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GORENSTEIN INJECTIVE COVERS AND ENVELOPES OVER NOETHERIAN RINGS

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ABSTRACT. We prove that if R is a commutative Noetherian ring such that the character modules of Gorenstein injective modules are Gorenstein flat, then the class of Gorenstein injective modules is closed under direct limits and it is covering.

We also prove that over such a ring the class of Gorenstein injective modules is enveloping. In particular this shows the existence of the Gorenstein injective envelopes over commutative Noetherian rings with dualizing complexes.

1. INTRODUCTION

The Gorenstein injective modules were introduced in [4] as a generalization of the injective modules. Together with the Gorenstein projective and the Gorenstein flat modules, they are the key ingredients of Gorenstein homological algebra. The existence of the Gorenstein (injective, projective, flat) covers and envelopes has been studied intensively in recent years (see for example [6], [10], [11], and [13]). Our results are part of this ongoing program in Gorenstein homological algebra.

We start by giving a sufficient condition for the existence of the Gorenstein injective covers over commutative Noetherian rings. We prove (Theorem 1) that if for any Gorenstein injective module M, the character module M^+ is Gorenstein flat, then the class of Gorenstein injective modules is closed under direct limits and it is covering. This is the case when the ring is Iwanaga-Gorenstein. More generally, the result holds over any commutative Noetherian ring with a dualizing complex.

In Section 4 we give a sufficient condition for the existence of Gorenstein injective envelopes over Noetherian rings. We prove (Proposition 1) that if R is a Noetherian ring such that the class of modules which are left orthogonal to all Gorenstein injective modules, ${}^{\perp}\mathcal{GI}$, is closed under pure quotients, then the class of Gorenstein injective modules is enveloping. The result holds for Gorenstein rings. It also holds for Artin algebras that are virtually Gorenstein.

We also prove (Proposition 2) that over a Noetherian ring the class of Gorenstein injective modules is enveloping if and only if the class ${}^{\perp}\mathcal{GI}$ is covering.

Then we prove (Theorem 2) the existence of Gorenstein injective envelopes over commutative Noetherian rings with the property that the character modules of Gorenstein injective modules are Gorenstein flat. In particular this shows that the

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class of Gorenstein injective modules is enveloping over commutative Noetherian rings with dualizing complexes.

2. Preliminaries

Throughout this section R denotes an associative ring with unity. By R-module we mean a left R-module.

Definition 1 ([5, Definition 10.1.1]). An R-module M is Gorenstein injective if there exists an exact and $Hom(Inj, -)$ exact sequence of injective R-modules

$$\ldots \to E_1 \to E_0 \to E_{-1} \to \ldots$$

such that $M = Ker(E_0 \to E_{-1})$.

We will use the notation $\mathcal{GI}$ for the class of Gorenstein injective modules.

A stronger notion is that of strongly Gorenstein injective modules:

Definition 2 ([2, Definition 2.1]). An R-module M is strongly Gorenstein injective if there exists an exact and $Hom(Inj, -)$ exact sequence

$$\ldots \to E \xrightarrow{f} E \xrightarrow{f} E \to \ldots$$

with E injective and with $M = Ker(f)$.

It is known [2, Theorem on page 3] that a module is Gorenstein injective if and only if it is a direct summand of a strongly Gorenstein injective one.

The dual notion of Gorenstein injective module is that of Gorenstein projective module.

Definition 3 ([5, Definition 10.2.1]). An R-module X is Gorenstein projective if there exists an exact and $Hom(-, Proj)$ exact sequence of projective R-modules

$$\ldots \to P_1 \to P_0 \to P_{-1} \to \ldots$$

such that $X = Ker(P_0 \to P_{-1})$.

We will use the notation $\mathcal{GP}$ for the class of Gorenstein projective modules.

The Gorenstein flat modules are defined in terms of the tensor product.

Definition 4 ([5, Definition 10.3.1]). An R-module G is Gorenstein flat if there exists an exact and $Inj \otimes -$ exact sequence of flat modules

$$\ldots \to F_1 \to F_0 \to F_{-1} \to \ldots$$

such that $G = Ker(F_0 \to F_{-1})$.

We will use the notation $\mathcal{GF}$ for the class of Gorenstein flat modules.

It is known [8, Theorem 3.6] that if the ring R is right coherent then a module G is Gorenstein flat if and only if its character module G^+ is Gorenstein injective.

Given a class of R-modules $\mathcal{F}$, we will denote as usual by $\mathcal{F}^\perp$ the class of all R-modules M such that $Ext^1(F, M) = 0$ for every $F \in \mathcal{F}$.

The left orthogonal class of $\mathcal{F}$, denoted ${}^\perp\mathcal{F}$, is the class of all ${}_RN$ such that $Ext^1(N, F) = 0$ for every $F \in \mathcal{F}$.

We recall that a pair $(\mathcal{L}, \mathcal{C})$ is a *cotorsion pair* if $\mathcal{L}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{L}$.

A cotorsion pair $(\mathcal{L}, \mathcal{C})$ is *complete* if for every ${}_RM$ there exist exact sequences $0 \to C \to L \to M \to 0$ and $0 \to M \to C' \to L' \to 0$ with C, C' in $\mathcal{C}$ and L, L' in $\mathcal{L}$.

Duality pairs were introduced by Holm and Jørgensen in [10]. We recall their definition (the opposite ring is denoted R^0):

Definition 5 ([10, Definition 2.1])**.** A duality pair over a ring R is a pair (M, C) where M is a class of R-modules and C is a class of R^o-modules, subject to the following conditions:

(1) For an R-module M, one has $M \in \mathrm{M}$ if and only if $M^+ \in \mathrm{C}$.
(2) C is closed under direct summands and finite direct sums.

A duality pair (M, C) is called (co)product-closed if the class M is closed under (co)products in the category of all R-modules.

A duality pair (M, C) is called *perfect* if it is coproduct-closed, if M is closed under extensions, and if R belongs to M.

We also recall the following result:

Theorem ([10, Theorem 3.1])**.** *Let* (M, C) *be a duality pair. Then* M *is closed under pure submodules, pure quotients, and pure extensions. Furthermore, the following hold:*

(a) *If* (M, C) *is product-closed then* M *is preenveloping.*
(b) *If* (M, C) *is coproduct-closed then* M *is covering.*
(c) *If* (M, C) *is perfect then* $(\mathrm{M}, \mathrm{M}^{\perp})$ *is a perfect cotorsion pair.*

3. Gorenstein injective covers

We prove (Theorem 1) that if R is a commutative Noetherian ring such that for any Gorenstein injective module M, its character module M^+ is Gorenstein flat, then, the class of Gorenstein injective modules is closed under direct limits and it is covering in $R - Mod$.

Our first result is the following:

Lemma 1. *Let R be a Noetherian ring, and assume that for every Gorenstein injective module G, the character module G^+ is Gorenstein flat. If X is any R-module such that the character module X^+ is Gorenstein flat, then X is Gorenstein injective.*

Proof. Let X be an R-module such that X^+ is Gorenstein flat. Since R is Noetherian, there exists an exact sequence $0 \to X \to G \to L \to 0$ with G Gorenstein injective and with $L \in {}^{\perp}\mathcal{GI}$ [12, 7.12]. Then the sequence $0 \to L^+ \to G^+ \to X^+ \to 0$ is exact with both X^+ and G^+ Gorenstein flat. It follows that L^+ is Gorenstein flat [1, Proposition 2.2 and Theorem 2.3].

Let B be Gorenstein flat. Then B^+ is Gorenstein injective and therefore $Ext^1(L, B^+) = 0$. So $Tor_1(L, B)^+ \simeq Ext^1(L, B^+) = 0$. Thus $Tor_1(L, B) = 0$ for any Gorenstein flat module B.

We also have that $Ext^1(B, L^+) \simeq Tor_1(L, B)^+ = 0$. Thus $L^+ \in \mathcal{GF}^{\perp}$. Since L^+ is Gorenstein flat and such that $L^+ \in \mathcal{GF}^{\perp}$, it follows that L^+ is a flat module.

The ring R is Noetherian, so we have $id_R(L) = fd_R(L^+) = 0$.

Thus there exists a short exact sequence $0 \to X \to G \to L \to 0$ with G Gorenstein injective and L injective. Consequently, X has finite Gorenstein injective dimension. By [3, Lemma 2.18], there exists an exact sequence $0 \to B \to H \to X \to 0$ with B Gorenstein injective and with $id_R H = Gid_R X < \infty$. It follows that $fd_R H^+$ is finite. In the exact sequence $0 \to X^+ \to H^+ \to B^+ \to 0$ the modules X^+ and B^+ are Gorenstein flat by our assumptions, and hence H^+ is Gorenstein flat as well. Since a Gorenstein flat module with finite flat dimension is flat [5, Corollary 10.3.4], the module H^+ is flat. Hence H is injective, and since $Gid_R X = id_R H$, we conclude that X is Gorenstein injective. □

Lemma 2. *Let $\mathcal{X}$ be a class of R-modules which is closed under pure submodules and pure quotients. If $\mathcal{X}$ is closed under products, then $\mathcal{X}$ is also closed under direct sums and direct limits.*

Proof. For a family $(X_i)_{i\in I}$ of R-modules, the direct sum is a pure submodule of the product $\prod_{i\in I} X_i$. For a direct system $X_i \to X_j$ of R-modules, the direct limit $\varinjlim X_i$ is a pure quotient of the direct sum $\oplus_{i\in I} X_i$. □

Theorem 1. *Let R be a commutative Noetherian ring such that for every Gorenstein injective module G, the character module G^+ is Gorenstein flat. Then the class of Gorenstein injective modules is covering and closed under direct limits.*

Proof. For every R-module M, Lemma 1 shows that M is Gorenstein injective if and only if M^+ is Gorenstein flat. As R is coherent, the class of Gorenstein flat modules, $\mathcal{GF}$, is closed under direct summands and direct sums [8, Theorem 3.7]. Then $(\mathcal{GI}, \mathcal{GF})$ is a duality pair in the sense of [10, Definition 2.1], and it follows from [10, Theorem 3.1] that $\mathcal{GI}$ is closed under pure submodules and pure quotients. Since $\mathcal{GI}$ is closed under products, Lemma 2 implies that $\mathcal{GI}$ is closed under direct sums and direct limits. Hence the duality pair $(\mathcal{GI}, \mathcal{GF})$ is coproduct-closed in the sense of [10, Definition 2.1]. Another application of [10, Theorem 3.1] yields that $\mathcal{GI}$ is covering. □

Examples. 1. Every commutative Iwanaga-Gorenstein ring satisfies Theorem 1 (by [5, Corollary 10.3.9]).

2. Every commutative Noetherian ring with a dualizing complex satisfies Theorem 1 (by [3, Theorem 6.9]). This is [10, Theorem 3.3].

3. The ring $R = K[[X^S]]$, with K a field and $S \subset N$ a symmetric submonoid of the monoid of natural numbers (here the Gorenstein injective modules are just the divisible modules).

4. Gorenstein injective envelopes

We give a sufficient condition for the existence of Gorenstein injective envelopes over left Noetherian rings. We prove (Proposition 1) that if R is a left Noetherian ring such that the class of modules which are left orthogonal to all Gorenstein injective modules, ${}^{\perp}\mathcal{GI}$, is closed under pure quotients, then the class of Gorenstein injective modules is enveloping. The result holds for Gorenstein rings. It also holds for Artin algebras that are virtually Gorenstein.

When the ring R is commutative Noetherian and such that the character modules of Gorenstein injective modules are Gorenstein flat, we prove that $({}^{\perp}\mathcal{GI}, \mathcal{GF}^{\perp})$ is a perfect duality pair. This implies that the class of Gorenstein injective modules is enveloping over such a ring (Theorem 2). In particular this shows the existence of Gorenstein injective envelopes over commutative Noetherian rings with dualizing complexes.

Lemma 3. *If R is a Noetherian ring, then $({}^{\perp}\mathcal{GI})^{\perp} = \mathcal{GI}$.*

Proof. We have $\mathcal{GI} \subseteq ({}^{\perp}\mathcal{GI})^{\perp}$.

Let $X \in ({}^{\perp}\mathcal{GI})^{\perp}$. By [12] there exists an exact sequence

$$0 \to X \to G \to K \to 0$$

with G Gorenstein injective and with $K \in^{\perp} \mathcal{GI}$. We have $Ext^1(K, X) = 0$, so the sequence is split exact. Since X is isomorphic to a direct summand of G, it is Gorenstein injective. □

We recall that a cotorsion pair $(\mathcal{L}, \mathcal{C})$ is hereditary if $Ext^i(L, C) = 0$ for any $L \in \mathcal{L}$, any $C \in \mathcal{C}$ and for all $i \geq 1$.

Corollary 1. *If R is Noetherian, then $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete hereditary cotorsion pair.*

Proof. By Lemma 3 $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a cotorsion pair. By [12, 7.12] and [5, 7.1.7] $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is complete.

Let $K \in^{\perp} \mathcal{GI}$ and let $G \in \mathcal{GI}$. There exists an exact sequence $0 \to G \to E \to G' \to 0$ with E injective and with G' Gorenstein injective. This gives an exact sequence $0 = Ext^1(K, E) \to Ext^1(K, G') \to Ext^2(K, G) \to Ext^2(K, E) = 0$. Since $Ext^1(K, G') = 0$ it follows that $Ext^2(K, G) = 0$. Similarly $Ext^i(K, G) = 0$ for all $i \geq 3$ and all Gorenstein injective modules G. □

Lemma 4. *If the class ${}^{\perp}\mathcal{GI}$ is closed under pure quotients, then it is also closed under pure submodules.*

Proof. Let $K \in^{\perp} \mathcal{GI}$.

Let $0 \to K' \to K \to K" \to 0$ be a pure exact sequence. By hypothesis $K" \in^{\perp} \mathcal{GI}$. By the above $Ext^i(K", G) = 0$ for all $i \geq 1$ and for all Gorenstein injective modules G.

The long exact sequence $0 = Ext^1(K", G) \to Ext^1(K, G) \to Ext^1(K', G) \to Ext^2(K", G) = 0$ gives that $Ext^1(K', G) = 0$ for any Gorenstein injective module G. □

We recall that a cotorsion pair $(\mathcal{L}, \mathcal{C})$ is perfect if every R-module has an $\mathcal{L}$-cover and a $\mathcal{C}$-envelope.

Proposition 1. *Let R be a Noetherian ring. Assume that the class ${}^{\perp}\mathcal{GI}$ is closed under pure quotients. Then $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a perfect cotorsion pair. In this case the class of Gorenstein injective modules is enveloping.*

Proof. By Lemma 4, ${}^{\perp}\mathcal{GI}$ is closed under pure quotients and pure submodules. By definition, the left orthogonal class ${}^{\perp}\mathcal{GI}$ is also closed under extensions and under direct sums. Since ${}^{\perp}\mathcal{GI}$ contains the ground ring R, it follows that $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a perfect cotorsion pair [9, Theorem 3.4]. In particular, $\mathcal{GI}$ is enveloping. □

Example 1. If R is an n Iwanaga-Gorenstein ring, then the class of Gorenstein injective modules is enveloping.

Proof. In this case ${}^{\perp}\mathcal{GI}$ is the class of modules of finite injective dimension.

Let $0 \to L' \to L \to L" \to 0$ be a pure exact sequence in ${}^{\perp}\mathcal{GI}$. Then $L^+ = (L')^+ \oplus (L")^+$. Since $fd_R(L^+) = id_R L \leq n$ it follows that $fd_R(L")^+ \leq n$. So $id_R L" \leq n$. By Proposition 1, $\mathcal{GI}$ is enveloping. □

Example 2. If R is an Artin algebra that is virtually Gorenstein, then the class of Gorenstein injective modules is enveloping.

Proof. By definition an Artin algebra is virtually Gorenstein if the class of modules which are right orthogonal to all Gorenstein projective modules, $\mathcal{GP}^{\perp}$, coincides with the class of modules that are left orthogonal to all Gorenstein injective modules, ${}^{\perp}\mathcal{GI}$.

By the proof of Lemma 4 the class ${}^{\perp}\mathcal{GI}$ is closed under kernels of epimorphisms. A dual argument shows that the class $\mathcal{GP}^{\perp}$ is closed under cokernels of epimorphisms. So in this case if $0 \to K' \to K \to K" \to 0$ is any exact sequence with $K \in^{\perp} \mathcal{GI}$, then both K' and $K"$ are in ${}^{\perp}\mathcal{GI}$. □

There exist complete cotorsion pairs $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{B}$ is enveloping but $\mathcal{A}$ is not covering; see [7, Corollary 4.4.2 and Remark 4.4.18]. The next result shows that this phenomenon cannot happen for the complete cotorsion pair $({}^{\perp}\mathcal{GI}, \mathcal{GI})$.

Proposition 2. *Let R be Noetherian. The following are equivalent:*

1. *The class of Gorenstein injective modules is enveloping.*
2. *$({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a perfect cotorsion pair.*
3. *The class ${}^{\perp}\mathcal{GI}$ is covering.*

Proof. $1 \Rightarrow 2$. By Corollary 1, $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a hereditary cotorsion pair. By [6, Theorem 1.4], this cotorsion pair is perfect if and only if $\mathcal{GI}$ is enveloping and every Gorenstein injective module has a ${}^{\perp}\mathcal{GI}$ cover.

Let $G \in \mathcal{GI}$, and let $0 \to G' \to E \to G \to 0$ be exact with $E \to G$ the injective cover of G. Then $E \in^{\perp} \mathcal{GI}$ and $G' \in \mathcal{GI}$, so the sequence is $Hom({}^{\perp}\mathcal{GI}, -)$ exact. Thus $E \to G$ is a ${}^{\perp}\mathcal{GI}$ cover of G.

$3 \Rightarrow 2$. By [6, Theorem 1.4], the hereditary cotorsion pair $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is perfect if and only if ${}^{\perp}\mathcal{GI}$ is covering and every $X \in^{\perp} \mathcal{GI}$ has a Gorenstein injective envelope.

Let $X \in^{\perp} \mathcal{GI}$. By [12] there exists an exact sequence $0 \to X \to G \to T \to 0$ with $G \in \mathcal{GI}$ and with $T \in^{\perp} \mathcal{GI}$. Then $G \in \mathcal{GI} \cap^{\perp} \mathcal{GI}$. So G is injective and $X \to G$ is an injective preenvelope of X. Let $0 \to X \to A \to Y \to 0$ be exact with $X \to A$ the injective envelope of X. Then A is a direct summand of G. There exists a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & A & \longrightarrow & Y & \longrightarrow & 0 \\ & & \Vert & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & G & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

This gives an exact sequence $0 \to A \to G \oplus Y \to T \to 0$. Since A is injective this sequence is split exact. It follows that Y is isomorphic to a direct summand of T, and therefore it is in ${}^{\perp}\mathcal{GI}$.

Since $0 \to X \to A \to Y \to 0$ is exact with $A \in \mathcal{GI}$ and with $Y \in^{\perp} \mathcal{GI}$ it follows that $X \to A$ is a Gorenstein injective preenvelope of X. Any $u \in Hom_R(A, A)$ that is the identity on X is an automorphism of A, so $X \to A$ is a Gorenstein injective envelope.

$2 \Rightarrow 1$ and $2 \Rightarrow 3$ follow from the definition of a perfect cotorsion pair. □

We prove that the class of Gorenstein injective modules is enveloping over a commutative Noetherian ring with the property that the character modules of Gorenstein injective modules are Gorenstein flat.

We prove first the following result:

Lemma 5. *Let R be a Noetherian ring such that the character modules of Gorenstein injective modules are Gorenstein flat. Then $K \in^{\perp} \mathcal{GI}$ if and only if $K^+ \in \mathcal{GF}^{\perp}$.*

Proof. $\Rightarrow$ Let $K \in^{\perp} \mathcal{GI}$. For any Gorenstein flat module B we have $B^+ \in \mathcal{GI}$. It follows that $Ext^1(K, B^+) = 0$. Then $Ext^1(B, K^+) \simeq Ext^1(K, B^+) = 0$. So $K^+ \in \mathcal{GF}^{\perp}$.

$\Leftarrow$ Assume that K is an R-module such that $K^+ \in \mathcal{GF}^{\perp}$.

Since R is Noetherian there exists an exact sequence $0 \to K \to G \to V \to 0$ with G Gorenstein injective and with $V \in^{\perp} \mathcal{GI}$. This gives an exact sequence $0 \to V^+ \to G^+ \to K^+ \to 0$ with G^+ Gorenstein flat. Since $V \in^{\perp} \mathcal{GI}$ we have that $V^+ \in \mathcal{GF}^{\perp}$. Since K^+ is also in $\mathcal{GF}^{\perp}$ it follows that $G^+ \in \mathcal{GF} \cap \mathcal{GF}^{\perp}$. Thus G^+ is flat and therefore G is injective.

So we have an exact sequence $0 \to K \to G \to V \to 0$ with $G \in Inj \subseteq^{\perp} \mathcal{GI}$ and with $V \in^{\perp} \mathcal{GI}$. Then for any Gorenstein injective module A we have an exact sequence $0 = Ext^1(V, A) \to Ext^1(G, A) \to Ext^1(K, A) \to Ext^2(V, A) = 0$ (by Corollary 1). Since $Ext^1(G, A) = 0$ it follows that $Ext^1(K, A) = 0$. Thus $K \in^{\perp} \mathcal{GI}$. □

Theorem 2. *Let R be a commutative Noetherian ring such that the character modules of Gorenstein injective modules are Gorenstein flat. Then $({}^{\perp}\mathcal{GI}, \mathcal{GF}^{\perp})$ is a perfect duality pair. In particular the class of Gorenstein injective modules is enveloping.*

Proof. By Lemma 5 we have that $K \in^{\perp} \mathcal{GI}$ if and only if $K^+ \in \mathcal{GF}^{\perp}$.

Since the class, $\mathcal{GF}^{\perp}$ is closed under direct summands and finite direct sums, it follows that the pair $({}^{\perp}\mathcal{GI}, \mathcal{GF}^{\perp})$ is a duality pair.

Since the class ${}^{\perp}\mathcal{GI}$ is closed under direct sums and extensions and it contains the ground ring R, the duality pair $({}^{\perp}\mathcal{GI}, \mathcal{GF}^{\perp})$ is perfect. Then by [10, Theorem 3.1], $({}^{\perp}\mathcal{GI}, ({}^{\perp}\mathcal{GI})^{\perp})$ is a perfect cotorsion pair. Since R is Noetherian, we have $({}^{\perp}\mathcal{GI})^{\perp}) = \mathcal{GI}$. So $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a perfect cotorsion pair. Therefore the class of Gorenstein injective modules is enveloping. □

Corollary 2. *If R is commutative Noetherian with a dualizing complex, then the class of Gorenstein injective modules is enveloping.*

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