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GROUPS AND FIELDS WITH NTP_2

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Abstract. NTP_2 is a large class of first-order theories defined by Shelah generalizing simple and NIP theories. Algebraic examples of NTP_2 structures are given by ultra-products of p-adics and certain valued difference fields (such as a non-standard Frobenius automorphism living on an algebraically closed valued field of characteristic 0). In this note we present some results on groups and fields definable in NTP_2 structures. Most importantly, we isolate a chain condition for definable normal subgroups and use it to show that any NTP_2 field has only finitely many Artin-Schreier extensions. We also discuss a stronger chain condition coming from imposing bounds on burden of the theory (an appropriate analogue of weight) and show that every strongly dependent valued field is Kaplansky.

1. Introduction

The class of NTP_2 theories (i.e. theories without the tree property of the second kind) was introduced by Shelah [30,31]. It generalizes both simple and NIP theories and turns out to be a good context for the study of forking and dividing, even if one is only interested in NIP theories: in [7,10,12] it is demonstrated that the theory of forking in simple theories [21] can be viewed as a special case of the theory of forking in NTP_2 theories over an extension base.

What are the known algebraic examples of NTP_2 theories?

Fact 1.1 ([10]). Let $\bar{K} = (K, \Gamma, k, v, \mathrm{ac})$ be a Henselian valued field of equicharacteristic 0 in the Denef-Pas language. Assume that k is NTP_2 (respectively, Γ and k are strong, of finite burden; see Section 4). Then $\bar{K}$ is NTP_2 (respectively strong, of finite burden).

Example 1.2. Let $\mathfrak{U}$ be a non-principal ultra-filter on the set of prime numbers P. Then:

(1) $\bar{K} = \prod_{p \in P} \mathbb{Q}_p / \mathfrak{U}$ is NTP_2. This follows from Fact 1.1 because:
  - The residue field is pseudo-finite, so of burden 1 (as burden is bounded by weight in a simple theory by [1]).
  - The value group is a $\mathbb{Z}$-group, thus dp-minimal, and burden equals dp-rank in NIP theories by [1].

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We remark that, while $\mathbb{Q}_p$ is dp-minimal for each p by [13], the field $\bar{K}$ is neither simple nor NIP even in the pure ring language (as the valuation ring is definable by [4]).

(2) $\bar{K} = \prod_{p \in P} F_p((t)) / \mathfrak{U}$ is NTP_2, of finite burden, as it has the same theory as the previous example by [3] (while each of $F_p((t))$ has TP_2 by Corollary 3.3).

Fact 1.3 ([11]). Let $\bar{K} = (K, \Gamma, k, v, \mathrm{ac}, \sigma)$ be a σ-Henselian contractive valued difference field of equicharacteristic 0; i.e. σ is an automorphism of the field K such that for all $x \in K$ with $v(x) > 0$ we have $v(\sigma(x)) > n \cdot v(x)$ for all $n \in \omega$ (see [5]). Assume that both (K, σ) and (Γ, σ), with the naturally induced automorphisms, are NTP_2. Then $\bar{K}$ is NTP_2.

Example 1.4. Let $(F_p, \Gamma, k, v, \sigma)$ be an algebraically closed valued field of characteristic p with σ interpreted as the Frobenius automorphism. Then $\prod_{p \in P} F_p / \mathfrak{U}$ is NTP_2. This case was studied by Hrushovski [17] and later by Durhan [5]. It follows from [17] that the reduct to the field language is a model of ACFA, hence simple but not NIP. On the other hand, this theory is not simple, as the valuation group is definable.

Moreover, certain valued difference fields with a value preserving automorphism are NTP_2. Of course, any simple or NIP field is NTP_2, and there are further conjectural examples of pure NTP_2 fields such as bounded pseudo-real closed or pseudo-p-adically closed fields (see Section 5.1).

But what does knowing that a theory is NTP_2 tell us about properties of algebraic structures definable in it? In this note we show some initial implications. In Section 2 we isolate a chain condition for normal subgroups uniformly definable in an NTP_2 theory. In Section 3 we use it to demonstrate that every field definable in an NTP_2 theory has only finitely many Artin-Schreier extensions, generalizing some of the results of [19]. In Section 4 we impose bounds on the burden, a quantitative refinement of NTP_2 similar to SU-rank in simple theories, and observe that some results for type-definable groups existing in the literature actually go through with a weaker assumption of bounded burden; e.g. every strong field is perfect, and every strongly dependent valued field is Kaplansky. The final section contains a discussion around the topics of the paper: we pose several conjectures about new possible examples (and non-examples) of NTP_2 fields and about definable envelopes of nilpotent/soluble groups in NTP_2 theories. We also remark how the stabilizer theorem of Hrushovski from [18] could be combined with properties of forking established in [12] and [7] in the NTP_2 context.

Preliminaries. Our notation is standard. As usual, we will be working in a monster model $\mathfrak{C}$ of the theory under consideration. Let G be a group, and H a subgroup of G. We write $[G : H] < \infty$ to denote that the index of H in G is bounded, which in the case of definable groups means finite. We assume that all groups (and fields) are finitary — contained in some finite Cartesian product of the monster.

Definition 1.5. We recall that a formula $\varphi(x, y)$ has TP_2 if there are tuples $(a_{i,j})_{i,j \in \omega}$ and $k \in \omega$ such that:

- $\{\varphi(x, a_{i,j}) \mid j < \omega\}$ is k-inconsistent for each $i \in \omega$,
- $\{\varphi(x, a_{i,f(i)}) \mid i < \omega\}$ is consistent for each $f : \omega \to \omega$.

A formula is NTP$_2$ otherwise, and a theory is called NTP$_2$ if no formula has TP$_2$.

Fact 1.6 ([10]). T is NTP$_2$ if and only if every formula $\varphi(x,y)$ with $|x|=1$ is NTP$_2$.

We note that every simple or NIP formula is NTP$_2$. See [10] for more on NTP$_2$ theories.

2. Chain conditions for groups with NTP$_2$

Lemma 2.1. *Let T be* NTP$_2$*, G a definable group and $(H_i)_{i\in\omega}$ a uniformly definable family of normal subgroups of G, with $H_i=\varphi(x,a_i)$. Let $H=\bigcap_{i\in\omega}H_i$ and $H_{\neq j}=\bigcap_{i\in\omega\setminus\{j\}}H_i$. Then there is some $i^*\in\omega$ such that $[H_{\neq i^*}:H]$ is finite.*

Proof. Let $(H_i)_{i\in\omega}$ be given and assume that the conclusion fails. Then for each $i\in\omega$ we can find $(b_{i,j})_{j\in\omega}$ with $b_{i,j}\in H_{\neq i}$ and such that $(b_{i,j}H)_{j\in\omega}$ are pairwise different cosets in $H_{\neq i}$. We have:

- $b_{i,j}H_i\cap b_{i,k}H_i=\emptyset$ for $j\neq k\in\omega$ and every i.
- For every $f:\omega\to\omega$, the intersection $\bigcap_{i\in\omega}b_{i,f(i)}H_i$ is non-empty. Indeed, fix f; by compactness it is enough to check that $\bigcap_{i\leq n}b_{i,f(i)}H_i\neq\emptyset$ for every $n\in\omega$. Take $b=\prod_{i\leq n}b_{i,f(i)}$ (the order of the product does not matter). As $b_{i,f(i)}\in H_j$ for all $i\neq j$, it follows by normality that $b\in b_{i,f(i)}H_i$ for all $i\leq n$.

But then $\psi(x;y,z)=\exists w\,(\varphi(w,y)\wedge x=z\cdot w)$ has TP$_2$ as witnessed by the array $(c_{i,j})_{i,j\in\omega}$ with $c_{i,j}=a_i\hat{\ }b_{i,j}$. ☐

Problem 2.2. Is the same result true without the normality assumption? See also Theorem 4.12.

Corollary 2.3. *Let T be* NTP$_2$ *and suppose that G is a definable group. Then for every $\varphi(x,y)$ there are $k_\varphi,n_\varphi\in\omega$ such that:*

- *If $(\varphi(x,a_i))_{i<K}$ is a family of normal subgroups of G and $k_\varphi\leq K$, then there is some $i^*<K$ such that $\left[\bigcap_{i<K,i\neq i^*}\varphi(x,a_i):\bigcap_{i<K}\varphi(x,a_i)\right]<n_\varphi$.*

Proof. Follows from Lemma 2.1 and compactness. ☐

Theorem 2.4. *Let G be* NTP$_2$ *and $\{\varphi(x,a)\,|\,a\in C\}$ be a family of normal subgroups of G. Then there is some $k\in\omega$ (depending only on φ) such that for every finite $C'\subseteq C$ there is some $C_0\subseteq C'$ with $|C_0|\leq k$ and such that*

$$\left[\bigcap_{a\in C_0}\varphi(x,a):\bigcap_{a\in C'}\varphi(x,a)\right]<\infty.$$

Proof. Let k_φ be as given by Corollary 2.3. If $|C'|>k_\varphi$, by Corollary 2.3 we find some $a_0\in C'$ such that $\left[\bigcap_{a\in C'\setminus\{a_0\}}\varphi(x,a):\bigcap_{a\in C'}\varphi(x,a)\right]<\infty$. If $|C'\setminus\{a_0\}|>k_\varphi$, by Corollary 2.3 again we find some $a_1\in C'\setminus\{a_0\}$ such that

$$\left[\bigcap_{a\in C'\setminus\{a_0,a_1\}}\varphi(x,a):\bigcap_{a\in C'\setminus\{a_0\}}\varphi(x,a)\right]<\infty.$$

Continuing in this way we end up with $a_0, \ldots, a_m \in C'$ such that for all $i < m$,

$$\left[\bigcap_{a \in C' \setminus \{a_0, \ldots, a_{i+1}\}} \varphi(x, a) : \bigcap_{a \in C' \setminus \{a_0, \ldots, a_i\}} \varphi(x, a)\right] < \infty,$$

and, letting $C_0 = C' \setminus \{a_0, \ldots, a_m\}$, we have that $|C_0| \le k_\varphi$. □

Corollary 2.5. *Let G be a torsion-free group with* NTP_2 *and assume that $\varphi(x, y)$ defines a divisible normal subgroup for every y. Then $\varphi(x, y)$ is* NIP.

Proof. Assume that $\varphi(x, y)$ has IP and let $\bar{a} = (a_i)_{i \in \mathbb{Z}}$ be an indiscernible sequence witnessing this. Taking $H_i = \varphi(\mathfrak{C}, a_i)$, $H_{\neq 0} \setminus H_0 \neq \emptyset$. Let $H = \bigcap_{i \in \mathbb{Z}} H_i$, so it is divisible (here we used the assumption that G is torsion-free), as is $H_{\neq 0}$. But then $H_{\neq 0}/H$ is a divisible non-trivial group, and thus infinite. By indiscernibility $[H_{\neq i} : H] = \infty$ for all i, contradicting Lemma 2.1. □

3. Fields with NTP_2

Let K be a field of characteristic $p > 0$. Recall that a field extension L/K is called an Artin-Schreier extension if $L = K(\alpha)$ for some $\alpha \in L \setminus K$ such that $\alpha^p - \alpha \in K$. L/K is an Artin-Schreier extension if and only if it is Galois and cyclic of degree p.

Theorem 3.1. *Let K be an infinite field definable in an* NTP_2 *theory. Then it has only finitely many Artin-Schreier extensions.*

Proof. We follow the proof of the fact that dependent fields have no Artin-Schreier extensions in [19].

We may assume that K is $\aleph_0$-saturated, and we put $k = K^{p^\infty} = \bigcap_{n \in \omega} K^{p^n}$, a type-definable perfect sub-field which is infinite by saturation (all contained in an algebraically closed $\mathcal{K}$).

For a tuple $\bar{a} = (a_0, \ldots, a_{n-1})$, let

$$G_{\bar{a}} = \left\{(t, x_0, \ldots, x_{n-1}) \in K^{n+1} : t = a_i \cdot \varrho(x_i) \text{ for } i < n\right\},$$

where $\varrho(x) = x^p - x$ is the Artin-Schreier polynomial. We consider it as an algebraic group (a subgroup of $(\mathcal{K}^{n+1}, +)$). As such, by [19, Lemma 2.8], when the elements of $\bar{a}$ are algebraically independent it is connected. If in addition $\bar{a}$ belong to some perfect field k, then $G_{\bar{a}}$ is isomorphic by an algebraic isomorphism over k to $(\mathcal{K}, +)$ by [19, Corollary 2.9].

By Theorem 2.4, there is some $n < \omega$, an algebraically independent $(n+1)$-tuple $\bar{a} \in k$ and an n-subtuple $\bar{a}'$, such that $\left[\bigcap_{a \in \bar{a}'} a \cdot \varrho(K) : \bigcap_{a \in \bar{a}} a \cdot \varrho(K)\right] < \infty$. It follows that the image of the projection map $\pi : G_{\bar{a}}(K) \to G_{\bar{a}'}(K)$ has finite index in $G_{\bar{a}'}(K)$.

We have algebraic isomorphisms $G_{\bar{a}} \to (\mathcal{K}, +)$ and $G_{\bar{a}'} \to (\mathcal{K}, +)$ over k. Hence we can find an algebraic map ρ over k (i.e. a polynomial) which makes the following diagram commute:

$$\begin{array}{ccc} G_{\bar{a}} & \xrightarrow{\pi} & G_{\bar{a}'} \\ \downarrow & & \downarrow \\ (\mathcal{K}, +) & \xrightarrow{\rho} & (\mathcal{K}, +) \end{array}$$

As all groups and maps are defined over $k \subseteq K$, we can restrict to K. We saw that $[G_{\bar{a}'} : \pi(G_{\bar{a}}(K))] < \infty$, so $[K : \rho(K)] < \infty$ as well (in the group $(K, +)$). In the proof of [19, Theorem 4.3], it is shown that there is some $c \in K$ such that, letting $\rho'(x) = \rho(c \cdot x)$, ρ' has the form $a \cdot \varrho(x)$ for some $a \in K^\times$. The way it is done there is by choosing any $0 \neq c \in \ker(\rho) \subseteq k$, and then since ρ' is additive with kernel $\mathbb{F}_p$ and degree p (as this is the degree of π), there exists such an $a \in k$. Since $\rho'(K) = \rho(K)$ has finite index in K, so does the image of $\varrho = a^{-1}\rho'$.

By [19, Remark 2.3], this index is finite if and only if the number of Artin-Schreier extensions is finite. ☐

Proposition 3.2. *Suppose (K, v, Γ) is a valued field of characteristic $p > 0$ that has finitely many Artin-Schreier extensions. Then the valuation group Γ is p-divisible.*

Proof. (This proof is similar to the proof of [19, Proposition 5.4].) Recall that ϱ is the Artin-Schreier polynomial. By Artin-Schreier theory (this is explained in [19, Remark 2.3]), the index $[K : \varrho(K)]$ in the additive group $(K, +)$ is finite. Suppose $\{a_i \mid i < l\}$ are representatives for the cosets of $\varrho(K)$ in $(K, +)$. Let $\alpha \in \Gamma$ be smaller than $\alpha_0 = \min\{v(a_i) \mid i < l\} \cup \{0\}$. Suppose $v(x) = \alpha$ for $x \in K$. But then there is some $i < l$ such that $x - a_i \in \varrho(K)$, and since $v(x) = v(x - a_i)$, we may assume that $x \in \varrho(K)$, so there is some y such that $y^p - y = x$. But then $v(y) < 0$, so $v(y^p) = p \cdot v(y) < v(y)$, and so

$$\alpha = v(x) = v(y^p - y) = v(y^p) = p \cdot v(y).$$

Thus α is p-divisible. Take any negative $\beta \in \Gamma$; then $\beta + p \cdot \alpha_0$ is p-divisible, so β is also p-divisible. Since this is true for all negative values, Γ is p-divisible. ☐

Corollary 3.3. *$\mathbb{F}_p((t))$ is not* NTP$_2$.

Proof. Follows from Theorem 3.1 and Proposition 3.2. ☐

4. Strong theories and bounded burden

In this section we are going to consider groups and fields whose theories satisfy quantitative refinements of NTP$_2$ in terms of a bound on its burden (similar to the bounds on the rank in simple theories).

For notational convenience we consider an extension Card* of the linear order on cardinals by adding a new maximal element ∞ and replacing every limit cardinal κ by two new elements κ_- and κ_+. The standard embedding of cardinals into Card* identifies κ with κ_+. In the following, whenever we take a supremum of a set of cardinals, we will be computing it in Card*.

Definition 4.1. Let T be a complete theory.

(1) An inp-pattern of depth κ consists of $(\bar{a}_\alpha, \varphi_\alpha(x, y_\alpha), k_\alpha)_{\alpha \in \kappa}$ with $\bar{a}_\alpha = (a_{\alpha,i})_{i \in \omega}$ and $k_\alpha \in \omega$ such that:
- $\{\varphi_\alpha(x, a_{\alpha,i}) \mid i < \omega\}$ is k_α-inconsistent for every $\alpha \in \kappa$,
- $\{\varphi_\alpha(x, a_{\alpha, f(\alpha)}) \mid \alpha < \kappa\}$ is consistent for every $f : \kappa \to \omega$.

(2) An inp^2-pattern of depth κ consists of $(\bar{a}_\alpha, b_\alpha, \phi_\alpha(x, y_\alpha, z_\alpha))_{\alpha < \kappa}$, where $\phi_\alpha \in L$, $\bar{a}_\alpha = (a_{\alpha,i})_{i < \omega}$ and $b_\alpha \subseteq \bigcup\{\bar{a}_\beta \mid \beta < \alpha\}$, such that:
- $(\bar{a}_\alpha)_{\alpha < \kappa}$ are mutually indiscernible,
- $\{\phi_\alpha(x, a_{\alpha,i}, b_\alpha) \mid i < \omega\}$ is inconsistent for every α,
- $\{\phi_\alpha(x, a_{\alpha,0}, b_\alpha) \mid \alpha < \kappa\}$ is consistent.

(3) The *burden (burden^2)* of T is the supremum (in Card*) of the depths of inp-patterns (resp. inp^2-patterns) with x a singleton.
(4) It is easy to see by compactness that T is NTP$_2$ if and only if its burden is $< \infty$, equivalently $< |T|^+$. The same is true for burden^2; see [7, Proposition 5.5(viii)].
(5) A theory T is called *strong (strong^2)* if its burden $\leq (\aleph_0)_-$ (resp. burden$^2 \leq (\aleph_0)_-$).

Strong theories were introduced by Adler [1] based on the notion of inp-patterns of Shelah [30, Ch. III], and were further studied in [10], where it was shown that burden is "sub-multiplicative". Strong^2 theories were introduced in [7] as a generalization of Shelah's strongly^2 dependent theories. Of course, every strong^2 theory is strong, and every strong theory is NTP$_2$.

Fact 4.2 ([10]). Burden is "sub-multiplicative": if there is an inp-pattern of depth κ^n with $|x| = n$, then there is an inp-pattern of depth κ with $|x| = 1$. In particular, in a strong theory there are no inp-patterns of infinite depth with x of arbitrary finite length (while the definition requires this only for $|x| = 1$).

Problem 4.3. Does the same hold for inp^2-patterns?

Remark 4.4. (1) For T simple, being strong corresponds to the fact that every finitary type has finite weight [1]. Also, every supersimple theory is strong^2 [7, Section 5].
(2) In [32], Shelah introduced strongly and strongly^2 dependent theories. For strong dependence, the definition is very similar to the one given: one asks that there is no pattern $(\bar{a}_\alpha, \varphi_\alpha(x, y_\alpha))_{\alpha<\omega}$ as above such that for every function $f : \omega \to \omega$, $\left\{\varphi_\alpha(x, a_{\alpha,\beta})^{\text{if } \beta = f(\alpha)} \mid \alpha < \kappa\right\}$ is consistent. One can easily show that T is strongly dependent if and only if it is strong and NIP.
(3) The definition of strongly^2 dependent is again similar to the definition of strong^2, allowing parameters from other rows in the definition of strong dependence. For T dependent, being strong^2 is the same as being strongly^2 dependent (sometimes called strongly$^+$ dependent) [7, Section 5].
(4) There are stable strong theories which are not strong^2, and there are stable strong^2 theories which are not superstable [7, Section 5].

4.1. **Strong groups and fields.** The following are taken from [20, Proposition 3.11, Corollary 3.12] with some easy modifications:

Proposition 4.5. *Let G be a type-definable group and $G_i \leq G$ type-definable normal subgroups for $i < \omega$.*

(1) *If T is strong, then there is some i_0 such that* $\left[\bigcap_{i\neq i_0} G_i : \bigcap_{i<\omega} G_i\right] < \infty$.
(2) *If T is of finite burden, then there is some $n \in \omega$ and $i_0 < n$ such that* $\left[\bigcap_{i\neq i_0, i<n} G_i : \bigcap_{i<n} G_i\right] < \infty$.

Proof. (1) Assume not. Then, for each $i < \omega$, we have an indiscernible sequence $(a_{i,j})_{j<\omega}$ (over the parameters defining all the groups) such that $a_{i,j} \in \bigcap_{k\neq i} G_k$ and for $j_1 < j_2 < \omega$, $a_{i,j_1}^{-1} \cdot a_{i,j_2} \notin G_i$. By compactness there is a formula $\psi_i(x)$ in the type defining G_i such that $\neg\psi_i\left(a_{i,j_1}^{-1} \cdot a_{i,j_2}\right)$ holds (by indiscernibility it is the same for all $j_1 < j_2$). We may assume, applying Ramsey, that the sequences

$\left\langle (a_{i,j})_{j<\omega} \mid i<\omega \right\rangle$ are mutually indiscernible. Let ψ_i' be another formula in the type defining G_i such that $\psi_i'(x) \wedge \psi_i'(y) \vdash \psi_i\left(x^{-1} \cdot y\right)$. Let $\varphi_i(x,y) = \psi_i'\left(x^{-1} \cdot y\right)$.

Now we check that the set $\{\varphi_i(x, a_{i,0}) \mid i<n\}$ is consistent for each $n<\omega$. Let $c = a_{0,0} \cdot \ldots \cdot a_{n-1,0}$ (the order does not really matter, but for the proof it is easier to fix one). So $\varphi_i(c, a_{i,0})$ holds if and only if $\psi_i'\left(a_{n-1,0}^{-1} \cdot \ldots \cdot a_{i,0}^{-1} \cdot \ldots \cdot a_{0,0}^{-1} \cdot a_{i,0}\right)$ holds. But since G_i is normal, $a_{i,0}^{-1} \cdot \ldots \cdot a_{0,0}^{-1} \cdot a_{i,0} \in G_i$, so the entire product is in G_i, so $\varphi_i(c, a_{i,0})$ holds. On the other hand, if for some c', $\varphi_i(c', a_{i,0}) \wedge \varphi_i(c', a_{i,1})$ holds, then $\psi_i\left(a_{i,0}^{-1} \cdot a_{i,1}\right)$ holds, a contradiction. So the rows are inconsistent, which contradicts strength.

(2) Follows from the proof of (1) using Fact 4.2. ☐

Corollary 4.6. *If G is an abelian group type-definable in a strong theory and $S \subseteq \omega$ is an infinite set of pairwise co-prime numbers, then for almost all (i.e. for all but finitely many) $n \in S$, $[G : G^n] < \infty$. In particular, if K is a definable field in a strong theory, then for almost all primes p, $\left[K^\times : (K^\times)^p\right] < \infty$.*

Proof. Let $K \subseteq S$ be the set of $n \in S$ such that $[G : G^n] < \infty$. If $S \backslash K$ is infinite, we replace S with $S \backslash K$.

For $i \in S$, let $G_i = G^i$ (so it is type-definable). By Proposition 4.5, there is some n such that $\left[\bigcap_{i\neq n} G_i : \bigcap_{i\in S} G_i\right] < \infty$. Now it is enough to show that $\bigcap_{i \neq n} G_i / \bigcap_{i\in S} G_i \cong G/G_n$. For this we show that the natural map $\bigcap_{i\neq n} G_i \to G/G_n$ is onto. To show that, we may assume by compactness that S is finite. Let $r = \prod S \backslash \{n\}$. Then since r and n are co-prime, there are some $a, b \in \mathbb{Z}$ such that $ar + bn = 1$, so for any $g \in G$, $g^{ar} \equiv g \pmod{G_n}$, and we are done. ☐

The proof of the following proposition is taken from [22, Proposition 2.3], so we observe that it goes through in larger generality.

Proposition 4.7. *Any infinite strong field is perfect.*

Proof. Let K be of characteristic $p > 0$, and suppose that $K^p \neq K$. Then there are $b_1, b_2 \in K$ linearly independent over K^p. Let $\langle a_i : i \in \mathbb{Q}\rangle$ be an indiscernible non-constant sequence over b_1, b_2. By compactness we can find a and $(c_i)_{i<\omega}$ from K such that $c_0 = a$ and $c_i = b_1 c_{i+1}^p + b_2 a_i^p$. Since b_1, b_2 are linearly independent over K^p, we get that $a_i \in \operatorname{dcl}(b_1 b_2 a)$ for every $i<\omega$. For each $i<\omega$, let $\varphi_i(y, b_1, b_2, a)$ be a formula defining a_i. We may assume that $\forall x, y_1, y_2 \bigwedge_{j=1,2} \varphi_i(y_j, b_1, b_2, x) \to y_1 = y_2$. So:

- the sequences $I_i = (a_j)_{i-1/2<j<i+1/2}$, where $i<\omega$ are mutually indiscernible over b_1, b_2,
- $\{\varphi_i(a_j, b_1, b_2, x) \mid i - 1/2 < j < i+1/2\}$ is 2-inconsistent,
- $\{\varphi_i(a_i, b_1, b_2, x) \mid i<\omega\}$ is consistent (realized by a),

which contradicts strength. ☐

Definition 4.8. A valued field (K, v) of characteristic $p > 0$ is *Kaplansky* if it satisfies:

(1) The valuation group Γ is p-divisible.
(2) The residue field k is perfect and does not admit a finite separable extension whose degree is divisible by p.

Corollary 4.9. *Every strongly dependent valued field is Kaplansky.*

Proof. Combine Proposition 4.7, Proposition 3.2 and [19, Corollary 4.4]. □

4.2. **Strong^2 theories.** The following is just a repetition of [20, Proposition 2.5]:

Proposition 4.10. *Suppose T is strong^2. Then it is impossible to have a sequence of type-definable groups $\langle G_i \mid i < \omega \rangle$ such that $G_{i+1} \leq G_i$ and $[G_i : G_{i+1}] = \infty$.*

Proof. Without loss of generality, we shall assume that all groups are type-definable over $\emptyset$. Suppose there is such a sequence $\langle G_i \mid i < \omega \rangle$. Let $\langle \bar{a}_i \mid i < \omega \rangle$ be mutually indiscernible, where $\bar{a}_i = \langle a_{i,j} \mid j < \omega \rangle$, such that for $i < \omega$, the sequence $\langle a_{i,j} \mid j < \omega \rangle$ is a sequence from G_i (in $\mathfrak{C}$) such that $a_{i,j'}^{-1} \cdot a_{i,j} \notin G_{i+1}$ for all $j < j' < \omega$. We can find such an array because of our assumption and Ramsey.

For each $i < \omega$, let $\psi_i(x)$ be in the type defining G_{i+1} such that $\neg\psi_i\left(a_{i,j'}^{-1} \cdot a_{i,j}\right)$ for $j' < j$. By compactness, there is a formula $\xi_i(x)$ in the type defining G_{i+1} such that for all $a, b \in \mathfrak{C}$, if $\xi_i(a) \wedge \xi_i(b)$, then $\psi_i\left(a \cdot b^{-1}\right)$ holds. Let $\varphi_i(x, y, z) = \xi_i\left(y^{-1} \cdot z^{-1} \cdot x\right)$. For $i < \omega$, let $b_i = a_{0,0} \cdot \ldots \cdot a_{i-1,0}$ (so $b_0 = 1$).

Let us check that the set $\{\varphi_i(x, a_{i,0}, b_i) \mid i < \omega\}$ is consistent. Let $i_0 < \omega$, and let $c = b_{i_0}$. Then for $i < i_0$, $\varphi_i(c, a_{i,0}, b_i)$ holds if and only if $\xi_i(a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0})$ holds, but the product $a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0}$ is an element of G_{i+1} and ξ_i is in the type defining G_{i+1}, so $\varphi_i(c, a_{i,0}, b_i)$ holds. Now, if $\varphi_i(c', a_{i,0}, b_i) \wedge \varphi_i(c', a_{i,0}, b_i)$ holds for some c', then $\xi_i\left(a_{i,0}^{-1} b_i^{-1} c'\right)$ and $\xi_i\left(a_{i,1}^{-1} b_i^{-1} c'\right)$ hold, so also $\psi_i\left(a_{i,0}^{-1} a_{i,1}\right)$ holds, a contradiction. So the rows are inconsistent, contradicting strength^2. □

We also get (exactly as [20, Proposition 2.6]):

Corollary 4.11. *Assume T is strong^2. If G is a type-definable group and h is a definable homomorphism $h : G \to G$ with finite kernel, then h is almost onto G; i.e. the index $[G : h(G)]$ is bounded (i.e. $< \infty$). If G is definable, then the index must be finite.*

Theorem 2.4 holds for type-definable subgroups without the normality assumption.

Theorem 4.12. *Let G be strong^2 and $\{\varphi(x, a) \mid a \in C\}$ be a family of definable subgroups of G. Then there is some $k \in \omega$ such that for every finite $C' \subseteq C$ there is some $C_0 \subseteq C'$ with $|C_0| \leq k$ and such that*

$$\left[\bigcap_{a \in C_0} \varphi(x, a) : \bigcap_{a \in C'} \varphi(x, a)\right] < \infty.$$

Proof. The proof of Theorem 2.4 relied on Lemma 2.1. Thus we only need to show that this lemma goes through. Let $H_i = \varphi(x, a_i)$ for $i < \omega$. Consider $H_i' = \bigcap_{j<i} H_j$. At some point $\left[H_j' : H_{j+1}'\right] < \infty$. But then also $\left[H_{\neq j} : \bigcap_{i<\omega} H_i\right] < \infty$. □

5. Questions, conjectures and further research directions

5.1. **More pure** NTP_2 **fields.** Recall that a field is pseudo-algebraically closed (or PAC) if every absolutely irreducible variety defined over it has a point in it. It is well-known [8] that the theory of a PAC field is simple if and only if it is bounded (i.e. for any integer n it has only finitely many Galois extensions of degree n). Moreover, if a PAC field is unbounded, then it has TP_2 [9, Section 3.5]. On the other hand, the following fields were studied extensively:

(1) Pseudo-real closed (or PRC) fields: a field F is PRC if every absolutely irreducible variety defined over F that has a rational point in every real closure of F also has an F-rational point [27–29].
(2) Pseudo-p-adically closed (or PpC) fields: a field F is PpC if every absolutely irreducible variety defined over F that has a rational point in every p-adic closure of F also has an F-rational point [14, 16, 23, 24].

Conjecture 5.1. *A* PRC *field is* NTP$_2$ *if and only if it is bounded. Similarly, a* PpC *field is* NTP$_2$ *if and only if it is bounded.*

We remark that if K is an unbounded PRC field, then it has TP$_2$. Indeed, since K is PRC, then $L = K\left(\sqrt{-1}\right)$ is PAC (because every finite extension of a PRC field is PRC and L has no real closures at all). By [15, Remark 16.10.3(b)] L is unbounded, and of course, L is interpretable in K. But by the result of Chatzidakis [9], L has TP$_2$; thus K also has TP$_2$.

5.2. **More valued fields with** NTP$_2$**.** Is there an analogue of Fact 1.1 in positive characteristic? A similar result for NIP was established in [6, Corollaire 7.6].

Conjecture 5.2. *Let* (K, v) *be a valued field of characteristic* $p > 0$*, Kaplansky and algebraically maximal. Then* (K, v) *is* NTP$_2$ *(strong) if and only if* k *is* NTP$_2$ *(resp. strong).*

The following is demonstrated in [19, Proposition 5.3].

Fact 5.3. Let (K, v) be an NIP valued field of characteristic $p > 0$. Then the residue field contains $\mathbb{F}_p^{\mathrm{alg}}$ (so in particular is infinite).

Hrushovski asked if the following is true:

Problem 5.4. Assume that (K, v) is an NTP$_2$ (Henselian) valued field of positive characteristic. Does it follow that the residue field is infinite?

We remark that the finite number of Artin-Schreier extensions alone is not sufficient to conclude that the residue field is infinite:

Example 5.5 (Due to Arno Fehm)**.** Let $\Omega = \left(\mathbb{F}_p\left((t)\right)\right)^{\mathrm{sep}}$, so the restriction map $\mathrm{Gal}\left(\Omega/\mathbb{F}_p\left((t)\right)\right) \to \mathrm{Gal}\left(\mathbb{F}_p^{\mathrm{alg}}/\mathbb{F}_p\right)$ is onto. Let $\sigma \in \mathrm{Gal}\left(\mathbb{F}_p^{\mathrm{alg}}/\mathbb{F}_p\right)$ be the Frobenius automorphism, and let $\tau \in \mathrm{Gal}\left(\Omega/\mathbb{F}_p\left((t)\right)\right)$ be such that $\tau \restriction \mathbb{F}_p^{\mathrm{alg}} = \sigma$. Let F be the fixed field of τ. Then F has exactly one Artin-Schreier extension (as $\mathrm{Gal}\left(\Omega/F\right)$ is pro-cyclic and F is a regular extension of $\mathbb{F}_p$). Since $\mathbb{F}_p\left((t)\right)$ is a Henselian valued field, its usual valuation extends uniquely to a Henselian valuation on F. Since every element of $\mathbb{F}_p^{\mathrm{alg}} \backslash \mathbb{F}_p$ is moved by σ, one can see that the residue field must be $\mathbb{F}_p$.

Example 5.6 (Due to the anonymous referee)**.** Let Ω be the generalized power series $\mathbb{F}_p^{\mathrm{alg}}\left(\left(t^{\mathbb{Q}}\right)\right)$ — the field of formal sums $\sum a_i t^i$ with well-ordered support where $i \in \mathbb{Q}$ and $a_i \in \mathbb{F}_p^{\mathrm{alg}}$. This field is algebraically closed. Let $\tau \in \mathrm{Aut}\left(\Omega\right)$ be the map $\sum a_i t^i \mapsto \sum a_i^p t^i$. Let F be the fixed field of τ, so $F = \mathbb{F}_p\left(\left(t^{\mathbb{Q}}\right)\right)$. Then F is Henselian with residue field $\mathbb{F}_p$ and (as in Example 5.5) has exactly one Artin-Schreier extension.

5.3. **Definable envelopes.** Assume that we are given a subgroup of an NTP_2 group. Is it possible to find a *definable* subgroup which is close to the subgroup we started with and satisfies similar properties?

Fact 5.7.

(1) [2,32] If G is a group definable in an NIP theory and H is a subgroup which is abelian (nilpotent of class n; normal and soluble of derived length n), then there is a definable group containing H which is also abelian (resp. nilpotent of class n; normal and soluble of derived length n).
(2) [25] Let G be a group definable in a simple theory and let H be a subgroup of G.
   (a) If H is nilpotent of class n, then there is a definable (with parameters from H) nilpotent group of class at most $2n$, finitely many translates of which cover H. If H is in addition normal, then there is a definable normal nilpotent group of class at most $3n$ containing H.
   (b) If H is a soluble of class n, then there is a definable (with parameters from H) soluble group of derived length at most $2n$, finitely many translates of which cover H. If H is in addition normal, then there is a definable normal soluble group of derived length at most $3n$ containing H.

Thus it seems very natural to make the following conjecture.

Conjecture 5.8. *Let G be an* NTP_2 *group and assume that H is a subgroup. If H is nilpotent (soluble), then there is a definable nilpotent (resp. soluble) group, finitely many translates of which cover H. If H is in addition normal, then there is a definable normal nilpotent (resp. soluble) group containing H.*

5.4. **Hrushovski's stabilizer theorem.** Let I be an ideal in the Boolean algebra of definable sets in a fixed variable x, with parameters from the monster model (i.e. $\emptyset \in I$;$\phi(x,a) \vdash \psi(x,b)$ and $\psi(x,b) \in I$ imply $\phi(x,a) \in I$;$\phi(x,a) \in I$ and $\psi(x,b) \in I$ imply $\phi(x,a) \vee \psi(x,b) \in I$). An ideal I is invariant over a set A if $\phi(x,a) \in I$ and $a \equiv_A b$ implies $\phi(x,b) \in I$. An A-invariant ideal is called S1 if for every sequence $(a_i)_{i\in\omega}$ indiscernible over A, $\phi(x,a_0) \wedge \phi(x,a_1) \in I$ implies $\phi(x,a_0) \in I$. A partial type $q(x)$ over A is called wide (or I-wide) if it implies no formula in I.

In the following, $\tilde{G}$ is a subgroup of some definable group, generated by some definable set X.

Fact 5.9 ([18, Theorem 3.5]). Let M be a model, and let μ be an M-invariant S1 ideal on definable subsets of $\widetilde{G}$, invariant under (left or right) translations by elements of $\widetilde{G}$. Let q be a wide type over M (contained in $\widetilde{G}$). Assume:

(F) There exist two realizations a,b of q such that $\mathrm{tp}(b/Ma)$ does not fork over M and $\mathrm{tp}(a/Mb)$ does not fork over M.

Then there is a wide type-definable over M subgroup S of G. We have $S=\left(q^{-1}q\right)^2$; the set $qq^{-1}q$ is a coset of S. Moreover, S is normal in $\widetilde{G}$, and $S\setminus q^{-1}q$ is contained in a union of non-wide M-definable sets.

In [12] it is proved that if M is a model of an NTP_2 theory and $q \in S(M)$, then it has a global strictly invariant extension $p \in S(\mathfrak{C})$ (meaning that p is an M-invariant type, and for every $N \supseteq M$ and $a \models p|_N$ we have that $\mathrm{tp}(N/Ma)$

does not fork over M). It thus follows that the assumption (F) is always satisfied in NTP_2 theories. In [7, Section 2 and the discussion before Proposition 3.5] it is proved that in an NTP_2 theory, the ideal of formulas forking over a model M is $S1$. However, in general the ideal of forking formulas is not invariant under the action of the definable group. By [18, Theorem 3.5, Remark (4)] the assumption of invariance under the action of $\widetilde{G}$ can be replaced by the existence of an f-generic extension of q. It seems interesting to find a correct version of this result generalizing the theory of stabilizers in simple theories [26].

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