows one to write the conformal welding

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equation as an operator equation on a Hilbert space $\hat{\Pi}_\phi g = f$ where g and f are unknown elements of the Sobolev space. It was observed by B. Penfound and E. Schippers [15] that the matrix representation of this operator equation is in some sense solvable in spite of having two unknown functions, because of the particular form of the equation. This observation, placed in the proper functional analytic context mentioned above, is the heart of the proof.

The role of symplectic/Kähler geometry was discovered by A. A. Kirillov [9] and A. A. Kirillov and D. V. Yur'ev [10], [11], as an example of the coadjoint orbit method of Kirillov/Kostant/Souriau. S. Nag and D. Sullivan [14] extended this picture to the general case of quasisymmetries, and observed in particular that composition by quasisymmetries is a bounded symplectomorphism, and acts on the set of positive polarizing Lagrangian subspaces of the Sobolev space. Thus the quasisymmetries (modulo Möbius transformations of the circle) could be represented as elements of an infinite Siegel disc of symmetric operators whose graphs were the polarizing subspaces. This resulted in a new model of the universal Teichmüller space with pleasing analogies with the finite-dimensional case.

This picture was further developed and clarified by L. Takhtajan and L-P. Teo in [20] in connection to their work on the Weil-Petersson metric on universal Teichmüller space. One of their many deep insights was the realization that the elements of the Siegel disc were in fact Grunsky matrices.

In the present work, we require the Kirillov-Yur'ev/Nag-Sullivan/Takhtajan-Teo picture for two reasons. First, we need to show that a certain block of the composition operator associated with a quasisymmetry is invertible. This fact follows easily from the conformal welding theorem, but since we are of course unable to use it, we give an alternate proof using the representation by Lagrangian subspaces described above. Second, in order to show that the functions f and g in the Sobolev space are in fact quasiconformally extendible conformal maps, we use the result of Takhtajan and Teo relating the quasisymmetry to the Grunsky matrix. By establishing a bound on the Grunsky matrix, we can conclude by a function theoretic result of Pommerenke that the maps f and g are univalent maps with quasiconformal extensions. We remark that our proof is in the spirit of Kirillov and Kirillov-Yur'ev's understanding of the role of symplectic group actions in representations of $\mathrm{Diff}(\mathbb{S}^1)$ (although we stress again that Kirillov's proof is the same as that of Pfluger).

We also like to mention that our proof is in some sense algebraic. The analytic issues are solved by placing the algebraic proof in the proper functional analytic setting. The idea of an algebraic proof of the conformal welding theorem is motivated by conformal field theory, where one can find a purely algebraic "sewing equation" which is closely related to the conformal welding equation $g \circ \phi = f$. Y.-Z. Huang [5] showed that this has a purely algebraic solution. See [15] for a discussion.

In order for our proof to be new, it is necessary to avoid using the well-posedness of the Beltrami equation, and we have therefore taken great care to ensure that this is the case. The proofs are of comparable lengths if one discounts the "preamble" (the action by symplectomorphisms and the Grunsky matrix in the present paper, and singular integral equations and the Beltrami equation in the standard proof).

It goes without saying that the purpose of this paper is not to advance our functional analytic proof (based on symplectic geometry) as somehow superior or preferable to the standard harmonic analytic one (based on singular integrals). Rather, we would like to shed new light on this fundamental theorem, in particular the role of the symplectic action by the quasisymmetries of the circle and the period matrix of Kirillov-Yuriev/Nag-Sullivan/Takhtajan-Teo; see [10], [11], [14], [20].

2. Representations of quasisymmetric mappings

2.1. **Preliminaries.** In this section we define the relevant function spaces and their relation to each other. For details see [14]. We shall sometimes use the shorthand notation $a \lesssim b$ instead of $a \leq Cb$ (where C is a constant).

Definition 2.1. Let $\mathcal{H}_{\mathbb{R}}$ be the set of real-valued L^2 functions on $\mathbb{S}^1$ with half-order derivatives in L^2. Denote the set of functions in $\mathcal{H}_{\mathbb{R}}$ with mean value 0 by $\mathcal{H}_{\mathbb{R}*}$.

Similarly, $\mathcal{H}$ will denote the set of complex-valued L^2 functions on $\mathbb{S}^1$ with half-order derivatives in L^2, and $\mathcal{H}_*$ will denote those with mean value zero.

Now given a function $f \in L^2(\mathbb{S}^1)$, its Fourier transform is defined by $\hat{f}(n) := \frac{1}{2\pi}\int_{\mathbb{S}^1} f(t)e^{-int}\,dt$. For $f \in \mathcal{H}_{\mathbb{R}}$, f has a Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta},$$

with $\hat{f}(n) = \overline{\hat{f}(-n)}$, which converges off a set of capacity zero on $\mathbb{S}^1$ to $f(e^{i\theta})$. Of course a given $f \in \mathcal{H}_{\mathbb{R}}$ is in $\mathcal{H}_{\mathbb{R}*}$ if and only if $\hat{f}(0) = 0$.

For any $f \in \mathcal{H}_{\mathbb{R}}$ we also have that

$$\sum_{n=-\infty}^{\infty} |n||\hat{f}(n)|^2 < \infty. \tag{2.1}$$

Conversely, a series satisfying equation (2.1) converges except on a set of capacity zero to a function in $\mathcal{H}_{\mathbb{R}}$. The complexification

$$\mathcal{H} = \mathcal{H}_{\mathbb{R}} \oplus i\,\mathcal{H}_{\mathbb{R}},$$

where $i = \sqrt{-1}$, is thus the set of complex-valued functions in $L^2(\mathbb{S}^1)$ with Fourier series satisfying $\sum_{n=-\infty}^{\infty} |n||\hat{f}(n)|^2 < \infty$ and no further restrictions on the coefficients. We define a norm on $\mathcal{H}$ by

$$\|f\|_{\mathcal{H}} = \{|\hat{f}(0)|^2 + \sum_{n=-\infty, n\neq 0}^{\infty} |n||\hat{f}(n)|^2\}^{\frac{1}{2}}.$$

Denote by $\mathcal{H}_*$ the set of those elements in $\mathcal{H}$ which have zero mean value.

For

$$f = \sum_{n=-\infty, n\neq 0}^{\infty} \hat{f}(n)e^{in\theta}, \quad g = \sum_{n=-\infty, n\neq 0}^{\infty} \hat{g}(n)e^{in\theta}$$

in $\mathcal{H}_*$ define the complex inner product

$$\langle f, g\rangle = \sum_{n=-\infty, n\neq 0}^{\infty} |n|\overline{\hat{f}(n)}\hat{g}(n). \tag{2.2}$$

With this inner product, $\mathcal{H}_*$ is a complex Hilbert space. We endow $\mathcal{H}$ with the complex inner product $(f, g) = \left\langle f - \hat{f}(0), g - \hat{g}(0)\right\rangle + \overline{\hat{f}(0)}\hat{g}(0)$.

For $f, g \in \mathcal{H}_{\mathbb{R}*}$ we have

$$\langle f, g\rangle = 2\,\mathrm{Re}\left(\sum_{n=1}^{\infty} n\overline{\hat{f}(n)}\hat{g}(n)\right),$$

and the restriction of the inner product to $\mathcal{H}_{\mathbb{R}\,*}$ is thus real. With this inner product $\mathcal{H}_{\mathbb{R}\,*}$ is a real Hilbert space.

The Hilbert space $\mathcal{H}$ has a natural decomposition

$$\mathcal{H}_* = \mathcal{H}_+ \oplus \mathcal{H}_-, \tag{2.3}$$

where $\mathcal{H}_+$ consists of those functions with non-zero Fourier coefficients only for $n > 0$, and $\mathcal{H}_-$ consists of those functions with non-zero Fourier coefficients only for $n < 0$.

We define a complex structure on $\mathcal{H}_{\mathbb{R}\,*}$ via the Hilbert transform

$$Jf(e^{i\theta}) = \sum_{n=-\infty, n\neq 0}^{\infty} -i\mathrm{sgn}(n)\hat{f}(n)e^{in\theta},$$

where $\mathrm{sgn}(n)$ is 1 for $n > 0$ and -1 for $n < 0$. The operator J extends complex linearly to $\mathcal{H}_*$. If we identify elements of $\mathcal{H}_*$ with column vectors $(f_+, f_-)^T$ where $\cdot^T$ denotes transpose and $f_\pm \in \mathcal{H}_\pm$, then J has the matrix

$$J = \begin{pmatrix} -i \cdot I & 0 \\ 0 & i \cdot I \end{pmatrix},$$

with respect to the basis z^n, where I is the identity matrix. Note that $\mathcal{H}_{\mathbb{R}\,*}$ and $i \cdot \mathcal{H}_{\mathbb{R}\,*}$ are invariant under J. Furthermore $\mathcal{H}_\pm$ are precisely the $\mp i$-eigenspaces of J so J induces the decomposition (2.3). It is immediate that the inner product on $\mathcal{H}$ is compatible with J in the sense that $\langle Jf, Jg\rangle = \langle f, g\rangle$ for all $f, g \in \mathcal{H}_*$.

There is a natural anti-symmetric form ω on $\mathcal{H}$ given by

$$\omega(f,g) = \langle f, Jg\rangle = \sum_{n=1}^{\infty}\left(-in\overline{\hat{f}(n)}\hat{g}(n) + in\overline{\hat{f}(-n)}\hat{g}(-n)\right).$$

By the Cauchy-Schwarz inequality ω is continuous in both entries. It is not hard to see using the decomposition $\mathcal{H}_* = \mathcal{H}_+ \oplus \mathcal{H}_-$ and the non-degeneracy of $\langle \cdot, \cdot\rangle$ that ω is non-degenerate. For $f, g \in \mathcal{H}_{\mathbb{R}\,*}$ we have that

$$\omega(f,g) = 2\,\mathrm{Im}\left(\sum_{n=1}^{\infty} n\overline{\hat{f}(n)}\hat{g}(n)\right),$$

and thus the restriction of ω to $\mathcal{H}_{\mathbb{R}\,*}$ is real. We will refer to ω as a symplectic form (real on $\mathcal{H}_{\mathbb{R}\,*}$, complex on $\mathcal{H}_*$).

Remark 2.2. In the case that $f, g \in \mathcal{H}_{\mathbb{R}\,*}$ are sufficiently regular, it is easily checked that

$$\omega(f,g) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f\,dg,$$

and hence we may interpret $\omega(f,g)$ as the signed area of the image of $\mathbb{S}^1$ under the map (f, g).

We say that $\Phi : \mathcal{H}_* \to \mathcal{H}_*$ is a symplectomorphism if Φ is a map such that $\omega(\Phi\mathbf{v}, \Phi\mathbf{w}) = \omega(\mathbf{v}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{H}_*$. The symplectomorphisms $\Phi : \mathcal{H}_{\mathbb{R}\,*} \to \mathcal{H}_{\mathbb{R}\,*}$ are defined in a similar way.

Finally we define quasisymmetric maps.

Definition 2.3. A map $\phi : \mathbb{R} \to \mathbb{R}$ is a quasisymmetry if there is a constant K such that

$$\frac{1}{K} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq K$$

for all x and h in $\mathbb{R}$. A map $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetry if there it is conjugate to a quasisymmetry of $\mathbb{R}$ by a Möbius transformation.

2.2. **Quasisymmetries as composition operators.** It was shown by Nag and Sullivan [14] that if ϕ is a quasisymmetric map, then the composition operator

$$\begin{aligned} \hat{\Pi}_\phi : \mathcal{H}_{\mathbb{R}\,*} &\to \mathcal{H}_{\mathbb{R}\,*} \\ f &\mapsto f\circ\phi - \frac{1}{2\pi}\int_{\mathbb{S}^1} f\circ\phi \end{aligned} \tag{2.4}$$

is a bounded symplectomorphism. Hence the same is true for the complex linear extension of $\hat{\Pi}_\phi$ to $\mathcal{H}_*$, which we will also denote by $\hat{\Pi}_\phi$. We will need to extend this result to $\mathcal{H}$. In order to do this it will be convenient to work with a Sobolev space with equivalent inner product.

Definition 2.4. The fractional Sobolev space $H^{\frac{1}{2}}(\mathbb{S}^1)$ is the space of functions $f \in L^2(\mathbb{S}^1)$ such that $\sum_{n=-\infty}^{\infty}(1+|n|^2)^{\frac{1}{2}}|\hat{f}(n)|^2 < \infty$. The norm of u in this space is given by

$$\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)} = \left\{\sum_{n=-\infty}^{\infty}(1+|n|^2)^{\frac{1}{2}}|\hat{f}(n)|^2\right\}^{\frac{1}{2}}.$$

It is well-known that $H^{1/2}(\mathbb{S}^1)$ is equivalent to $\mathcal{H}$.

Proposition 2.5. *The spaces $\mathcal{H}$ and $H^{1/2}(\mathbb{S}^1)$ are equal as sets, and there are constants K_1 and K_2 such that for any $f \in \mathcal{H}$,*

$$K_1\|f\|_{H^{1/2}} \le \|f\|_{\mathcal{H}} \le K_2\|f\|_{H^{1/2}}.$$

The same claims hold for the real subspaces.

Proof. Clearly if $f \in H^{1/2}(\mathbb{S}^1)$, then by the definition of $H^{1/2}$ it is in $L^2(\mathbb{S}^1)$ and

$$|\hat{f}(0)|^2 + \sum_{n=-\infty, n\neq 0}^{\infty}|n||\hat{f}(n)|^2 \le \sum_{n=-\infty}^{\infty}(1+|n|^2)^{1/2}|\hat{f}(n)|^2,$$

which proves the right hand inequality, and thus that $H^{1/2}(\mathbb{S}^1) \subseteq \mathcal{H}$. On the other hand, if $f \in \mathcal{H}$, then by the definition of $\mathcal{H}$, it belongs to $L^2(\mathbb{S}^1)$ and since $(1+|n|^2)^{1/2} \le 1+|n| \le 2|n|$, one has

$$\|f\|^2_{H^{1/2}} = \sum_{n=-\infty}^{\infty}(1+|n|^2)^{1/2}|\hat{f}(n)|^2 \le 2|\hat{f}(0)|^2 + \sum_{n=-\infty, n\neq 0}^{\infty}2|n||\hat{f}(n)|^2 = 2\|f\|^2_{\mathcal{H}},$$

which completes the proof. ☐

Definition 2.6. Let $H^1(\mathbb{D})$ denote the Sobolev space of functions in $L^2(\mathbb{D})$ with $\|F\|_{H^1(\mathbb{D})} := \{\|F\|^2_{L^2(\mathbb{D})} + \|\nabla F\|^2_{L^2(\mathbb{D})}\}^{\frac{1}{2}} < \infty$.

Theorem 2.7. *Let ϕ be a quasisymmetry on $\mathbb{S}^1$. Then the composition map $\Pi_\phi(f) := f\circ\phi$ is bounded from $H^{\frac{1}{2}}(\mathbb{S}^1)$ to $H^{\frac{1}{2}}(\mathbb{S}^1)$.*

Proof. It is a well-known fact that the elements of $H^1(\mathbb{D})$ have a well-defined trace or restriction to the boundary of the disk, i.e., to $\mathbb{S}^1$. Given $f \in H^1(\mathbb{D})$, its trace $f|_{\mathbb{S}^1}$ exists as a function in the Sobolev space $H^{\frac{1}{2}}(\mathbb{S}^1)$; moreover for some $C > 0$ one has $\|f|_{\mathbb{S}^1}\|_{H^{\frac{1}{2}}(\mathbb{S}^1)} \lesssim \|f\|_{H^1(\mathbb{D})}$. Conversely, there is an extension $E : H^{1/2}(\mathbb{S}^1) \to H^1(\mathbb{D})$ such that $\|Ef\|_{H^1(\mathbb{D})} \lesssim \|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}$.

First we extend the quasisymmetry ϕ to a quasi-isometry Φ on $\mathbb{D}$. For this, one uses an explicit construction of extension of a quasisymmetry on the line to a quasi-isometry on the upper half plane due to Z. Ibragimov [7], and thereafter the conformal equivalence of the half plane and the disk. Next we extend f to a function $F = Ef$ in $H^1(\mathbb{D})$. The boundary value of $\Pi_\phi(F)$ is $f \circ \phi$. So by the continuity of the restriction to the boundary it is enough to show the continuity of the composition on the $H^1(\mathbb{D})$. But it is well-known that $\|F \circ \Phi\|_{H^1(\mathbb{D})} \lesssim \|F\|_{H^1(\mathbb{D})}$, for every quasi-isometric homeomorphism Φ; see e.g. [4, Theorem 4.4′] and its Corollary 1. Thus

$$\|f \circ \phi\|_{H^{\frac{1}{2}}(\mathbb{S}^1)} \lesssim \|F \circ \Phi\|_{H^1(\mathbb{D})} \lesssim \|F\|_{H^1(\mathbb{D})} \lesssim \|f\|_{H^{1/2}(\mathbb{S}^1)},$$

which completes the proof. □

Remark 2.8. Using orthogonal projection onto the subspaces with mean value zero, this gives an alternate proof of one direction in Nag and Sullivan's theorem. Their proof uses the Dirichlet principle and quasi-invariance of Dirichlet energy.

3. The conformal welding theorem

3.1. **The infinite Siegel disc.** The composition operator $\hat{\Pi}_\phi$ acts by symplectomorphisms on $\mathcal{H}_*$, by Nag and Sullivan's theorem. They showed that the quasisymmetries parametrize the set of polarizations in $\mathcal{H}_*$, giving an infinite-dimensional version of the so-called Siegel disc, which can be seen as a representation of the universal Teichmüller space. The results in Nag and Sullivan's paper were clarified and strengthened by Takhtajan and Teo [20] as part of their far-reaching work on the Weil-Petersson metric on universal Teichmüller space. They further showed that the polarizing Lagrangian subspaces are graphs of Grunsky matrices. This picture forms an essential part of our proof of the conformal welding theorem. However, those papers assume the validity conformal welding theorem, so we must avoid using results which rely on it. Thus particular care is necessary in presenting our argument.

We follow [21] for the general theory of Lagrangian subspaces. We will fix a basis and treat the operators as matrices in this fixed basis, since overall this yields a simpler presentation. For the same reason we will not distinguish between operators and matrices notationally; since we work in a fixed basis this will cause no confusion.

Choosing the orthonormal basis $\{z^n/\sqrt{\pi|n|}\}$, $n \neq 0$, with the ordering $n = 1, 2, \ldots, -1, -2, \ldots$ and using the decomposition $\mathcal{H}_* = \mathcal{H}_+ \oplus \mathcal{H}_-$, we can see that the inner product on $\mathcal{H}$ has the form

$$(\mathbf{v}, \mathbf{w}) = \overline{\mathbf{v}}^T \mathbf{w}.$$

The fact that an operator $\Phi : \mathcal{H}_* \to \mathcal{H}_*$ preserves $\mathcal{H}_{\mathbb{R}*}$ means that the matrix of Φ in this basis has the form

$$[\Phi] = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix},$$

where $\overline{B}$ denotes the complex conjugate of the matrix B. As an operator, $\overline{B}$: $\mathcal{H}_+ \to \mathcal{H}_-$ should be written as follows. Let $R : \mathcal{H}_* \to \mathcal{H}_*$ be the bounded linear operator such that $R(e^{in\theta}) = e^{-in\theta}$ for $n \in \mathbb{Z}$, $n \neq 0$. The matrix of $\overline{B}$ is obviously the complex conjugate of B. As noted above we will make no notational distinction between the matrix and the operator. Similarly for $\overline{A}$.

Since the change of basis matrix from z^n to $z^n/\sqrt{\pi|n|}$ is diagonal, the operator J has the same matrix in both bases, and we can write

$$\omega(v,w) = \overline{\mathbf{v}}^T J \mathbf{w},$$

where $\mathbf{v}$, $\mathbf{w}$ denote the vector of coefficients representing v and w. In the orthonormal basis, the fact that Φ is a symplectomorphism can thus be written as

$$\overline{\Phi}^T J \Phi = \overline{\begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}}^T J \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} = J.$$

This is equivalent to the identities

$$\overline{A}^T A - B^T \overline{B} = I, \quad \overline{A}^T B = B^T \overline{A}. \tag{3.1}$$

Assuming that Φ is invertible, it is easily checked that Φ^{-1} is also a symplectomorphism. In that case the following relations also hold:

$$A\overline{A}^T - B\overline{B}^T = I, \quad AB^T = BA^T. \tag{3.2}$$

In fact, equations (3.2) and (3.1) are equivalent under the assumption that Φ is invertible.

We define a set of operators, which parametrizes the set of Lagrangian subspaces of $\mathcal{H}_*$, following [18] and [21]. This will be used to give a description of the infinite Siegel disc.

Definition 3.1. Let $\mathcal{LF}$ be the set of operators

$$(P,Q) : \mathcal{H}_- \to \mathcal{H}_*,$$

where $P : \mathcal{H}_- \to \mathcal{H}_+$ and $Q : \mathcal{H}_- \to \mathcal{H}_-$ are bounded operators satisfying $\overline{P}^T P - \overline{Q}^T Q > 0$ (that is, it is positive definite) and $Q^T P = P^T Q$.

The important thing for our purposes is that $\mathcal{LF}$ is invariant under bounded symplectomorphisms.

Proposition 3.2. *$\mathcal{LF}$ is invariant under bounded symplectomorphisms, in the sense that if Ψ is a bounded symplectomorphism which preserves $\mathcal{H}_{\mathbb{R}*}$, then*

$$\Psi \begin{pmatrix} P \\ Q \end{pmatrix} \in \mathcal{LF}.$$

Proof. Ψ is a symplectomorphism if and only if $\Psi^* J \Psi = J$. We may characterize the conditions that $Q^T P = P^T Q$ by

$$(P^T \, Q^T) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} J \begin{pmatrix} P \\ Q \end{pmatrix} = 0.$$

If Ψ is a bounded symplectomorphism that preserves $\mathcal{H}_{\mathbb{R}*}$, then

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \Psi = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} = \begin{pmatrix} \overline{A} & \overline{B} \\ B & A \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \overline{\Psi} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned}(P^T\,Q^T)\,\Psi^T\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}J\Psi\begin{pmatrix}P\\ Q\end{pmatrix} &= (P^T\,Q^T)\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\overline{\Psi}^T J\Psi\begin{pmatrix}P\\ Q\end{pmatrix}\\ &= (P^T\,Q^T)\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}J\begin{pmatrix}P\\ Q\end{pmatrix} = 0,\end{aligned}$$

which proves that $\Psi\begin{pmatrix}P\\ Q\end{pmatrix} \in \mathcal{LF}$.

The condition that $\overline{P}^T P - \overline{Q}^T Q > 0$ can be written as

$$\frac{1}{i}\left(\overline{P}^T, \overline{Q}^T\right) J \begin{pmatrix}P\\ Q\end{pmatrix} > 0,$$

which is clearly invariant under symplectomorphisms. □

We will need the fact that Q has a left inverse.

Proposition 3.3. *If $(P, Q) \in \mathcal{LF}$, then Q has a left inverse.*

Proof. If $Q\mathbf{v} = 0$, then by the positive-definiteness of $\overline{Q}^T Q - \overline{P}^T P$,

$$0 \leq \overline{\mathbf{v}}^T\left(\overline{Q}^T Q - \overline{P}^T P\right)\mathbf{v} = -\overline{\mathbf{v}}^T \overline{P}^T P\mathbf{v} = -\|P\mathbf{v}\|^2.$$

Thus $Pv = 0$. This implies that $\overline{\mathbf{v}}^T\left(\overline{Q}^T Q - \overline{P}^T P\right)\mathbf{v} = 0$ so $\mathbf{v} = 0$. Thus Q is injective, or equivalently Q has a left inverse. □

Remark 3.4. Note that we are only claiming that Q has a left inverse as a linear map.

Theorem 3.5. *If Φ is a bijective bounded symplectomorphism with block matrix*

$$[\Phi] = \begin{pmatrix}A & B\\ \overline{B} & \overline{A}\end{pmatrix}, \tag{3.3}$$

then the operators A and $\overline{A}$ have bounded inverses.

Proof. Since Φ is bounded and orthogonal projection is bounded, A and $\overline{A}$ are bounded. By the open mapping theorem, it is enough to show that $\overline{A}$ is invertible as a linear map. Now, Proposition 3.2 and the facts that $(0, I) \in \mathcal{LF}$ and Φ is a bounded symplectomorphism yield that

$$\begin{pmatrix}B\\ \overline{A}\end{pmatrix} = \begin{pmatrix}A & B\\ \overline{B} & \overline{A}\end{pmatrix}\begin{pmatrix}0\\ I\end{pmatrix}$$

is in $\mathcal{LF}$. Thus by Proposition 3.3 $\overline{A}$ has a left inverse (and hence so does A).

Using the invertibility of the operator Φ, we may apply the symplectic relations (3.1) (3.2) to show that the matrix of Φ^{-1} is

$$[\Phi^{-1}] = \begin{pmatrix}\overline{A}^T & -B^T\\ -\overline{B}^T & A^T\end{pmatrix}. \tag{3.4}$$

Since Φ^{-1} is also a bounded symplectomorphism, we can apply the reasoning in the previous paragraph to see that A^T has a left inverse. Thus A and $\overline{A}$ are invertible. □

At this point we shall define the infinite Siegel disc, and refer the reader to Siegel [18] for the Siegel upper half plane in the finite-dimensional case, and [14] and [20] for the infinite-dimensional disc case.

Definition 3.6. The infinite Siegel disc $\mathfrak{S}$ is the set of maps $Z : \mathcal{H}_- \to \mathcal{H}_+$ such that $Z^T = Z$ and $I - Z\overline{Z}$ is positive definite.

The following two consequences play a key role in the proof of the conformal welding theorem.

Proposition 3.7. *If $\Psi : \mathcal{H}_* \to \mathcal{H}_*$ is an invertible real bounded symplectomorphism and A, B are the blocks of Ψ as in equation (3.3), then $B\overline{A}^{-1} \in \mathfrak{S}$.*

Proof. The invertibility of A follows from Theorem 3.5.

Set $Z = B\overline{A}^{-1}$. The fact that $Z^T = \left(B\overline{A}^{-1}\right)^T = B\overline{A}^{-1} = Z$ follows from the second relation of (3.1). Since $(B, \overline{A})^T \in \mathcal{LF}$ it follows that $A^T\overline{A} - \overline{B}^T B > 0$, so $I - Z\overline{Z}$ is positive definite. This proves the second claim. □

In particular, given a quasisymmetry $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ this applies to the blocks of the matrix $\hat{\Pi}_\phi$, since the inverse of a quasisymmetry is a quasisymmetry [12].

Corollary 3.8. *Let $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ be a quasisymmetry and $\hat{\Pi}_\phi$ be the corresponding operator as in* (2.4)*. If $P_\pm$ denotes the projections onto $\mathcal{H}_\pm$, then $P_-\hat{\Pi}_\phi\Big|_{\mathcal{H}_-}$ and $P_+\hat{\Pi}_\phi\Big|_{\mathcal{H}_+}$ are bounded invertible operators.*

Remark 3.9. The matrix $B\overline{A}^{-1}$ will be seen to be the Grunsky matrix of one of the conformal welding maps associated to ϕ.

Note that invertibility of the matrix A is established in the course of the proof of the relations concerning $B\overline{A}^{-1}$ in [20, Chapter II, Proposition 5.1]. This proof assumed the existence of the conformal welding maps, so we were obligated to give a different proof above.

3.2. **The conformal welding theorem.** In this section we prove the conformal welding theorem. Since the Grunsky matrix plays a central role in the proof, we proceed by defining it and refer the reader to [8], [17] or [20] for details.

Let

$$g(z) = g_{-1}z + g_0 + \frac{g_1}{z} + \frac{g_2}{z^2} + \cdots \tag{3.5}$$

be a meromorphic function in a neighbourhood of ∞, where $g_{-1} \neq 0$. Let

$$f(z) = f_1 z + f_2 z^2 + \cdots \tag{3.6}$$

be a holomorphic function in a neighbourhood of 0, where $f_1 \neq 0$. The nth Faber polynomial of f, denoted Q_n^-, is defined to be the principal part of $f^{-1}(z)^{-n}$ where $f^{-1}(z)$ denotes the inverse of f. The nth Faber polynomial of g, denoted Q_n^+, is the polynomial part of $g^{-1}(z)^n$. Note that the definition depends on the existence of a local inverse for f and g near 0 and ∞, respectively, which is guaranteed by the conditions on the first coefficient.

Define the Grunsky coefficients of f to be $b_{-n\,-m}$ where

$$Q_n^-(f(z)) = z^{-n} + n\sum_{m=1}^{\infty} b_{-n\,-m} z^m. \tag{3.7}$$

Similarly the Grunsky coefficients of g are b_{nm} where

$$Q_n^+(g(z)) = z^n + n \sum_{m=1}^{\infty} b_{nm} z^{-m}. \tag{3.8}$$

It is a classical result that the Grunsky coefficients have the generating function

$$\log \frac{g(z) - g(\zeta)}{z - \zeta} = \log g'(\infty) - \sum_{n,m=1}^{\infty} b_{nm} z^{-n} \zeta^{-m} \tag{3.9}$$

(this is sometimes given as the definition). For f we have

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} - \log \frac{f(z)}{z} - \log \frac{f(\zeta)}{\zeta} = -\log f'(0) + \sum_{n,m=1}^{\infty} b_{-n,-m} z^n \zeta^m. \tag{3.10}$$

Observe that the Grunsky coefficients of αg and g are equal for any non-zero complex number α. We will define the Grunsky matrix of f to be $\mathcal{C}_-(f) = (c_{nm})$ where

$$c_{nm} = (\sqrt{nm} b_{-n\,-m})$$

for $n > 0$ and $m > 0$ and the Grunsky matrix of g to be $\mathcal{C}_+(g) = (c_{nm})$ where

$$c_{nm} = (\sqrt{nm} b_{nm})$$

for $n > 0$ and $m > 0$. It is well-known that $\mathcal{C}_-(f)$ and $\mathcal{C}_+(g)$ are symmetric, as can be read from the expressions (3.9) and (3.10).

Remark 3.10. There are many different choices in the literature for “Grunsky matrix”; for example b_{nm} also bears that name. The different choices correspond to different choices of basis of the underlying function space.

The transformation

$$h(z) \mapsto \tilde{h}(z) = \frac{1}{\overline{h(1/\bar{z})}}$$

takes functions of the type (3.6) to the type (3.5) and vice versa. This transformation is useful in that it has a simple effect on the conformal welding equation and the Grunsky matrices. It is easily checked that

$$\mathcal{C}_+(\tilde{f}) = \overline{\mathcal{C}_-(f)}. \tag{3.11}$$

We shall also need the Dirichlet spaces

$$\mathcal{D}(\mathbb{D}) = \left\{ h : \mathbb{D} \to \mathbb{C} \,:\, h(0) = 0, \quad h \text{ holomorphic}, \quad \iint_{\mathbb{D}} |h'|^2 < \infty \right\}$$

and

$$\mathcal{D}(\mathbb{D}^*) = \left\{ h : \mathbb{D}^* \to \mathbb{C} \,:\, h(\infty) = 0, \quad h \text{ holomorphic}, \quad \iint_{\mathbb{D}^*} |h'|^2 < \infty \right\}.$$

It is classical (see e.g. [1] and [19]) that any $h \in \mathcal{H}_\pm$ has an extension to $\mathcal{D}(\mathbb{D})$ or $\mathcal{D}(\mathbb{D}^*)$, respectively, by replacing $e^{in\theta}$ by z^n, and conversely elements of $\mathcal{D}(\mathbb{D})$ or $\mathcal{D}(\mathbb{D}^*)$ have traces on $\mathbb{S}^1$ which are in $\mathcal{H}_\pm$. Furthermore extension and restriction are isometries.

We will need the following theorem, which was proven in [20, Chapter II, Proposition 5.1]. Note that although their stated hypotheses were more restrictive (they require that f and g are the conformal welding maps), in fact their proof carries through in the generality of the theorem below with only trivial changes.

Theorem 3.11 (Takhtajan and Teo)**.** *Let $f(z) = f_1 z + f_2 z^2 + \cdots \in \mathcal{D}(\mathbb{D})$ and $g = g_{-1} z + g_-$ where $g_- \in \mathcal{D}(\mathbb{D}^*)$, and let $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ be a quasisymmetry. Assume that $g \circ \phi = f$ on $\mathbb{S}^1$. Let*

$$\hat{\Pi}_\phi = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \overline{\mathfrak{B}} & \overline{\mathfrak{A}} \end{pmatrix} \quad \textit{and} \quad \hat{\Pi}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}. \tag{3.12}$$

(1) *If $g_{-1} \neq 0$, then $\mathcal{C}_+(g) = \overline{B} A^{-1}$.*
(2) *If $f_1 \neq 0$, then $\mathcal{C}_-(f) = \overline{\mathfrak{B}} \mathfrak{A}^{-1}$.*

Proof. Assume that $g_{-1} \neq 0$. Then, g has an inverse in a neighbourhood of ∞, and we can define the Faber polynomials and Grunsky coefficients (3.8). Let $\tilde{Q}_n^+(z) = Q_n^+(z) - Q_n^+(0)$. Then

$$\begin{aligned} P_- \hat{\Pi}_\phi \left(\tilde{Q}_n^+(g(e^{i\theta})) - e^{in\theta} \right) &= P_- \left(\tilde{Q}_n^+(f(e^{i\theta})) - \phi(e^{i\theta})^n \right) \\ &= -P_- \hat{\Pi}_\phi \left(e^{in\theta} \right) \\ &= -\sum_{k=1}^\infty \sqrt{\frac{n}{k}} \overline{\mathfrak{B}}_{kn} e^{-ik\theta}, \end{aligned}$$

where the first subscript denotes the row number and the second the column number. On the other hand,

$$\begin{aligned} P_- \hat{\Pi}_\phi \left(\tilde{Q}_n^+(g(e^{i\theta})) - e^{in\theta} \right) &= P_- \hat{\Pi}_\phi \left(n \sum_{m=1}^\infty b_{nm} z^m \right) \\ &= n \sum_{m=1}^\infty \sum_{k=1}^\infty \sqrt{\frac{m}{k}} \overline{\mathfrak{A}}_{kn} b_{nm} e^{-ik\theta}, \end{aligned}$$

that is,

$$c_{ln} = -\sum_{k=1}^\infty \overline{\mathfrak{A}}_{lk}^{-1} \overline{\mathfrak{B}}_{kn}.$$

Thus

$$\mathcal{C}_+(g) = \mathcal{C}_+(g)^T = -\left(\overline{\mathfrak{A}}^{-1} \overline{\mathfrak{B}} \right)^T = -\overline{\mathfrak{B}}^T \overline{\mathfrak{A}}^{T^{-1}} = \overline{B} A^{-1}$$

by (3.4).

The proof that $\mathcal{C}_-(f) = \overline{\mathfrak{B}} \mathfrak{A}^{-1}$ if $f_1 \neq 0$ is similar. □

Remark 3.12. It is important for our proof of the conformal welding theorem that the assumption $g_{-1} \neq 0$ suffices for showing that $\mathcal{C}(g) = \overline{B} A^{-1}$. This means that the assumption $f_1 \neq 0$ is not required.

Theorem 3.13 (Conformal Welding Theorem)**.** *Let $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ be a quasisymmetric map and $\alpha \neq 0$ be a fixed complex number. There exist unique maps f and g such that*

(1) *$f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $g : \mathbb{D}^* \to \mathbb{C}^*$ is holomorphic except for a simple pole at ∞.*
(2) *f and g have quasiconformal extensions to $\bar{\mathbb{C}}$.*
(3) *$f(0) = 0$, $g(\infty) = \infty$ and $g'(\infty) = \alpha$.*
(4) *$g^{-1} \circ f = \phi$ on $\mathbb{S}^1$.*

Proof. First we observe that Theorem 2.7 yields that Π_ϕ and $\Pi_{\phi^{-1}}$ are bounded operators on $\mathcal{H}$. Denote the block matrix of $\Pi_{\phi^{-1}}$ in the orthonormal basis by

$$\Pi_{\phi^{-1}} = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix},$$

where if n denotes the row number and m the column number, then M_{++} has entries with $n > 0$ and $m > 0$, M_{+-} has entries with $n > 0$ and $m \leq 0$, M_{-+} has entries with $n \leq 0$ and $m > 0$, and M_{--} has entries with $n \leq 0$ and $m \leq 0$. Note that we are arranging the matrices so that n runs first from 1 to ∞ and then from 0 to $-\infty$ and similarly for the columns, as above. Let (g_+) denote the column vector, with rows ranging from 1 to ∞ and all entries zero except for the row $n = 1$ which has entry α. By Corollary 3.8,

$$M_{++} = P_+ \Pi_{\phi^{-1}}\big|_{\mathcal{H}_+} = P_+ \hat{\Pi}_{\phi^{-1}}\Big|_{\mathcal{H}_+}$$

is invertible. Thus we may set $(f_+) = M_{++}^{-1}(g_+)$, and $(f_+, 0)$ represents an element $f \in \mathcal{H}_+$, which is the trace of an element of $\mathcal{D}(\mathbb{D})$. Next, setting $(g_-) = M_{-+}(f_+)$, it is easily seen that for the element g of $\hat{H}$ represented by (g_+, g_-) and f as above, we have that $\Pi_{\phi^{-1}} f = g$. Furthermore, the proof shows that f and g are the *unique* solutions in $\mathcal{H}$ of this equation such that $g_{-1} = \alpha$.

Now f has an extension to $\mathcal{D}(\mathbb{D})$ which we will also denote by f. Also g_- has an extension to $\mathcal{D}(\mathbb{D}^*)$, say

$$g_- = g_0 + \frac{g_1}{z} + \frac{g_2}{z^2} + \cdots,$$

and let

$$g(z) = \alpha z + g_0 + \frac{g_1}{z} + \cdots.$$

This function is thus meromorphic on $\mathbb{D}$, and we will not distinguish these functions notationally. Now it follows that f and g satisfy $\Pi_{\phi^{-1}} f = g$ on $\mathbb{S}^1$ as required. However, we must show that f and g are quasiconformally extendible univalent functions satisfying the required normalizations.

Since $g_{-1} = \alpha \neq 0$, Theorem 3.11 yields that $\mathcal{C}_+(g) = \overline{B}A^{-1}$. We will denote $\overline{B}A^{-1}$ by Z. By Proposition 3.7, $I - Z\overline{Z} > 0$ so $\|Z\| = k$ for some $k < 1$. Therefore [17, Theorem 9.12] implies that g is univalent and maps onto a quasicircle. (Note that this theorem applies to g normalized such that $g'(\infty) = 1$; our claim follows by rescaling.) Thus using [12, Lemma I.6.2] we deduce that g has a quasiconformal extension to the sphere.

Since g and ϕ are continuous on $\mathbb{S}^1$ and $f = g \circ \phi$, f is continuous on $\mathbb{S}^1$. Furthermore, the curve $f(e^{it})$ has winding number one around 0. The curve $H(r,t) = f(re^{it})$ is continuous in both r and t; hence for $(r,t) \in [R,1] \times [0,2\pi]$ it does not contain 0. So for $r \in [R,1]$ the curve $t \mapsto f(re^{it})$ has winding number one with respect to zero, and by the argument principle f takes the value 0 precisely once in $|z| < r$. Therefore $f_1 \neq 0$, and Theorem 3.11 yields that $\mathcal{C}_-(f) = \mathfrak{B}\overline{\mathfrak{A}}^{-1}$. Now since $\mathcal{C}_+(\tilde{f}) = \overline{\mathcal{C}_-(f)}$ by (3.11), we can apply [17, Theorem 9.12] to see that $\tilde{f}$ is univalent and has a quasiconformal extension to $\mathbb{C}$, and the same holds for f.

It remains to show uniqueness. If (f,g) is any pair of maps satisfying (1)-(4), then since f omits ∞ it is in $\mathcal{D}(\mathbb{D})$ and thus its trace on $\mathbb{S}^1$ is in $\mathcal{H}_*$. Since Π_ϕ preserves $\mathcal{H}$, the same holds for the trace g. It was observed above that the solution in $\mathcal{H}$ is unique. This completes the proof. □

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