

THE MYHILL PROPERTY FOR CELLULAR AUTOMATA ON AMENABLE SEMIGROUPS

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Dedicated to Slava Grigorchuk on his 60th birthday

ABSTRACT. Let S be a cancellative left-amenable semigroup and let A be a finite set. We prove that every pre-injective cellular automaton $\tau: A^S \rightarrow A^S$ is surjective.

1. INTRODUCTION

Let S be a semigroup, i.e., a set equipped with an associative binary operation.

Given $s \in S$, we denote by L_s and R_s the left and right multiplication by s , that is, the maps $L_s: S \rightarrow S$ and $R_s: S \rightarrow S$ defined by $L_s(t) = st$ and $R_s(t) = ts$ for all $t \in S$.

Let A be a set, called the *alphabet*. The set A^S , consisting of all maps $x: S \rightarrow A$, is called the set of *configurations*. Given an element $s \in S$ and a configuration $x \in A^S$, we define the configuration $sx \in A^S$ by $sx := x \circ R_s$. Thus, we have $sx(t) = x(ts)$ for all $t \in S$. The map $(s, x) \mapsto sx$ defines a left action of the semigroup S on A^S , that is, it satisfies $s_1(s_2x) = (s_1s_2)x$ for all $s_1, s_2 \in S$ and $x \in A^S$. This action is called the (left) S -*shift* on A^S .

We say that a map $\tau: A^S \rightarrow A^S$ is a *cellular automaton* over the semigroup S and the alphabet A if there exist a finite subset $M \subset S$ and a map $\mu: A^M \rightarrow A$ such that

$$(1.1) \quad \tau(x)(s) = \mu((sx)|_M) \quad \text{for all } x \in A^S \text{ and } s \in S,$$

where $(sx)|_M \in A^M$ is the restriction of the configuration $sx = x \circ R_s$ to M . Such a set M is called a *memory set* for τ and one says that μ is a *local defining map* for τ relative to M .

Two configurations $x_1, x_2 \in A^S$ are said to be *almost equal* if they coincide outside a finite subset of S . A cellular automaton $\tau: A^S \rightarrow A^S$ is called *pre-injective* if $\tau(x_1) = \tau(x_2)$ implies $x_1 = x_2$ whenever $x_1, x_2 \in A^S$ are almost equal.

Let $\ell^\infty(S)$ denote the vector space consisting of all bounded real-valued maps $f: S \rightarrow \mathbb{R}$. A *mean* on S is an \mathbb{R} -linear map $m: \ell^\infty(S) \rightarrow \mathbb{R}$ such that $\inf_{s \in S} f(s) \leq m(f) \leq \sup_{s \in S} f(s)$ for all $f \in \ell^\infty(S)$. One says that a mean m on S is *left-invariant* (resp. *right-invariant*) if it satisfies $m(f \circ L_s) = m(f)$ (resp. $m(f \circ R_s) = m(f)$)

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for all $f \in \ell^\infty(S)$ and $s \in S$. The semigroup S is called *left-amenable* (resp. *right-amenable*) if it admits a left (resp. right) invariant mean. One says that S is *amenable* if it is both left- and right-amenable. All commutative semigroups, all finite groups, all solvable groups, and all finitely generated groups of subexponential growth are amenable. For groups, it turns out that left-amenability is equivalent to right-amenability. Also, every subgroup of an amenable group is itself amenable. As non-abelian free groups are non-amenable, it follows that every group that contains a non-abelian free subgroup is non-amenable. On the other hand, there are semigroups that are left-amenable but not right-amenable. There are finite semigroups that are neither left-amenable nor right-amenable and amenable groups containing subsemigroups that are neither left-amenable nor right-amenable. For more on amenable groups and semigroups, see for example [5], [8], [13], [14].

Fifty years ago, Moore and Myhill proved the Garden of Eden theorem for cellular automata over \mathbb{Z}^2 . This theorem states that if A is a finite set and $G = \mathbb{Z}^2$, then a cellular automaton $\tau: A^G \rightarrow A^G$ is surjective if and only if it is pre-injective. In fact, Moore [11] first proved that surjectivity implies pre-injectivity and, shortly after, Myhill [12] proved the converse implication. The Garden of Eden theorem was subsequently extended to all finitely generated groups of subexponential growth in [10] and to all amenable groups in [4]. Actually it follows from a result in [1] that the class of amenable groups is the larger class of groups for which the Moore implication holds true. It is unknown whether the Myhill implication characterizes group amenability, i.e., if every non-amenable group admits a pre-injective but not surjective cellular automaton with finite alphabet (this is known to be true for groups containing non-abelian free subgroups).

The Garden of Eden theorem does not extend to all amenable semigroups. For example, the additive monoid \mathbb{N} of non-negative integers is amenable since it is commutative. However, the shift map $\tau: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$, defined by $\tau(x)(n) = x(n+1)$ for all $x \in \{0, 1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, yields an example of a cellular automaton with finite alphabet over \mathbb{N} that is surjective but not pre-injective (see Example 8.1 below for a generalization). Thus, the Moore implication is not true for the monoid \mathbb{N} .

Recall that an element s in a semigroup S is called *left-cancellable* (resp. *right-cancellable*) if the map L_s (resp. R_s) is injective. One says that s is cancellable if it is both left-cancellable and right-cancellable. The semigroup S is called *left-cancellative* (resp. *right-cancellative*, resp. *cancellative*) if every element in S is left-cancellable (resp. right-cancellable, resp. cancellable).

The main result in the present paper is that the Myhill implication remains valid for all cancellative left-amenable semigroups. More precisely, we shall establish the following:

Theorem 1.1. *Let S be a cancellative left-amenable semigroup and let A be a finite set. Then every pre-injective cellular automaton $\tau: A^S \rightarrow A^S$ is surjective.*

As injectivity implies pre-injectivity, an immediate consequence of Theorem 1.1 is the following result.

Corollary 1.2. *Let S be a cancellative left-amenable semigroup and let A be a finite set. Then every injective cellular automaton $\tau: A^S \rightarrow A^S$ is surjective. \square*

Let us say that a semigroup S is *surjunctive* if every injective cellular automaton with finite alphabet over S is surjective. Then Corollary 1.2 may be rephrased by

saying that every cancellative left-amenable semigroup is surjunctive. In the final section, we will see that the bicyclic monoid is not surjunctive. As the bicyclic monoid is amenable, this shows that the hypothesis of non-cancellativity cannot be removed either from Theorem 1.1 or even from Corollary 1.2. In contrast, the question whether every group is surjunctive, which is known as the *Gottschalk conjecture* [7], remains open. However, it is known to be true for sofic groups [9], [15] and the class of sofic groups is very large. It includes in particular all residually amenable groups and no examples of non-sofic groups have been found up to now.

The paper is organized as follows. Sections 2, 3, 4, and 5 contain preliminary material on boundaries, Følner nets, cellular automata, and tilings in semigroups. In Section 6, given a left-cancellative semigroup S and a finite set A , we define the entropy of a subset of the configuration space A^S with respect to a Følner net. This entropy is always bounded above by the logarithm of the cardinality of the alphabet A . Moreover, for closed invariant subsets $X \subset A^S$ and S cancellative, equality holds if and only if $X = A^S$. Theorem 1.1 is established in Section 7 by showing that the image of a pre-injective cellular automaton is a closed invariant subset of the configuration space with maximal entropy. In Section 8, we describe examples of cellular automata with finite alphabet over cancellative amenable semigroups that are surjective but not pre-injective. These examples generalize the shift map on $\{0, 1\}^{\mathbb{N}}$ mentioned above. Finally, we show that the bicyclic monoid is not surjunctive.

2. BOUNDARIES

Let S be a semigroup. Given two non-empty subsets K and Ω of S , we define the K -interior $\text{Int}_K(\Omega)$ and the K -adherence $\text{Adh}_K(\Omega)$ of Ω by

$$\begin{aligned}\text{Int}_K(\Omega) &:= \{s \in \Omega : Ks \subset \Omega\}, \\ \text{Adh}_K(\Omega) &:= \{s \in S : Ks \cap \Omega \neq \emptyset\}.\end{aligned}$$

Note that

$$\text{Int}_K(\Omega) \subset \Omega \cap \text{Adh}_K(\Omega).$$

We define the K -boundaries $\text{partial}_K(\Omega)$ and $\text{tial}_K^*(\Omega)$ of Ω by

$$\text{tial}_K(\Omega) := \Omega \setminus \text{Int}_K(\Omega) \quad \text{and} \quad \partial_K^*(\Omega) := \text{Adh}_K(\Omega) \setminus \text{Int}_K(\Omega).$$

Proposition 2.1. *Let S be a semigroup. Let Ω and K be two non-empty subsets of S and suppose that every element of K is left-cancellable. Then one has*

$$(2.1) \quad \partial_K(\Omega) = \bigcup_{k \in K} L_k^{-1}(k\Omega \setminus \Omega)$$

and

$$(2.2) \quad \partial_K^*(\Omega) \subset \partial_K(\Omega) \amalg \left(\bigcup_{k \in K} L_k^{-1}(\Omega \setminus k\Omega) \right)$$

(here \amalg denotes disjoint union).

Proof. By definition, an element $s \in S$ is in $\partial_K(\Omega)$ if and only if $s \in \Omega$ and there exists $k \in K$ such that $ks \notin \Omega$. As L_k is injective for each $k \in K$, this is equivalent to the existence of $k \in K$ such that $s \in L_k^{-1}(k\Omega \setminus \Omega)$. This shows (2.1).

Now let $s \in \partial_K^*(\Omega) = \text{Adh}_K(\Omega) \setminus \text{Int}_K(\Omega)$. If $s \in \Omega$, then we have $s \in \Omega \setminus \text{Int}_K(\Omega) = \partial_K(\Omega)$. Suppose now that $s \in S \setminus \Omega$. As $s \in \text{Adh}_K(\Omega)$, there exists

$k \in K$ such that $ks \in \Omega$. We have $ks \notin k\Omega$ since $s \notin \Omega$ and k is left-cancellable. Thus $s \in L_k^{-1}(\Omega \setminus k\Omega)$. This shows (2.2). \square

In the sequel, we shall use $|\cdot|$ to denote cardinality of finite sets.

Corollary 2.2. *Let S be a semigroup. Suppose that K and Ω are non-empty finite subsets of S and that every element of K is left-cancellable. Then the sets $\partial_K(\Omega)$ and $\partial_K^*(\Omega)$ are finite. Moreover, one has*

$$(2.3) \quad |\partial_K(\Omega)| \leq \sum_{k \in K} |k\Omega \setminus \Omega| \quad \text{and} \quad |\partial_K^*(\Omega)| \leq 2 \sum_{k \in K} |k\Omega \setminus \Omega|.$$

Proof. Let $k \in K$. We first observe that by left-cancellability of k , we have $|k\Omega| = |\Omega|$ and therefore $|\Omega \setminus k\Omega| = |k\Omega \setminus \Omega|$. Also, the injectivity of L_k implies that the sets $L_k^{-1}(k\Omega \setminus \Omega)$ and $L_k^{-1}(\Omega \setminus k\Omega)$ are finite of cardinality $|L_k^{-1}(k\Omega \setminus \Omega)| = |k\Omega \setminus \Omega|$ and $|L_k^{-1}(\Omega \setminus k\Omega)| \leq |k\Omega \setminus \Omega| = |k\Omega \setminus \Omega|$. Thus, taking cardinalities in (2.1), we get

$$\begin{aligned} |\partial_K(\Omega)| &= \left| \bigcup_{k \in K} L_k^{-1}(k\Omega \setminus \Omega) \right| \\ &\leq \sum_{k \in K} |L_k^{-1}(k\Omega \setminus \Omega)| \\ &= \sum_{k \in K} |k\Omega \setminus \Omega|, \end{aligned}$$

which yields the first inequality in (2.3). On the other hand, we deduce from (2.2) that

$$\begin{aligned} |\partial_K^*(\Omega)| &\leq |\partial_K(\Omega)| \prod \left(\bigcup_{k \in K} L_k^{-1}(\Omega \setminus k\Omega) \right) \\ &\leq |\partial_K(\Omega)| + \sum_{k \in K} |L_k^{-1}(\Omega \setminus k\Omega)| \\ &\leq \sum_{k \in K} |k\Omega \setminus \Omega| + \sum_{k \in K} |\Omega \setminus k\Omega| \\ &= 2 \sum_{k \in K} |k\Omega \setminus \Omega|, \end{aligned}$$

which gives the second inequality in (2.3). \square

Suppose that K and Ω are non-empty finite subsets of a semigroup S and that every element of K is left-cancellable. We then define the *relative amenability constants* $\alpha(\Omega, K)$ and $\alpha^*(\Omega, K)$ of Ω with respect to K by

$$\alpha(\Omega, K) := \frac{|\partial_K(\Omega)|}{|\Omega|}$$

and

$$\alpha^*(\Omega, K) := \frac{|\partial_K^*(\Omega)|}{|\Omega|}.$$

3. FØLNER NETS

For left-cancellative semigroups, we have the following characterizations of left-amenability.

Theorem 3.1 (Følner-Frey-Namioka). *Let S be a left-cancellative semigroup. Then the following conditions are equivalent:*

- (a) S is left-amenable;
- (b) for every finite subset $K \subset S$ and every real number $\varepsilon > 0$, there exists a non-empty finite subset $F \subset S$ such that $|kF \setminus F| \leq \varepsilon|F|$ for all $k \in K$;
- (c) there exists a directed net $(F_j)_{j \in J}$ of non-empty finite subsets of S such that

$$(3.1) \quad \lim_j \frac{|sF_j \setminus F_j|}{|F_j|} = 0 \quad \text{for all } s \in S.$$

Proof. The equivalence of conditions (a) and (b) follows from [13, Corollary 4.3]. On the other hand, the equivalence of (b) and (c) is straightforward (see for example the discussion in [3, Section 1]). □

Remark 3.2. If we drop the left-cancellativity hypothesis in the preceding theorem, the equivalence between (b) and (c), as well as the fact that (a) implies (b), remain true. However, every finite semigroup S trivially satisfies (b) by taking $F = S$. As there exist finite semigroups that are not left-amenable, it follows that the implication (b) \Rightarrow (a) becomes false if we remove the left-cancellativity hypothesis in Theorem 3.1.

A directed net $(F_j)_{j \in J}$ of non-empty finite subsets of a semigroup S satisfying (3.1) is called a *Følner net* for S .

Proposition 3.3. *Let S be a left-cancellative and left-amenable semigroup. Let $(F_j)_{j \in J}$ be a Følner net for S and let K be a non-empty finite subset of S . Then one has $\lim_j \alpha(F_j, K) = \lim_j \alpha^*(F_j, K) = 0$.*

Proof. Let $\varepsilon > 0$. Since $(F_j)_{j \in J}$ is a Følner net for S , there exists $j_0 \in J$ such that $|sF_j \setminus F_j|/|F_j| \leq \varepsilon$ for all $s \in K$ and $j \geq j_0$. This implies $\alpha(F_j, K) \leq |K|\varepsilon$ and $\alpha^*(F_j, K) \leq 2|K|\varepsilon$ for all $j \geq j_0$ by using the inequalities in Corollary 2.2. Consequently, we have $\lim_j \alpha(F_j, K) = \lim_j \alpha^*(F_j, K) = 0$. □

4. CELLULAR AUTOMATA

If E is a set equipped with a left action of a semigroup S , one says that a map $f: E \rightarrow E$ is *S-equivariant* if it commutes with the S -action, i.e., if one has $f(sx) = sf(x)$ for all $s \in S$ and $x \in E$.

Proposition 4.1. *Let S be a semigroup and let A be a set. Then every cellular automaton $\tau: A^S \rightarrow A^S$ is S -equivariant.*

Proof. Let $\tau: A^S \rightarrow A^S$ be a cellular automaton with memory set $M \subset S$ and local defining map $\mu: A^M \rightarrow A$. Let $s, t \in S$ and $x \in A^S$. By applying (1.1), we get

$$\tau(tx)(s) = \mu((s(tx))|_M) = \mu(((st)x)|_M) = \tau(x)(st) = (\tau(x))(s).$$

Consequently, we have $\tau(tx) = t\tau(x)$. This shows that τ is S -equivariant. □

Proposition 4.2. *Let S be a semigroup and let A be a set. Let $\tau: A^S \rightarrow A^S$ be a cellular automaton with memory set $M \subset S$. Let $x_1, x_2 \in A^S$, $s \in S$, and $\Omega \subset S$. Then the following hold.*

- (i) if the configurations x_1 and x_2 coincide on Ms , then $\tau(x_1)(s) = \tau(x_2)(s)$;
- (ii) if the configurations x_1 and x_2 coincide on Ω , then the configurations $\tau(x_1)$ and $\tau(x_2)$ coincide on $\text{Int}_M(\Omega)$;
- (iii) if the configurations x_1 and x_2 coincide outside Ω , then the configurations $\tau(x_1)$ and $\tau(x_2)$ coincide outside $\text{Adh}_M(\Omega)$.

Proof. Assertion (i) immediately follows from formula (1.1). Assertion (i) gives us (ii) since $Ms \subset \Omega$ for all $s \in \text{Int}_M(\Omega)$. We also deduce (iii) from (i) since Ms does not meet Ω for all $s \in S \setminus \text{Adh}_M(\Omega)$. □

Let S be a semigroup and let A be a set. We equip the set $A^S = \prod_{s \in S} A$ with its *prodiscrete* topology, that is, with the product topology obtained by taking the discrete topology on each factor A of A^S . The space A^S is Hausdorff and totally disconnected. Moreover, A^S is metrizable if S is countable, and it follows from the Tychonoff product theorem that it is compact if A is finite.

Proposition 4.3. *Let S be a semigroup and let A be a set. Then every cellular automaton $\tau: A^S \rightarrow A^S$ is continuous with respect to the prodiscrete topology.*

Proof. Let $\tau: A^S \rightarrow A^S$ be a cellular automaton with memory set $M \subset S$ and local defining map $\mu: A^M \rightarrow A$. Let $x \in A^S$ and let $N \subset A^S$ be a neighborhood of $\tau(x)$. By definition of the prodiscrete topology on A^S , there exists a finite subset $\Omega \subset S$ such that N contains all configurations in A^S that coincide with $\tau(x)$ on Ω . By applying Proposition 4.2.(i), we deduce that $\tau^{-1}(N)$ contains all configurations in A^S that coincide with x on $M\Omega = \bigcup_{s \in \Omega} Ms$. Since $M\Omega$ is finite, it follows that $\tau^{-1}(N)$ is a neighborhood of x . This shows that τ is continuous for the prodiscrete topology. □

5. TILINGS

Let S be a semigroup and K a subset of S . We say that a subset $T \subset S$ is a *K-tiling* of S if it satisfies the following conditions:

- (T-1) if $t_1, t_2 \in T$ and $t_1 \neq t_2$, then $Kt_1 \cap Kt_2 = \emptyset$;
- (T-2) for every $s \in S$, there exists $t \in T$ such that $Ks \cap Kt \neq \emptyset$.

Proposition 5.1. *Let S be a semigroup and let K be a non-empty subset of S . Then S admits a K -tiling.*

Proof. Consider the set \mathcal{T} consisting of all non-empty subsets $T \subset S$ satisfying condition (T-1) above. The set \mathcal{T} is not empty since $\{s_0\} \in \mathcal{T}$ for any $s_0 \in S$. On the other hand, the set \mathcal{T} , partially ordered by inclusion, is inductive. Indeed, if \mathcal{T}' is a totally ordered subset of \mathcal{T} , then $M = \bigcup_{T' \in \mathcal{T}'} T'$ belongs to \mathcal{T} and is an upper bound for \mathcal{T}' . By applying Zorn's lemma, we deduce that \mathcal{T} admits a maximal element T . Then T satisfies (T-1) since $T \in \mathcal{T}$. On the other hand, T also satisfies (T-2) by maximality. Consequently, T is a K -tiling. □

Proposition 5.2. *Let S be a left-cancellative and left-amenable semigroup. Let $(F_j)_{j \in J}$ be a Følner net for S . Let K be a non-empty finite subset of S and suppose that $T \subset S$ is a K -tiling of S . Let us set, for each $j \in J$,*

$$T_j := \{t \in T : Kt \subset F_j\}.$$

Then there exist a real number $\delta > 0$ and an element $j_0 \in J$ such that

$$|T_j| \geq \delta |F_j| \quad \text{for all } j \geq j_0.$$

Proof. Define $T_j^* \subset T$ by

$$T_j^* := \{t \in T : Kt \cap F_j \neq \emptyset\} = T \cap \text{Adh}_K(F_j).$$

Consider an element $s \in \text{Int}_K(F_j)$. Since T is a K -tiling, it follows from condition (T-2) that we can find $t \in T$ such that $Ks \cap Kt \neq \emptyset$. As $Ks \subset F_j$, we have $t \in T_j^*$. We deduce that

$$\text{Int}_K(F_j) \subset \bigcup_{k \in K} L_k^{-1}(KT_j^*).$$

This implies

$$\begin{aligned} |\text{Int}_K(F_j)| &\leq \left| \bigcup_{k \in K} L_k^{-1}(KT_j^*) \right| \\ &\leq \sum_{k \in K} |L_k^{-1}(KT_j^*)| \\ &\leq \sum_{k \in K} |KT_j^*| \quad (\text{since } L_k \text{ is injective for each } k \in K) \\ &\leq |K|^2 |T_j^*| \end{aligned}$$

and hence

$$(5.1) \quad \frac{|T_j^*|}{|F_j|} \geq \frac{|\text{Int}_K(F_j)|}{|K|^2 |F_j|} = \frac{|F_j| - |\partial_K(F_j)|}{|K|^2 |F_j|} = \frac{1 - \alpha(F_j, K)}{|K|^2}.$$

On the other hand, since $T_j \subset T_j^* = T \cap \text{Adh}_K(F_j)$ and $T \cap \text{Int}_K(F_j) \subset T_j$, we have

$$T_j^* \setminus T_j \subset \text{Adh}_K(F_j) \setminus \text{Int}_K(F_j) = \partial_K^*(F_j)$$

and therefore

$$(5.2) \quad |T_j^* \setminus T_j| \leq |\partial_K^*(F_j)|.$$

Finally, we get

$$\begin{aligned} \frac{|T_j|}{|F_j|} &= \frac{|T_j^*|}{|F_j|} - \frac{|T_j^* \setminus T_j|}{|F_j|} \\ &\geq \frac{|T_j^*|}{|F_j|} - \frac{|\partial_K^*(F_j)|}{|F_j|} \quad (\text{by (5.2)}) \\ &= \frac{|T_j^*|}{|F_j|} - \alpha^*(F_j, K) \\ &\geq \frac{1 - \alpha(F_j, K)}{|K|^2} - \alpha^*(F_j, K) \quad (\text{by (5.1)}). \end{aligned}$$

By virtue of Proposition 3.3, we have $\lim_j \alpha(F_j, K) = \lim_j \alpha^*(F_j, K) = 0$. Therefore we can find $j_0 \in J$ such that $\alpha(F_j, K) \leq 1/2$ and $\alpha^*(F_j, K) \leq 1/(4|K|^2)$ for all $j \geq j_0$. Setting $\delta = 1/(4|K|^2)$, we then get

$$|T_j| \geq \delta |F_j|$$

for all $j \geq j_0$. □

6. ENTROPY

Let S be a left-cancellative and left-amenable semigroup. Let A be a finite set. For $\Omega \subset S$, we denote by $\pi_\Omega: A^S \rightarrow A^\Omega$ the restriction map, i.e., the map defined by $\pi_\Omega(x) = x|_\Omega$ for all $x \in A^S$. Let $\mathcal{F} = (F_j)_{j \in J}$ be a Følner net for S . We define the entropy $\text{ent}_{\mathcal{F}}(X)$ of a subset $X \subset A^S$ by

$$(6.1) \quad \text{ent}_{\mathcal{F}}(X) := \limsup_j \frac{\log |\pi_{F_j}(X)|}{|F_j|}.$$

Note that one has $\text{ent}_{\mathcal{F}}(X) \leq \log |A| = \text{ent}_{\mathcal{F}}(A^S)$ and that $\text{ent}_{\mathcal{F}}(X) \leq \text{ent}_{\mathcal{F}}(Y)$ if $X \subset Y \subset A^S$.

Remark 6.1. When S is a cancellative left-amenable semigroup and $X \subset A^S$ is S -invariant (i.e. $sx \in X$ for all $s \in S$ and $x \in X$), it immediately follows from the version of the Ornstein-Weiss lemma given in [3, Theorem 1.1] that the lim sup in the definition of $\text{ent}_{\mathcal{F}}(X)$ is a true limit which is independent of the choice of the Følner net \mathcal{F} . However, we will not use this fact in the proof of our main result.

A fundamental property of entropy is that the entropy of a set of configurations cannot be increased by a cellular automaton. More precisely, we have the following result.

Proposition 6.2. *Let S be a left-cancellative and left-amenable semigroup, $\mathcal{F} = (F_j)_{j \in J}$ a Følner net for S , and A a finite set. Let $\tau: A^S \rightarrow A^S$ be a cellular automaton and let $X \subset A^S$. Then one has*

$$\text{ent}_{\mathcal{F}}(\tau(X)) \leq \text{ent}_{\mathcal{F}}(X).$$

Proof. Let $Y := \tau(X) \subset A^S$ denote the image of X by τ . Suppose that $M \subset S$ is a memory set for τ . By Proposition 4.2.(ii), if two configurations $x_1, x_2 \in X$ coincide on F_j , then $\tau(x_1)$ and $\tau(x_2)$ coincide on $\text{Int}_M(F_j)$. It follows that

$$(6.2) \quad |\pi_{\text{Int}_M(F_j)}(Y)| \leq |\pi_{F_j}(X)|.$$

On the other hand, as F_j is the disjoint union of $\text{Int}_M(F_j)$ and $\partial_M(F_j)$, we have

$$\pi_{F_j}(Y) \subset \pi_{\text{Int}_M(F_j)}(Y) \times A^{\partial_M(F_j)}$$

and hence

$$\log |\pi_{F_j}(Y)| \leq \log |\pi_{\text{Int}_M(F_j)}(Y)| + |\partial_M(F_j)| \log |A|.$$

After dividing by $|F_j|$, this gives us

$$\begin{aligned} \frac{\log |\pi_{F_j}(Y)|}{|F_j|} &\leq \frac{\log |\pi_{\text{Int}_M(F_j)}(Y)|}{|F_j|} + \alpha(F_j, M) \log |A| \\ &\leq \frac{\log |\pi_{F_j}(X)|}{|F_j|} + \alpha(F_j, M) \log |A| \end{aligned} \quad (\text{by (6.2)}).$$

Since $\lim_j \alpha(F_j, M) = 0$ by Proposition 3.3, we finally get

$$\text{ent}_{\mathcal{F}}(Y) = \limsup_j \frac{\log |\pi_{F_j}(Y)|}{|F_j|} \leq \limsup_j \frac{\log |\pi_{F_j}(X)|}{|F_j|} = \text{ent}_{\mathcal{F}}(X).$$

□

The following result may be used to show that certain sets of configurations do not have maximal entropy.

Proposition 6.3. *Let S be a left-cancellative and left-amenable semigroup, $\mathcal{F} = (F_j)_{j \in J}$ a Følner net for S , and A a finite set. Suppose that a subset $X \subset A^S$ satisfies the following condition: there exist a non-empty finite subset $K \subset S$ and a K -tiling $T \subset S$ of S such that $\pi_{Kt}(X) \subsetneq A^{Kt}$ for all $t \in T$. Then one has $\text{ent}_{\mathcal{F}}(X) < \log |A|$.*

Proof. For each $j \in J$, consider the subset $T_j \subset T$ defined by $T_j := \{t \in T : Kt \subset F_j\}$ and the subset $F_j^* \subset F_j$ given by

$$F_j^* = F_j \setminus \prod_{t \in T_j} Kt.$$

As the sets F_j^* and $Kt, t \in T_j$, form a partition of F_j , we have

$$(6.3) \quad |F_j| = |F_j^*| + \sum_{t \in T_j} |Kt|.$$

On the other hand, by our hypothesis, we have

$$(6.4) \quad |\pi_{Kt}(X)| \leq |A^{Kt}| - 1 = |A|^{|Kt|} - 1 \quad \text{for all } t \in T.$$

As

$$\pi_{F_j}(X) \subset A^{F_j^*} \times \prod_{t \in T_j} \pi_{Kt}(X),$$

it follows that

$$\begin{aligned} \log |\pi_{F_j}(X)| &\leq \log |A^{F_j^*} \times \prod_{t \in T_j} \pi_{Kt}(X)| \\ &= |F_j^*| \log |A| + \sum_{t \in T_j} \log |\pi_{Kt}(X)| \\ &\leq |F_j^*| \log |A| + \sum_{t \in T_j} \log(|A|^{|Kt|} - 1) \quad (\text{by (6.4)}) \\ &= |F_j^*| \log |A| + \sum_{t \in T_j} |Kt| \log |A| + \sum_{t \in T_j} \log(1 - |A|^{-|Kt|}) \\ &= |F_j| \log |A| + \sum_{t \in T_j} \log(1 - |A|^{-|Kt|}) \quad (\text{by (6.3)}) \\ &\leq |F_j| \log |A| + |T_j| \log(1 - |A|^{-|K|}) \\ &\quad (\text{since } |Kt| \leq |K| \text{ for all } t \in T_j). \end{aligned}$$

By introducing the constant $c := -\log(1 - |A|^{-|K|}) > 0$, this gives us

$$\log |\pi_{F_j}(X)| \leq |F_j| \log |A| - c|T_j| \quad \text{for all } j \in J.$$

Now, by Proposition 5.2, there exist $\delta > 0$ and $j_0 \in J$ such that $|T_j| \geq \delta|F_j|$ for all $j \geq j_0$. Thus

$$\frac{\log |\pi_{F_j}(X)|}{|F_j|} \leq \log |A| - c\delta \quad \text{for all } j \geq j_0.$$

This implies that

$$\text{ent}_{\mathcal{F}}(X) = \limsup_j \frac{\log |\pi_{F_j}(X)|}{|F_j|} \leq \log |A| - c\delta < \log |A|.$$

□

Recall that if E is a set equipped with a left action of a semigroup S , one says that a subset $X \subset E$ is S -invariant if one has $sx \in X$ for all $s \in S$ and $x \in X$.

Corollary 6.4. *Let S be a cancellative and left-amenable semigroup, $\mathcal{F} = (F_j)_{j \in J}$ a Følner net for S , and A a finite set. Suppose that $X \subset A^S$ is an S -invariant subset satisfying the following condition: there exists a finite subset $K \subset S$ such that*

$$(6.5) \quad \pi_K(X) \subsetneq A^K.$$

Then one has $\text{ent}_{\mathcal{F}}(X) < \log |A|$.

Proof. First observe that

$$(6.6) \quad \pi_{Ks}(X) \subsetneq A^{Ks} \quad \text{for all } s \in S.$$

Indeed, by (6.5), we can find $u \in A^K$ such that $u \notin \pi_K(X)$. Then, given $s \in S$, we can define $v \in A^{Ks}$ by setting $v(ts) = u(t)$ for all $t \in K$ (the right-cancellability of s implies that v is well defined). Now, there is no $x \in X$ such that $\pi_{Ks}(x) = v$ since otherwise the configuration sx , which is in X by our S -invariance hypothesis, would satisfy $\pi_K(sx) = u$. This proves (6.6).

Since we can find a K -tiling of S by Proposition 5.1, we deduce from (6.6) that $\text{ent}_{\mathcal{F}}(X) < \log |A|$ by applying Proposition 6.3. □

Corollary 6.5. *Let S be a cancellative and left-amenable semigroup, $\mathcal{F} = (F_j)_{j \in J}$ a Følner net for S , and A a finite set. Suppose that $X \subset A^S$ is a closed (for the prodiscrete topology) and S -invariant subset of A^S . Then one has $\text{ent}_{\mathcal{F}}(X) = \log |A|$ if and only if $X = A^G$.*

Proof. The fact that $\text{ent}_{\mathcal{F}}(A^G) = \log |A|$ has already been observed and is trivial. Conversely, suppose that $X \subsetneq A^G$. As X is closed in A^S , this means that we can find a subset $K \subset S$ such that $\pi_K(X) \subsetneq A^K$. It then follows from Corollary 6.4 that $\text{ent}_{\mathcal{F}}(X) < \log |A|$. □

7. ENTROPY AND CELLULAR AUTOMATA

In this section, we give the proof of Theorem 1.1. We shall use the following auxiliary result.

Lemma 7.1. *Let S be a left-cancellative and left-amenable semigroup, $\mathcal{F} = (F_j)_{j \in J}$ a Følner net for S , and A a finite set. Suppose that $\tau: A^S \rightarrow A^S$ is a cellular automaton such that*

$$(7.1) \quad \text{ent}_{\mathcal{F}}(\tau(A^S)) < \log |A|.$$

Then τ is not pre-injective.

Proof. Let $Y := \tau(A^S)$ denote the image of τ and let $M \subset S$ be a memory set for τ . Recall that $\text{Adh}_M(F_j)$ is the disjoint union of $\text{Int}_M(F_j)$ and $\partial_M^*(F_j)$. Therefore we have

$$\pi_{\text{Adh}_M(F_j)}(Y) \subset \pi_{\text{Int}_M(F_j)}(Y) \times A^{\partial_M^*(F_j)}.$$

This implies

$$\begin{aligned} \log |\pi_{\text{Adh}_M(F_j)}(Y)| &\leq \log |\pi_{\text{Int}_M(F_j)}(Y)| + |\partial_M^*(F_j)| \log |A| \\ &\leq \log |\pi_{F_j}(Y)| + |\partial_M^*(F_j)| \log |A| \quad (\text{since } \text{Int}_M(F_j) \subset F_j). \end{aligned}$$

After dividing by $|F_j|$, we get

$$(7.2) \quad \frac{\log |\pi_{\text{Adh}_M(F_j)}(Y)|}{|F_j|} \leq \frac{\log |\pi_{F_j}(Y)|}{|F_j|} + \alpha^*(F_j, M) \log |A|$$

for all $j \in J$. As

$$\limsup_j \frac{\log |\pi_{F_j}(Y)|}{|F_j|} = \text{ent}_{\mathcal{F}}(Y) < \log |A|$$

by our hypothesis, and

$$\lim_j \alpha^*(F_j, M) = 0$$

by Proposition 3.3, we deduce from inequality (7.2) that

$$\limsup_j \frac{\log |\pi_{\text{Adh}_M(F_j)}(Y)|}{|F_j|} < \log |A|.$$

Consequently, there exists $j_0 \in J$ such that

$$(7.3) \quad \frac{\log |\pi_{\text{Adh}_M(F_{j_0})}(Y)|}{|F_{j_0}|} < \log |A|.$$

Now let us fix an arbitrary element $a_0 \in A$ and consider the set $Z \subset A^S$ consisting of all the configurations $z \in A^S$ such that $z(s) = a_0$ for all $s \in S \setminus F_{j_0}$. Note that the set Z is finite of cardinality

$$|Z| = |A|^{|F_{j_0}|}.$$

Inequality (7.3) gives us

$$(7.4) \quad |\pi_{\text{Adh}_M(F_{j_0})}(Y)| < |Z|.$$

Observe that if $z_1, z_2 \in Z$, then z_1 and z_2 coincide outside F_{j_0} so that the image configurations $\tau(z_1)$ and $\tau(z_2)$ coincide outside $\text{Adh}_M(F_{j_0})$ by Proposition 4.2.(iii). Thus we have

$$|\tau(Z)| = |\pi_{\text{Adh}_M(F_{j_0})}(\tau(Z))| \leq |\pi_{\text{Adh}_M(F_{j_0})}(Y)|$$

and hence, by using (7.4),

$$|\tau(Z)| < |Z|.$$

This last inequality implies that we can find two distinct configurations z_1 and z_2 in Z such that $\tau(z_1) = \tau(z_2)$. Since z_1 and z_2 are almost equal (they coincide outside the finite set F_{j_0}), this shows that τ is not pre-injective. \square

Proof of Theorem 1.1. Let $\tau: A^S \rightarrow A^S$ be a cellular automaton and suppose that τ is pre-injective. Choose a Følner net $\mathcal{F} = (F_j)_{j \in J}$ for S . Then the image set $Y := \tau(X)$ satisfies $\text{ent}_{\mathcal{F}}(Y) = \log |A|$ by Lemma 7.1. On the other hand, as A^S is a compact Hausdorff space and τ is continuous for the prodiscrete topology by Proposition 4.3, the set Y is closed in A^S . Since Y is S -invariant by Proposition 4.1, it then follows from Corollary 6.5 that $Y = A^S$. This shows that τ is surjective. \square

Remark 7.2. Recall that a semigroup S is said to be *left-reversible* if any two left-principal ideals in S intersect, i.e., $aS \cap bS \neq \emptyset$ for all $a, b \in S$. As every left-amenable semigroup is clearly left-reversible, one deduces from Ore’s theorem that if S is a cancellative left-amenable semigroup, then S embeds in an amenable group, its group of left-quotients $G := \{st^{-1} : s, t \in S\}$ (see [16, Corollary 3.6]). When S is a cancellative commutative semigroup, e.g., $S = \mathbb{N}$ for which $G = \mathbb{Z}$, given any finite subset $F \subset G$, we can always find $t \in S$ such that $t + F \subset S$ (if

$F = \{s_i - t_i : s_i, t_i \in S, 1 \leq i \leq n\}$, we can take $t = \sum_{1 \leq i \leq n} t_i$. It follows that the Myhill property for cellular automata over S may be easily deduced from the Myhill property for cellular automata over G in that particular case. Indeed, suppose that $\tau: A^S \rightarrow A^S$ is a cellular automaton with memory set $M \subset S$ and local defining map $\mu: A^M \rightarrow A$. Consider the cellular automaton $\sigma: A^G \rightarrow A^G$ that admits M as a memory set and μ as a local defining map. If two configurations $x_1, x_2 \in A^G$ coincide outside F , then their shifts by $-t$ coincide outside $t + F \subset S$. We deduce that the pre-injectivity of τ implies the pre-injectivity of σ . As the surjectivity of σ clearly implies the surjectivity of τ , this proves our claim.

8. SOME EXAMPLES OF CELLULAR AUTOMATA

Example 8.1 (Surjective but not pre-injective cellular automata). Let S be a semi-group admitting a left-cancellable element s_0 such that $s_0S \neq S$ (i.e., an element s_0 such that the left-multiplication map $L_{s_0}: S \rightarrow S$ is injective but not surjective). Take $A = \{0, 1\}$, and consider the map $\tau: A^S \rightarrow A^S$ defined by $\tau(x)(s) = x(s_0s)$ for all $x \in A^S$ and $s \in S$. Clearly τ is a cellular automaton over the semigroup S admitting $M = \{s_0\}$ as a memory set. Let $x_0 \in A^S$ be the configuration defined by $x_0(s) = 0$ for all $s \in S$. Choose an arbitrary element $s_1 \in S \setminus s_0S$ and let $x_1 \in A^S$ be the configuration defined by $x_1(s) = 0$ for all $s \neq s_1$ and $x_1(s_1) = 1$. Then we have $x_0 \neq x_1$ but $\tau(x_0) = \tau(x_1) = x_0$. As the configurations x_0 and x_1 are almost equal, this shows that τ is not pre-injective. On the other hand, let $y \in A^S$. Consider the configuration $x \in A^S$ defined by $x(s) = 0$ if $s \notin s_0S$ and $x(s) = y(t)$ if $s = s_0t$ for some $t \in S$ (the left-cancellability of s_0 guarantees that x is well defined). Then we have $\tau(x) = y$. Consequently, τ is surjective. Any free semigroup, any free monoid, any free commutative semigroup, and any free commutative monoid satisfies our hypothesis on S as soon as it is non-trivial. Note that all free commutative semigroups and all free commutative monoids are cancellative and amenable.

Example 8.2 (Non-surjectivity of the bicyclic monoid). We recall that the *bicyclic monoid* is the monoid B with presentation $B = \langle p, q : pq = 1 \rangle$ and that every element $s \in B$ can be uniquely written in the form $s = q^a p^b$, where $a = a(s)$ and $b = b(s)$ are non-negative integers. It is known (see for example [6, Example 2, page 311]) that the bicyclic monoid is an amenable inverse semigroup.

Take $A = \{0, 1\}$, and consider the map $\tau: A^B \rightarrow A^B$ defined by $\tau(x)(s) = x(ps)$ for all $x \in A^B$ and $s \in B$. Clearly τ is a cellular automaton over B admitting $M = \{p\}$ as a memory set. Observe that τ is not surjective since $1_B \neq qp$ and $\tau(x)(1_B) = \tau(x)(qp)$ for all $x \in A^B$. On the other hand, τ is injective since $pB = B$.

Remark 8.3. It would be interesting to give an example of a cancellative semigroup that is not surjunctive.

Note added in proof. We thank Laurent Bartholdi who pointed out to us that our argument in Remark 7.2 can be extended to all cancellative left-amenable semi-groups, thus yielding an alternative proof of Theorem 1.1. Suppose indeed that S is a cancellative left-amenable semigroup and denote by $G := \{st^{-1} : s, t \in S\}$ the amenable group of its left-quotients. Let $F = \{s_i t_i^{-1} : s_i, t_i \in S, 1 \leq i \leq n\} \subset G$ be a finite subset. Since S is left-reversible, we have $\bigcap_{i=1}^n t_i S \neq \emptyset$. Taking $t \in \bigcap_{i=1}^n t_i S$

gives $Ft \subset S$ and the remaining arguments (based on the Myhill property for cellular automata over G) in Remark 7.2 apply verbatim.

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