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HADAMARD DIFFERENTIABILITY VIA GÂTEAUX DIFFERENTIABILITY

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Abstract. Let X be a separable Banach space, Y a Banach space and $f : X \to Y$ a mapping. We prove that there exists a σ-directionally porous set $A \subset X$ such that if $x \in X \setminus A$, f is Lipschitz at x, and f is Gâteaux differentiable at x, then f is Hadamard differentiable at x. If f is Borel measurable (or has the Baire property) and is Gâteaux differentiable at all points, then f is Hadamard differentiable at all points except for a set which is a σ-directionally porous set (and so is Aronszajn null, Haar null and Γ-null). Consequently, an everywhere Gâteaux differentiable $f : \mathbb{R}^n \to Y$ is Fréchet differentiable except for a nowhere dense σ-porous set.

1. Introduction

The Hadamard derivative of a mapping $f : X \to Y$ between Banach spaces, which is stronger than the Gâteaux derivative but weaker than the Fréchet derivative, has been applied many times in the literature. For three formally different but equivalent definitions of the Hadamard derivative, see Lemma 2.1 below. If X is finite-dimensional, then the Hadamard derivative coincides with the Fréchet derivative, but for infinite-dimensional spaces the Fréchet derivative is "much stronger", even for Lipschitz f.

On the other hand, if f is Lipschitz, then the Hadamard derivative coincides with the Gâteaux derivative; in this sense these two types of derivatives are rather close. Using this fact, we obtain the well-known result that if f is everywhere Gâteaux differentiable and has the Baire property, then f is Hadamard differentiable at all points except for a nowhere dense set. Indeed, [12, Corollary 3.11] (see Proposition S after Proposition 3.8 below) easily implies that, under the above assumptions, f is locally Lipschitz on a dense open set V. However, an easy example (see Remark 3.10) shows that in general (even if $X = Y = \mathbb{R}$) we cannot find V with "measure null" complement.

We prove (Theorem 3.9) that if X is separable and f has the Baire property and is everywhere Gâteaux differentiable, then f is Hadamard differentiable at all points except for a σ-directionally porous set. This is interesting additional information, since each σ-directionally porous subset of a separable Banach spaces X is

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"measure null": it is Aronszajn (= Gauss) null, and so also Haar null (see [2, p. 164 and Chap. 6]) and also Γ-null (see [8, Remark 5.2.4]).

As an easy consequence of Theorem 3.9, we obtain (see Theorem 3.12) that an everywhere Gâteaux differentiable $f : \mathbb{R}^n \to Y$ is Fréchet differentiable except for a σ-porous set.

The main ingredient of the proof of Theorem 3.9 is Theorem 3.4. It asserts that if X is separable and $f : X \to Y$, then there exists a σ-directionally porous set $A \subset X$ such that if $x \in X \setminus A$, f is Lipschitz at x, and f is Gâteaux differentiable at x, then f is Hadamard differentiable at x.

Lemma 3.6 implies that if X is separable and f has the Baire property and is everywhere Gâteaux differentiable, then f is Lipschitz at all points except for a σ-directionally porous set. So, Theorem 3.4, together with Lemma 3.6, implies Theorem 3.9.

Further, Theorem 3.4 shows that the infinite-dimensional version of the Stepanoff theorem on Gâteaux differentiability from [4] also holds for Hadamard differentiability (see Corollary 3.14).

2. Preliminaries

In the following, by a Banach space we mean a real Banach space. If X is a Banach space, we set $S_X := \{x \in X : \|x\| = 1\}$. The symbol $B(x,r)$ will denote the open ball with center x and radius r.

In a metric space (X,ρ), the system of all sets with the Baire property is the smallest σ-algebra containing all open sets and all first category sets. We will say that a *mapping* $f : (X,\rho_1) \to (Y,\rho_2)$ *has the Baire property* if f is measurable with respect to the σ-algebra of all sets with the Baire property. In other words, f has the Baire property if and only if $f^{-1}(B)$ has the Baire property for all Borel sets $B \subset Y$ (see [6, Section 32]).

Let X, Y be Banach spaces, $G \subset X$ an open set, and $f : G \to Y$ a mapping.

We say that f *is Lipschitz at* $x \in G$ if $\limsup_{y\to x} \frac{\|f(y)-f(x)\|}{\|y-x\|} < \infty$. We say that f is *pointwise Lipschitz* if f is Lipschitz at all points of G.

The directional and one-sided directional derivatives of f at $x \in G$ in the direction $v \in X$ are defined respectively by

$$f'(x,v) := \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \quad \text{and} \quad f'_+(x,v) := \lim_{t\to 0+} \frac{f(x+tv)-f(x)}{t}.$$

The *Hadamard directional and one-sided directional derivatives* of f at $x \in G$ in the direction $v \in X$ are defined respectively by

$$f'_H(x,v) := \lim_{z\to v, t\to 0} \frac{f(x+tv)-f(x)}{t} \quad \text{and} \quad f'_{H+}(x,v) := \lim_{z\to v, t\to 0+} \frac{f(x+tv)-f(x)}{t}.$$

It is easy to see that $f'(x,v)$ (resp. $f'_H(x,v)$) exists if and only if $f'_+(x,-v) = -f'_+(x,v)$ (resp. $f'_{H+}(x,-v) = -f'_{H+}(x,v)$).

It is well known and easy to prove that if f is locally Lipschitz on G, then $f'(x,v) = f'_H(x,v)$ (resp. $f'_+(x,v) = f'_{H+}(x,v)$) whenever one of these two derivatives exists.

The usual modern definition of the Hadamard derivative is the following.

A continuous linear operator $L : X \to Y$ is said to be a *Hadamard derivative* of f at a point $x \in X$ if

$$\lim_{t \to 0} \frac{f(x+tv)-f(x)}{t} = L(v) \quad \text{for each} \quad v \in X$$

and the limit is uniform with respect to $v \in C$ whenever $C \subset X$ is a compact set. In this case we set $f'_H(x) := L$.

The following fact is well known (see [11]):

Lemma 2.1. *Let X, Y be Banach spaces, $\emptyset \neq G \subset X$ an open set, $x \in G$, $f : G \to Y$ a mapping and $L : X \to Y$ a continuous linear operator. Then the following conditions are equivalent:*

(i) $f'_H(x) := L$,
(ii) $f'_H(x, v) = L(v)$ *for each* $v \in X$,
(iii) *if* $\varphi : [0, 1] \to X$ *is such that* $\varphi(0) = x$ *and* $\varphi'_+(0)$ *exists, then* $(f \circ \varphi)'_+(0) = L(\varphi'_+(0))$.

Recall (see, e.g., [11]) that if f is Hadamard differentiable at $a \in G$, then f is Gâteaux differentiable at a, and

(2.1) if f is locally Lipschitz on G, then the opposite implication also holds.

Further,
(2.2)
if $X = \mathbb{R}^n$, then Hadamard differentiability is equivalent to Fréchet differentiability.

Definition 2.2. Let X be a Banach space. We say that $A \subset X$ is *directionally porous at a point* $x \in X$, if there exist $0 \neq v \in X$, $p > 0$ and a sequence $t_n \to 0$ of positive real numbers such that $B(x + t_n v, pt_n) \cap A = \emptyset$. (In this case we say that A is *porous at x in direction v*.)

We say that $A \subset X$ is *directionally porous* if A is directionally porous at each point $x \in A$.

We say that $A \subset X$ is *σ-directionally porous* if it is a countable union of directionally porous sets.

Recall that directional porosity is stronger than (upper) porosity, but

(2.3) if X is finite-dimensional, then these two notions are equivalent.

Let X be a Banach space, $x \in X$, $v \in S_X$ and $\delta > 0$. Then we define the open cone $C(x, v, \delta)$ as the set of all $y \neq x$ for which $\|v - \frac{y-x}{\|y-x\|}\| < \delta$.

The following easy inequality is well known (see, e.g., [9, Lemma 5.1]):

$$\text{(2.4)} \qquad \text{if } u, v \in X \setminus \{0\}, \text{ then } \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq \frac{2}{\|u\|} \|u - v\|.$$

Because of the lack of a reference we supply the proof of the following easy fact.

Lemma 2.3. *Let X be a Banach space, Y a Banach space, $G \subset X$ an open set, $a \in G$, and $f : G \to Y$ a mapping. Then the following are equivalent:*

(i) $f'_{H+}(a, 0)$ *exists,*
(ii) $f'_H(a, 0)$ *exists,*
(iii) $f'_H(a, 0) = 0$,
(iv) f *is Lipschitz at* a.

Proof. The implications $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are trivial.

To prove $(iv) \Rightarrow (iii)$, suppose that $K > 0$, $\delta > 0$ and $\|f(x) - f(a)\| \le K\|x - a\|$ for each $x \in B(a, \delta)$. To prove (iii), let $\varepsilon \in (0, K)$ be given. Then, for $z \in X$ with $\|z\| < \varepsilon/K$ and $0 < |t| < \delta$, we obtain

$$\|f(a + tz) - f(a)\| \le K|t|\frac{\varepsilon}{K}, \qquad \left\|\frac{f(a + tz) - f(a)}{t}\right\| \le \varepsilon,$$

and (iii) follows.

To prove $(i) \Rightarrow (iv)$, suppose that (iv) does not hold. Then, for each $n \in \mathbb{N}$, choose $h_n \in X$ such that $0 < \|h_n\| < n^{-2}$ and $\|f(a + h_n) - f(a)\| \ge n^2\|h_n\|$. Set $t_n := n\|h_n\|$ and $z_n := n^{-1}\|h_n\|^{-1}h_n$. Then $z_n \to 0$, $t_n \to 0$, $t_n > 0$, and

$$\left\|\frac{f(a + t_n z_n) - f(a)}{t_n}\right\| = \frac{\|f(a + h_n) - f(a)\|}{n\|h_n\|} \ge n,$$

which clearly implies that $f'_{H+}(a, 0)$ does not exist. □

We will also need the following special case of [12, Lemma 2.4]. It can be proved by the Kuratowski-Ulam theorem (as is noted in [12]), but the proof given in [12] is more direct.

Lemma 2.4. *Let U be an open subset of a Banach space X. Let $M \subset U$ be a set residual in U and $z \in U$. Then there exists a line $L \subset X$ such that z is a point of accumulation of $M \cap L$.*

3. Results

Proposition 3.1. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \to Y$ a mapping. Let A be the set of all points $x \in G$ for which there exists a cone $C = C(x, v, \delta)$ such that $\limsup_{y \to x, y \in C} \frac{\|f(y) - f(x)\|}{\|y - x\|} < \infty$ and f is not Lipschitz at x. Then A is a σ-directionally porous set.*

Proof. Let $\{v_n : n \in \mathbb{N}\}$ be a dense subset of S_X. For natural numbers k, p, n, m denote by $A_{k,p,n,m}$ the set of all points $x \in A$ such that

$$\frac{\|f(y) - f(x)\|}{\|y - x\|} < k \ \text{ whenever } \ 0 < \|y - x\| < \frac{1}{p} \ \text{ and } \ \left\|\frac{y - x}{\|y - x\|} - v_n\right\| < \frac{1}{m}. \tag{3.1}$$

Since clearly A is the union of all sets $A_{k,p,n,m}$, it is sufficient to prove that each set $A_{k,p,n,m}$ is directionally porous. So, suppose that natural numbers k, p, n, m and $x \in A_{k,p,n,m}$ are given. It is sufficient to prove that $A_{k,p,n,m}$ is directionally porous at x.

To this end, find a sequence $y_i \to x$ such that

$$\frac{\|f(y_i) - f(x)\|}{\|y_i - x\|} > k(12m + 4) \text{ for each } i \in \mathbb{N}. \tag{3.2}$$

Set $r_i := \|y_i - x\|$ and $x_i := x - 6mr_i v_n$. It is sufficient to prove that there exists $i_0 \in \mathbb{N}$ such that

$$B(x_i, r_i) \cap A_{k,p,n,m} = \emptyset \quad \text{for each} \quad i \ge i_0. \tag{3.3}$$

To this end, consider $i \in \mathbb{N}$ and $z_i \in B(x_i, r_i) \cap A_{k,p,n,m}$.

Observe that

$$6mr_i - r_i \le \|x - x_i\| - \|x_i - z_i\| \le \|x - z_i\| \le \|x - x_i\| + \|x_i - z_i\| \le 6mr_i + r_i,$$
$$6mr_i - 2r_i \le \|x - z_i\| - r_i \le \|y_i - z_i\| \le \|x - z_i\| + r_i \le 6mr_i + 2r_i.$$

These inequalities imply that there exists $i_0 \in \mathbb{N}$ (independent on i) such that

$$0 < \|x - z_i\| < \frac{1}{p} \text{ and } 0 < \|y_i - z_i\| < \frac{1}{p}, \quad \text{if } i \ge i_0. \tag{3.4}$$

Applying (2.4), we obtain

$$\left\| \frac{x - z_i}{\|x - z_i\|} - v_n \right\| = \left\| \frac{x - z_i}{\|x - z_i\|} - \frac{x - x_i}{\|x - x_i\|} \right\| \le 2\frac{\|x_i - z_i\|}{6mr_i - r_i} < \frac{2r_i}{6mr_i - r_i} \le \frac{1}{m} \tag{3.5}$$

and

$$\begin{aligned} \left\| \frac{y_i - z_i}{\|y_i - z_i\|} - v_n \right\| = \left\| \frac{y_i - z_i}{\|y_i - z_i\|} - \frac{x - x_i}{\|x - x_i\|} \right\| &\le 2\frac{\|y_i - x\| + \|x_i - z_i\|}{\|y_i - z_i\|} \\ &< 2\frac{2r_i}{6mr_i - 2r_i} \le \frac{1}{m}. \end{aligned} \tag{3.6}$$

Since $z_i \in A_{k,p,n,m}$, conditions (3.1), (3.4), (3.5) and (3.6) imply that if $i \ge i_0$, then

$$\frac{\|f(x) - f(z_i)\|}{\|x - z_i\|} < k \quad \text{and} \quad \frac{\|f(y_i) - f(z_i)\|}{\|y_i - z_i\|} < k.$$

So, also using (3.2), we obtain that if $i \ge i_0$, then

$$\begin{aligned} r_i k(12m + 4) \le \|f(x) - f(y_i)\| &\le \|f(x) - f(z_i)\| + \|f(y_i) - f(z_i)\| \\ &< k\|x - z_i\| + k\|y_i - z_i\| \le k(6mr_i + r_i + 6mr_i + 2r_i), \end{aligned}$$

which is impossible. So we have proved (3.3). □

Proposition 3.2. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \to Y$ a mapping. Let M be the set of all $x \in G$ at which f is Lipschitz and there exists $v \in X$ such that $f'_+(x, v)$ exists but $f'_{H+}(x, v)$ does not exist. Then M is σ-directionally porous.*

Proof. For each $k \in \mathbb{N}$, set

$$M_k := \{x \in M : \ \|f(y) - f(x)\| \le k\|y - x\| \quad \text{whenever} \quad \|y - x\| < 1/k\}.$$

It is clearly sufficient to prove that each M_k is directionally porous. So suppose that $k \in \mathbb{N}$ and $x \in M_k$. Choose $v \in X$ such that $f'_+(x, v) =: y$ exists but $f'_{H+}(x, v)$ does not exist. Lemma 2.3 gives that $v \ne 0$. We will prove that M_k is porous at x in direction v. Since $f'_{H+}(x, v)$ does not exist, we can choose $\varepsilon > 0$ such that for each $\delta > 0$ there exists $w \in B(v, \delta)$ and $t \in (0, \delta)$ with $\|t^{-1}(f(x + tw) - f(x)) - y\| > \varepsilon$.

For each $n \in \mathbb{N}$, we choose $w_n \in B(v, 1/n)$ and $t_n \in (0, 1/n)$ such that

$$\|(t_n)^{-1}(f(x + t_n w_n) - f(x)) - y\| > \varepsilon. \tag{3.7}$$

Set $x_n := x + t_n v$ and $p := \varepsilon(3k)^{-1}$. It is sufficient to prove that there exists $n_0 \in \mathbb{N}$ such that

$$B(x_n, pt_n) \cap M_k = \emptyset \ \text{ for each } \ n \ge n_0. \tag{3.8}$$

To this end consider $n \in \mathbb{N}$ and $z_n \in B(x_n, pt_n) \cap M_k$. Denote $x_n^* := x + t_n w_n$. We have

$$\|z_n - x_n\| < pt_n, \quad \|z_n - x_n^*\| \le \|x_n - x_n^*\| + \|z_n - x_n\| \le t_n/n + pt_n. \tag{3.9}$$

Thus there exists $\widetilde{n_0} \in \mathbb{N}$ (independent on n) such that the inequality $n \ge \widetilde{n_0}$ implies $\|z_n - x_n\| < 1/k$, $\|z_n - x_n^*\| < 1/k$, and thus, since $z_n \in M_k$, also

$$\|f(x_n) - f(z_n)\| \le k\|x_n - z_n\| \ \text{ and } \ \|f(x_n^*) - f(z_n)\| \le k\|x_n^* - z_n\|. \tag{3.10}$$

If $n \ge \widetilde{n_0}$, then (3.7), (3.9) and (3.10) imply

$$\begin{aligned} \varepsilon &< \left\| \frac{f(x_n^*) - f(x)}{t_n} - y \right\| \\ &\le \left\| \frac{f(x_n) - f(x)}{t_n} - y \right\| + \frac{\|f(x_n) - f(z_n)\|}{t_n} + \frac{\|f(x_n^*) - f(z_n)\|}{t_n} \\ &\le \left\| \frac{f(x_n) - f(x)}{t_n} - y \right\| + kp + k(1/n + p) \\ &= \left\| \frac{f(x_n) - f(x)}{t_n} - y \right\| + (2/3)\varepsilon + k/n. \end{aligned}$$

Since $\left\| \frac{f(x_n)-f(x)}{t_n} - y \right\| \to 0$, we obtain a contradiction if $n > n_0$, where $n_0 \ge \widetilde{n_0}$ is sufficiently large (independent on n). So (3.8) is proved. □

Remark 3.3. The corresponding "two-sided" result which works with $f'(x, v)$ and $f'_H(x, v)$ clearly follows from Proposition 3.2.

Using Proposition 3.2 (together with Remark 3.3) and Lemma 2.1, we immediately obtain:

Theorem 3.4. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set and $f : G \to Y$ a mapping. Then the set of all points at which f is Lipschitz and Gâteaux differentiable but is not Hadamard differentiable is σ-directionally porous.*

Theorem 3.4 together with [12, Corollary 3.9] has the following consequence.

Corollary 3.5. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \to Y$ a pointwise Lipschitz mapping. Then the set of all points from G at which f is Gâteaux differentiable but is not Hadamard differentiable is nowhere dense and σ-directionally porous.*

Lemma 3.6. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \to Y$ a mapping. Suppose that f has the Baire property and, for each line $L \subset X$ with $G \cap L \neq \emptyset$, the restriction of f to $G \cap L$ is continuous. Denote by A the set of all points $x \in G$ for which there exists a set $B_x \subset S_X$ of the second category in S_X such that*

$$\limsup_{t \to 0+} \frac{\|f(x + tv) - f(x)\|}{t} < \infty \quad \textit{for each} \quad v \in B_x \tag{3.11}$$

and f is not Lipschitz at x. Then A is a σ-directionally porous set.

Proof. By Proposition 3.1 it is sufficient to prove that for each $x \in A$ there exist $v \in S_X$ and $\delta > 0$ such that

$$\limsup_{y \to x, y \in C(x,v,\delta)} \frac{\|f(y) - f(x)\|}{\|y - x\|} < \infty. \tag{3.12}$$

So fix a point $x \in A$. Find $p_0 \in \mathbb{N}$ with $B(x, 1/p_0) \subset G$ and, for natural numbers $p \geq p_0$, k, denote

$$S_{p,k} := \{v \in S_X : \frac{\|f(x+tv) - f(x)\|}{t} \leq k \quad \text{for each} \quad 0 < t < 1/p\}.$$

Since B_x is clearly covered by all sets $S_{p,k}$, we can find k and $p \geq p_0$ such that $S_{p,k}$ is a second category set (in S_X).

Without any loss of generality, we can clearly suppose that $x = 0$ and $f(x) = f(0) = 0$. We will show that $S := S_{p,k}$ has the Baire property (in S_X). Set $E := B(0, 1/p) \setminus \{0\}$ and $E^* := S_X \times (0, 1/p)$ (equipped with the product topology). For each $(v,t) \in E^*$ set $\varphi((v,t)) := tv$. Then $\varphi : E^* \to E$ is clearly a homeomorphism (with $\varphi^{-1}(z) = (z/\|z\|, \|z\|)$ for $z \in E$). Set

$$M := \{z \in E : \frac{\|f(z)\|}{\|z\|} \leq k\} = \{z \in E : \|f(z)\| - k\|z\| \leq 0\}.$$

Since f has the Baire property, using continuity of norms we easily obtain that the real function $z \mapsto \|f(z)\| - k\|z\|$ has the Baire property on E, and so M has the Baire property in E. Consequently $M^* := \varphi^{-1}(M)$ and $C := E^* \setminus M^*$ have the Baire property in E^*.

For any set $A \subset E^*$ and $v \in S_X$ define the section $A_v := \{t \in (0, 1/p) : (v,t) \in A\}$ and the projection $\pi(A) := \{v \in S_X : \ A_v \neq \emptyset\}$. It is easy to see that

$$S = S_X \setminus \pi(C). \tag{3.13}$$

Using the continuity of f on the set $\{tv : t \in (0, 1/p)\}$, we obtain that

$$C_v \ \text{ is open for each } \ v \in S_X. \tag{3.14}$$

Since C has the Baire property, we can write $C = H \cup T$, where H is a G_δ set (in E^*) and T is a first category set (in E^*). We have

$$\pi(C) = \pi(H) \cup \pi(T) = \pi(H) \cup (\pi(T) \setminus \pi(H)). \tag{3.15}$$

The set $\pi(H)$ is analytic (see, e.g., [5, Exercise 14.3]), and so (see [5, Theorem 21.6]) it has the Baire property. By the Kuratowski-Ulam theorem (see, e.g., [5, Theorem 8.41]) there exists a first category set $Z \subset S_X$ such that T_v is a first category subset of $(0, 1/p)$ for each $v \in S_X \setminus Z$.

The inclusion $\pi(T) \setminus \pi(H) \subset Z$ holds. Indeed, suppose on the contrary that there exists $v \in (\pi(T) \setminus \pi(H)) \setminus Z$. Then T_v is a nonempty first category set and $H_v = \emptyset$. Thus $C_v = H_v \cup T_v = T_v$. Using (3.14), we obtain that C_v is a nonempty open first category set, which is a contradiction.

Consequently $\pi(T) \setminus \pi(H)$ is a first category set and thus S has the Baire property by (3.13) and (3.15). Since S is also a second category set, we can choose $v \in S_X$ and $\delta > 0$ such that $S = S_{p,k}$ is residual in $S_X \cap B(v, \delta)$. Set $U^* := (S_X \cap B(v, \delta)) \times (0, 1/p)$, $\psi := \varphi \restriction_{U^*}$ and $U := C(0, v, \delta) \cap B(0, 1/p)$. Then $\psi : U^* \to U$ is clearly a homeomorphism and $\psi(S \times (0, 1/p)) \subset M$. Since $S \times (0, 1/p)$ is residual in U^* (see [5, Lemma 8.43]), we obtain that M is residual in U. Now consider an arbitrary

$z \in U$. By Lemma 2.4 there exist a line $L \subset X$ and points $z_n \in M \cap L \cap U$ with $z_n \to z$. Since the restriction of f to $L \cap U$ is continuous, we obtain $\frac{\|f(z_n)\|}{\|z_n\|} \to \frac{\|f(z)\|}{\|z\|}$, and consequently $z \in M$. So $U \subset M$, which implies (3.12). □

In fact, we will use only the following immediate consequence of Lemma 3.6.

Lemma 3.7. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \to Y$ a mapping. Suppose that f has the Baire property and, for each line $L \subset X$ with $G \cap L \neq \emptyset$, the restriction of f to $G \cap L$ is continuous. Denote by B the set of all points $x \in G$ at which f is Gâteaux differentiable and f is not Lipschitz at x. Then B is a σ-directionally porous set.*

As an immediate application of Lemma 3.7 and Theorem 3.4 we obtain

Proposition 3.8. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \to Y$ a mapping. Suppose that f has the Baire property and, for each line $L \subset X$ with $G \cap L \neq \emptyset$, the restriction of f to $G \cap L$ is continuous. Then the set of all points $x \in G$ at which f is Gâteaux differentiable but is not Hadamard differentiable is σ-directionally porous.*

Shkarin's result [12, Corollary 3.11] reads as follows.

Proposition S. *Let E, F be normed linear spaces such that E^2 is a Baire topological space. Let $\emptyset \neq U \subset E$ be an open set, let $f : U \to F$ have the Baire property and let $f'_+(x, v)$ exist for every $x \in U$ and $v \in E$. Then there exists an open ball $B \subset U$ such that f is Lipschitz on B.*

Shkarin's result together with Proposition 3.8 gives the following theorem.

Theorem 3.9. *Let X be a separable Banach space, Y a Banach space, and $G \subset X$ an open set. Suppose that $f : G \to Y$ has the Baire property and is everywhere Gâteaux differentiable. Then f is Hadamard differentiable at all points of G except for a nowhere dense σ-directionally porous set.*

Proof. Let V be the union of all open sets $H \subset G$ on which f is locally Lipschitz. Proposition S clearly implies that V is dense in G. By (2.1) and Proposition 3.8, the assertion of the theorem easily follows. □

Remark 3.10. The assumptions of Theorem 3.9 do not imply that f is locally Lipschitz on the complement of a closed σ-directionally porous set even in the case $X = Y = \mathbb{R}$.

Indeed, let (a_n, b_n), $n \in \mathbb{N}$ be the system of bounded intervals such that $G := \bigcup_{n\in\mathbb{N}}(a_n, b_n)$ is dense in $\mathbb{R}$ and $F := \mathbb{R} \setminus G$ has positive Lebesgue measure. We can clearly define a function f on $\mathbb{R}$ such that

(i) $f(x) = 0$ for $x \in F$,
(ii) f has continuous derivative on each (a_n, b_n),
(iii) $|f(x)| \le (x - a_n)^2(x - b_n)^2$ for $x \in (a_n, b_n)$,
(iv) for each $n \in \mathbb{N}$ and $\delta > 0$, the derivative f' is unbounded both on $(a_n, a_n + \delta)$ and on $(b_n - \delta, b_n)$.

Then f is everywhere differentiable, G is the largest open set on which f is locally Lipschitz and F is not σ-directionally porous. So Theorem 3.9 cannot be proved simply using only the fact that Gâteaux and Hadamard differentiability coincide for locally Lipschitz functions.

Remark 3.11. In Theorem 3.9, we cannot omit the assumption that f has the Baire property. This follows from Lemma 2.3 and Shkarin's [12] example which shows that on each separable infinite-dimensional Banach space there exists an everywhere Gâteaux differentiable real function which is discontinuous at all points.

However, if X is finite-dimensional, the assumption that f has the Baire property can be omitted. Indeed (as is also noted without a reference in [12, Remark 3.5]), if f is everywhere Gâteaux differentiable on $\mathbb{R}^n$, then f has the Baire property. This follows, e.g., from the old well-known fact (see [7]) that a partially continuous real function on $\mathbb{R}^n$ is in the $(n-1)^{st}$ Baire class (whose proof also works for a Banach space valued function). So, also using (2.2) and (2.3), we obtain the following result.

Theorem 3.12. *Let G be an open subset of $\mathbb{R}^n$, Y a Banach space, and $f : G \to Y$ an everywhere Gâteaux differentiable mapping. Then f is Fréchet differentiable at all points of G except for a nowhere dense σ-porous set.*

Using Theorem 3.4 and the main result of [4] (which generalizes the result of [10] and improves the result of [3]), we easily obtain Theorem 3.13 below. In its formulation, the system $\tilde{\mathcal{A}}$ of subsets of a separable Banach space is used. This system, which was defined in [10], is strictly smaller (see [10, Proposition 13]) than the well-known system of Aronszajn null sets (and also than the system of Γ-null sets; see [13]). We will not recall the (slightly technical) definition of the system $\tilde{\mathcal{A}}$. Note only that all members of $\tilde{\mathcal{A}}$ are Borel by definition and that it is easy to see that $\tilde{\mathcal{A}}$ is stable with respect to countable unions.

Theorem 3.13. *Let X be a separable Banach space, Y a Banach space with the Radon-Nikodým property, $G \subset X$ an open set, and $f : G \to Y$ a mapping. Then there exists a set $A \in \tilde{\mathcal{A}}$ such that if $x \in G \setminus A$ and f is Lipschitz at x, then f is Hadamard differentiable at x.*

Proof. The main result of [4] says that there exists a set $A_1 \in \tilde{\mathcal{A}}$ such that if $x \in G \setminus A_1$ and f is Lipschitz at x, then f is Gâteaux differentiable at x. Let A_2^* be the set of all $x \in G$ at which f is Lipschitz, Gâteaux differentiable, but not Hadamard differentiable. Then A_2^* is σ-directionally porous by Theorem 3.4. Using [10, Proposition 14] and [10, Theorem 12] we easily obtain that there exists $A_2 \in \tilde{\mathcal{A}}$ which contains A_2^*. Now it is clearly sufficient to set $A := A_1 \cup A_2$. □

As a corollary, we obtain an infinite-dimensional analogue of the Stepanoff theorem on Hadamard differentiability.

Corollary 3.14. *Let X be a separable Banach space, Y a Banach space with the Radon-Nikodým property, $G \subset X$ an open set, and $f : G \to Y$ a pointwise Lipschitz mapping. Then f is Hadamard differentiable at all points of G except for a set belonging to $\tilde{\mathcal{A}}$.*

Remark 3.15. Corollary 3.14 and (2.2) imply the following result.

Let Y be a Banach space with the Radon-Nikodým property, $G \subset \mathbb{R}^n$ an open set, and $f : G \to Y$ a pointwise Lipschitz mapping. Then f is Fréchet differentiable at all points of G except for a set belonging to $\tilde{\mathcal{A}}$.

However, it is probably not an improvement of the classical Stepanoff theorem, since from some unpublished results it follows that each Borel Lebesgue null subset of $\mathbb{R}^n$ belongs to $\tilde{\mathcal{A}}$. For $n = 2$ this follows from [1, Theorem 7.5] (via [10, Theorem 12]), which is stated without a proof, and for $n \geq 3$ from the corresponding theorem recently announced by M. Csörnyei and P. Jones.

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