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GENERALIZED CROSSING CHANGES IN SATELLITE KNOTS

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Abstract. We show that if K is a satellite knot in the 3-sphere S^3 which admits a generalized cosmetic crossing change of order q with $|q| \geq 6$, then K admits a pattern knot with a generalized cosmetic crossing change of the same order. As a consequence of this, we find that any prime satellite knot in S^3 which admits a torus knot as a pattern cannot admit a generalized cosmetic crossing change of order q with $|q| \geq 6$. We also show that if there is any knot in S^3 admitting a generalized cosmetic crossing change of order q with $|q| \geq 6$, then there must be such a knot which is hyperbolic.

1. Introduction

One of the many easily stated yet still unanswered questions in knot theory is the following: when does a crossing change on a diagram for an oriented knot K yield a knot which is isotopic to K in S^3? We wish to study this and similar questions without restricting ourselves to any particular diagram for a given knot, so we will consider crossing changes in terms of crossing disks. A *crossing disk* for an oriented knot $K \subset S^3$ is an embedded disk $D \subset S^3$ such that K intersects $\text{int}(D)$ twice with zero algebraic intersection number (see Figure 1). A crossing change on K can be achieved by performing (± 1)-Dehn surgery of S^3 along the *crossing circle* $L = \partial D$. (See [12] for details on crossing changes and Dehn surgery.) More generally, if we perform $(-1/q)$-Dehn surgery along the crossing circle L for some $q \in \mathbb{Z} - \{0\}$, we twist K q times at the crossing circle in question. We will call this an *order-q generalized crossing change*. Note that if q is positive, then we give K q right-hand twists when we perform $(-1/q)$-surgery, and if q is negative, we give K q left-hand twists.

A crossing of K and its corresponding crossing circle L are called *nugatory* if L bounds an embedded disk in $S^3 - \eta(K)$, where $\eta(K)$ denotes a regular neighborhood of K in S^3. Obviously, a generalized crossing change of any order at a nugatory crossing of K yields a knot isotopic to K.

Definition 1.1. A (generalized) crossing change on K and its corresponding crossing circle are called *cosmetic* if the crossing change yields a knot isotopic to K and is performed at a crossing of K which is *not* nugatory.

The following question, often referred to as the nugatory crossing conjecture, is Problem 1.58 on Kirby's list [1].

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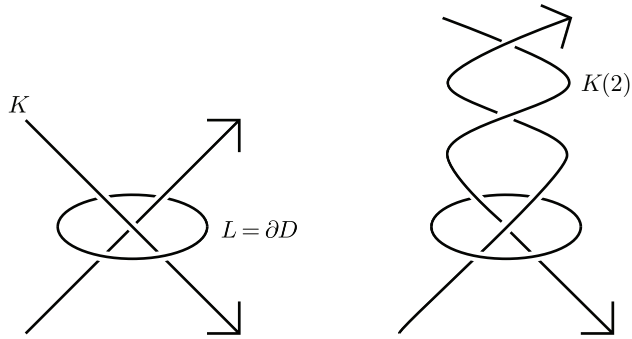

FIGURE 1. On the left is a crossing circle L bounding a crossing disk D. On the right is the knot resulting from an order-2 generalized crossing change at L.

Problem 1.2. Does there exist a knot K which admits a cosmetic crossing change? Conversely, if a crossing change on a knot K yields a knot isotopic to K, must the crossing be nugatory?

Note that a crossing change (in the traditional sense) is the same as an order-(± 1) generalized crossing change. Hence, one can ask the following stronger question concerning cosmetic generalized crossing changes.

Problem 1.3. Does there exist a knot K which admits a cosmetic generalized crossing change of any order?

Problem 1.2 was answered for the unknot by Scharlemann and Thompson when they showed that the unknot admits no cosmetic crossing changes in [13] using work of Gabai [4]. Progress was made towards answering Problem 1.2 for knots of braid index three by Wiley [17]. Obstructions to cosmetic crossing changes in genus-one knots were found by the author with Friedl, Kalfagianni and Powell in [2] using Seifert forms and the first homology of the 2-fold cyclic cover of S^3 over the knot. In particular, it was shown there that twisted Whitehead doubles of non-cable knots do not admit cosmetic crossing changes.

With regard to the stronger question posed in Problem 1.3, it has been shown by Kalfagianni that the answer to this question is no for fibered knots [7] and by Torisu that the answer is no for 2-bridge knots [16]. The answer is also shown to be no for closed 3-braids and certain Whitehead doubles by the author and Kalfagianni in [3]. Torisu reduces Problem 1.3 to the case where K is a prime knot in [16].

In this paper we will primarily be concerned with satellite knots, which are defined in the following way.

Definition 1.4. A knot K is a *satellite knot* if $M_K = \overline{S^3 - \eta(K)}$ contains a torus T which is incompressible and not boundary-parallel in M_K and such that K is contained in the solid torus V bounded by T in S^3. Such a torus T is called a *companion torus* for K. Further, there exists a homeomorphism $f : (V', K') \to (V, K)$ where V' is an unknotted solid torus in S^3 and K' is contained in $\text{int}(V')$. The knot K' is called a *pattern knot* for the satellite knot K. Finally, we require that K is neither the core of V nor contained in a 3-ball $B^3 \subset V$, and likewise for K' in V'.

We may similarly define a *satellite link*. Note that a patten knot for a given satellite knot may not be unique, but it is unique once we have specified the companion torus T and the map f. In general, a torus T in any orientable 3-manifold N is called *essential* if T is incompressible and not boundary-parallel in N.

The purpose of this paper is to prove the following.

Theorem 3.3. *Suppose K is a satellite knot which admits a cosmetic generalized crossing change of order q with $|q| \geq 6$. Then K admits a pattern knot K' which also has an order-q cosmetic generalized crossing change.*

This leads to the following two corollaries with regards to cosmetic generalized crossing changes.

Corollary 3.5. *Suppose K' is a torus knot. Then no prime satellite knot with pattern K' admits an order-q cosmetic generalized crossing change with $|q| \geq 6$.*

Corollary 3.7. *If there exists a knot admitting a cosmetic generalized crossing change of order q with $|q| \geq 6$, then there must be such a knot which is hyperbolic.*

Thus we have reduced Problem 1.3 to the cases where either the knot is hyperbolic or the crossing change has order q with $|q| < 6$.

2. Preliminaries

Given a 3-manifold N and submanifold $F \subset N$ of co-dimension 1 or 2, $\eta(F)$ will denote a closed regular neighborhood of F in N. If N is a 3-manifold containing a surface Σ, then by *N cut along Σ* we mean $\overline{N - \eta(\Sigma)}$. For a knot or link $\mathcal{L} \subset S^3$, we define $M_{\mathcal{L}} = \overline{S^3 - \eta(\mathcal{L})}$.

Given a knot K with a crossing circle L, let $K_L(q)$ denote the knot obtained via an order-q generalized crossing change at L. We may simply write $K(q)$ for $K_L(q)$ when there is no danger of confusion about the crossing circle in question.

Lemma 2.1. *Let K be an oriented knot with a crossing circle L. If L is not nugatory, then $M_{K \cup L}$ is irreducible.*

Proof. We will prove the contrapositive. Suppose $M_{K\cup L}$ is reducible. Then $M_{K\cup L}$ contains a separating 2-sphere S which does not bound a 3-ball $B \subset M_{K\cup L}$. So S must separate K and L in S^3, and consequently, $L \subset S^3$ lies in a 3-ball disjoint from K. Since L is unknotted, L bounds a disc in this 3-ball which is in the complement of K, and hence L is nugatory. □

Recall that a knot K is called *algebraically slice* if it admits a Seifert surface S such that the Seifert form $\theta : H_1(S) \times H_1(S) \to \mathbb{Z}$ vanishes on a half-dimensional summand of $H_1(S)$. In particular, if K is algebraically slice, then the Alexander polynomial $\Delta_K(t)$ is of the form $\Delta_K(t) \doteq f(t)f(t^{-1})$, where $f(t) \in \mathbb{Z}[t]$ is a linear polynomial and $\doteq$ denotes equality up to multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$. (See [9] for more details.) With this in mind, we have the following lemma from [2], which we will need in the proof of Corollary 3.4.

Lemma 2.2. *Suppose K is a genus-one knot which admits a cosmetic generalized crossing change of any order $q \in \mathbb{Z} - \{0\}$. Then K is algebraically slice.*

Proof. For $q = \pm 1$, this is Theorem 1.1(1) of [2]. The proof given there is easily adapted to generalized crossing changes of any order. □

Fix a knot K and let L be a crossing circle for K. Let $M(q)$ denote the 3-manifold obtained from $M_{K\cup L}$ via a Dehn filling of slope $(-1/q)$ along $\partial\eta(L)$. So, for $q \in \mathbb{Z} - \{0\}$, $M(q) = M_{K(q)}$, and $M(0) = M_K$. We will sometimes use $K(0)$ to denote $K \subset S^3$ when we want to be clear that we are considering $K \subset S^3$ rather than $K \subset M_L$.

Suppose there is some $q \in \mathbb{Z}$ for which $K_L(q)$ is a satellite knot. Then there is a companion torus T for $K(q)$ and, by definition, T must be essential in $M(q)$. This essential torus T must occur in one of the following two ways.

Definition 2.3. Let $T \subset M(q)$ be an essential torus. We say T is *Type 1* if T can be isotoped into $M_{K\cup L} \subset M(q)$. Otherwise, we say T is *Type 2.* If T is Type 2, then T is the image of a punctured torus $(P, \partial P) \subset (M_{K\cup L}, \partial\eta(L))$ and each component of ∂P has slope $(-1/q)$ on $\partial\eta(L)$.

In general, let $\mathcal{L}$ be any knot or link in S^3 and let Σ be a boundary component of $M_{\mathcal{L}}$. If $(P, \partial P) \subset (M_{\mathcal{L}}, \Sigma)$ is a punctured torus, then every component of ∂P has the same slope on Σ, which we call the *boundary slope* of P.

Suppose C_1 and C_2 are two non-separating simple closed curves (or boundary slopes) on a torus Σ. Let s_i be the slope of C_i on Σ, and let $[C_i]$ denote the isotopy class of C_i for $i = 1, 2$. Then $\Delta(s_1, s_2)$ is the minimal geometric intersection number of $[C_1]$ and $[C_2]$. It is known that if s_i is the rational slope $(1/q_i)$ for some $q_i \in \mathbb{Z}$ for $i = 1, 2$, then $\Delta(s_1, s_2) = |q_1 - q_2|$. (See [5] for more details.) Note that we consider $\infty = (1/0)$ to be a rational slope.

Gordon [5] proved the following theorem relating the boundary slopes of punctured tori in link complements. In fact, Gordon proved a more general result, but we state the theorem here only for the case which we will need later in Section 3.

Theorem 2.4 (Gordon, Theorem 1.1 of [5])**.** *Let $\mathcal{L}$ be a knot or link in S^3 and let Σ be a boundary component of $M_{\mathcal{L}}$. Suppose $(P_1, \partial P_1)$ and $(P_2, \partial P_2)$ are punctured tori in $(M_{\mathcal{L}}, \Sigma)$ such that the boundary slope of P_i on Σ is s_i for $i = 1, 2$. Then $\Delta(s_1, s_2) \leq 5$.*

Now suppose K is a knot contained in a solid torus $V \subset S^3$. An embedded disk $D \subset V$ is called a *meridian disk* of V if $\partial D = D \cap \partial V$ is a meridian of ∂V. We call K *geometrically essential* (or simply *essential*) in V if every meridian disk of V meets K at least once. With this in mind, we have the following lemma of Kalfagianni and Lin [8].

Lemma 2.5 (Lemma 4.6 of [8])**.** *Let $V \subset S^3$ be a knotted solid torus such that $K \subset \mathrm{int(V)}$ is a knot which is essential in V and K has a crossing disk D with $D \subset \mathrm{int(V)}$. If K is isotopic to $K(q)$ in S^3, then $K(q)$ is also essential in V. Further, if K is not the core of V, then $K(q)$ is also not the core of V.*

Proof. Suppose, by way of contradiction, that $K(q)$ is not essential in V. Then there is a 3-ball $B \subset V$ such that $K(q) \subset B$. This means that the winding number of $K(q)$ in V is 0.

Let S_1 be a Seifert surface for K which is of minimal genus in M_L, where $L = \partial D$. We may isotope S_1 so that $S_1 \cap D$ consists of a single curve α connecting the two points of $K \cap D$. Then twisting S_1 at L via a $(-1/q)$-Dehn filling on $\partial\eta(L)$ gives rise to a Seifert surface S_2 for $K(q)$. Since $M_{K\cup L}$ is irreducible by Lemma 2.1, we may apply Gabai's Corollary 2.4 of [4] to see that S_1 and S_2 are minimal-genus Seifert surfaces in S^3 for K and $K(q)$, respectively.

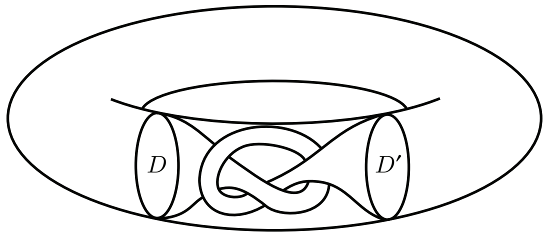

FIGURE 2. A knotted 3-ball B inside of a solid torus with disks $D, D' \subset \partial B$.

Since the winding number of $K(q)$ in V is 0, $S_1 \cap \partial V = S_2 \cap \partial V$ is homologically trivial in ∂V. For $i = 1, 2$, we can surger S_i along disks and annuli in ∂V which are bounded by curves in $S_i \cap \partial V$ to get new minimal genus Seifert surfaces $S_i' \subset \text{int}(V)$. Then S_2' is incompressible and V is irreducible, so we can isotope S_2' into $\text{int}(B)$. Hence α and therefore D can also be isotoped into $\text{int}(B)$. But then K must not be essential in V, which is a contradiction.

Finally, if K is not the core of V, then ∂V is a companion torus for the satellite knot K. Since a satellite knot cannot be isotopic to the core of its companion torus, $K(q)$ cannot be the core of V. □

Given a compact, irreducible, orientable 3-manifold N, let $\mathcal{T}$ be a collection of disjointly embedded, pairwise non-parallel, essential tori in N, which we will call an *essential torus collection* for N. By Haken's Finiteness Theorem (Lemma 13.2 of [6]) the number

$$\tau(N) = \max\{|\mathcal{T}| \mid \mathcal{T} \text{ is an essential torus collection for } N\}$$

is well-defined and finite, where $|\mathcal{T}|$ denotes the number of tori in $\mathcal{T}$. We will call such a collection $\mathcal{T}$ with $|\mathcal{T}| = \tau(N)$ a *Haken system* for N. Note that any essential torus $T \subset N$ is part of some Haken system $\mathcal{T}$.

Before moving on to the proofs of Theorem 3.3 and its corollaries, we state the following results of Motegi [11] (see also [14]) and McCullough [10], which we will need in the next section.

Lemma 2.6 (Motegi, Lemma 2.3 of [11]). *Let K be a knot embedded in S^3 and let V_1 and V_2 be knotted solid tori in S^3 such that the embedding of K is essential in V_i for $i = 1, 2$. Then there is an ambient isotopy $\phi : S^3 \to S^3$ leaving K fixed such that one of the following holds:*

(1) $\partial V_1 \cap \phi(\partial V_2) = \emptyset$.
(2) *There exist meridian disks D and D' for both V_1 and V_2 such that some component of V_1 cut along $(D \sqcup D')$ is a knotted* 3*-ball in some component of V_2 cut along $(D \sqcup D')$.*

By a *knotted 3-ball* we mean a ball B for which there is no isotopy which takes B to the standardly embedded 3-ball while leaving D and D' fixed. (See Figure 2.)

Before stating the result of McCullough, we recall the definition of a Dehn twist. Let C be a simple closed curve on a surface Σ and let $A \subset \Sigma$ be an open annular neighborhood of C. Then a *Dehn twist* of Σ at C is a diffeomorphism $\phi : \Sigma \to \Sigma$ such that ϕ is the identity map on $\Sigma - A$, and $\phi|_A$ is given by a full twist around

C. More specifically, if $A = S^1 \times (0,1)$ is given the coordinates $(e^{i\theta}, x)$, then $\phi|_A : (e^{i\theta}, x) \to (e^{i(\theta+2\pi x)}, x)$. The following result concerns diffeomorphisms of 3-manifolds that restrict to Dehn twists on the boundary of the manifold.

Theorem 2.7 (McCullough, Theorem 1 of [10]). *Let N be a compact, orientable 3-manifold that admits a homeomorphism which restricts to Dehn twists on the boundary of N along a simple closed curve in $C \subset \partial N$. Then C bounds a disk in N.*

3. Proofs of the main results

The goal of this section is to prove Theorem 3.3 and its corollaries. We begin with the following lemma.

Lemma 3.1 (Compare to Proposition 4.7 of [8]). *Let K be a prime satellite knot with a cosmetic crossing circle L of order q. Then at least one of the following must be true:*

(1) *$M(q)$ contains no Type 2 tori.*
(2) $|q| \leq 5$.

Proof. Suppose $M(q)$ contains a Type 2 torus. We claim that $M(0)$ must also contain a Type 2 torus. Assuming this is true, $M(0)$ and $M(q)$ each contain a Type 2 torus, and hence there are punctured tori $(P_0, \partial P_0)$ and $(P_q, \partial P_q)$ in $(M, \partial\eta(L))$ such that P_0 has boundary slope $\infty = (1/0)$ and P_q has boundary slope $(-1/q)$ on $\partial\eta(L)$. Then, by Theorem 2.4, $\Delta(\infty, -1/q) = |q| \leq 5$, as desired. Thus it remains to show that there is a Type 2 torus in $M(0)$.

Let $M = M_{K\cup L}$. Since L is not nugatory, Lemma 2.1 implies that M is irreducible and hence $\tau(M)$ is well-defined. First assume that $\tau(M) = 0$. Since K is a satellite knot, $M(0)$ must contain an essential torus, and it cannot be Type 1. Hence $M(0)$ contains a Type 2 torus.

Now suppose that $\tau(M) > 0$ and let T be an essential torus in M. Then T bounds a solid torus $V \subset S^3$. Let $\mathrm{ext}(V)$ denote $S^3 - V$. If $K \subset \mathrm{ext}(V)$, then L must be essential in V. If V were knotted, then either L is the core of V or L is a sattelite knot with companion torus V. This contradicts the fact that L is unknotted. Hence T is an unknotted torus. By definition, L bounds a crossing disk D. Since D meets K twice, $D \cap \mathrm{ext}(V) \neq \emptyset$. We may assume that D has been isotoped (rel boundary) to minimize the number of components in $D \cap T$. Since an innermost component of $D - (D \cap T)$ is a disk and L is essential in the unknotted solid torus V, $D \cap T$ consists of standard longitudes on the unknotted torus T. Hence $D \cap \mathrm{ext}(V)$ consists of either one disk which meets K twice or two disks each of which meets K once. In the first case, L is isotopic to the core of V, which contradicts T being essential in M. In the latter case, the linking number $lk(K, V) = \pm 1$. So K can be considered as the trivial connect sum $K\#U$, where U is the unknot and the crossing change at L takes place in the unknotted summand U. (See Figure 3.) The unknot does not admit cosmetic crossing changes of any order, so $K_L(q) = K\#K'$ where $K' \neq U$. This contradicts the fact that $K_L(q) = K$. Hence, we may assume that T is knotted and K is contained in the solid torus V bounded by T.

If $L \subset \mathrm{ext}(V)$ and cannot be isotoped into V, then $D \cap T$ has a component C that is both homotopically non-trivial and not boundary-parallel in $D - (K \cap D)$. So C must encircle exactly one of the two points of $K \cap D$. This means that the

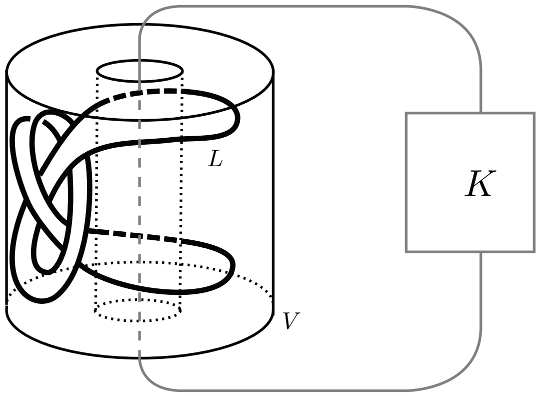

FIGURE 3. An example of an unknotted torus V containing a crossing circle L which bounds a crossing disk for the knot $K = K\#U$.

winding number of K in V is ± 1. Since T cannot be boundary parallel in M, K is not the core of T, and hence T is a "follow-swallow" torus for K and K is composite. But this contradicts the assumption that K is prime. Hence we may assume that L and D are contained in $\text{int}(V)$.

Since V is knotted and $D \subset \text{int}(V)$, Lemma 2.5 implies that T is also a companion torus for $K(q)$. This means every Type 1 torus in $M(0)$ is also a Type 1 torus in $M(q)$. Since $K(0)$ and $K(q)$ are isotopic, $\tau(M(0)) = \tau(M(q))$. By assumption, $M(q)$ contains a Type 2 torus, which must give rise to a Type 2 torus in $M(0)$, as desired. □

The following is an immediate corollary of Lemma 3.1.

Corollary 3.2. *Let K be a prime satellite knot with a cosmetic crossing circle L of order q with $|q| \geq 6$. Then $\tau(M_{K\cup L}) > 0$.*

We are now ready to prove our main theorem.

Theorem 3.3. *Suppose K is a satellite knot which admits a cosmetic generalized crossing change of order q with $|q| \geq 6$. Then K admits a pattern knot K' which also has an order-q cosmetic generalized crossing change.*

Proof. Let K be such a satellite knot with a crossing circle L bounding a crossing disk D which corresponds to a cosmetic generalized crossing change of order q. Let $M = M_{K\cup L}$.

If K is a composite knot, then Torisu [16] showed that the crossing change in question must occur within one of the summands of $K = K_1 \# K_2$, say K_1. We may assign to K the "follow-swallow" companion torus T, where the core of T is isotopic to K_2. Then the pattern knot corresponding to T is K_1 and the theorem holds.

Now assume K is prime. By Corollary 3.2, $\tau(M) > 0$. Let T be an essential torus in M and let $V \subset S^3$ be the solid torus bounded by T in S^3. As shown in the proof of Lemma 3.1, V is knotted in M and D can be isotoped to lie in $\text{int}(V)$. This means T is a companion torus for the satellite link $K \cup L$. Let $K' \cup L'$ be a pattern link for $K \cup L$ corresponding to T. So there is an unknotted solid torus $V' \subset S^3$ such that $(K' \cup L') \subset V'$, and there is a homeomorphism $f : (V', K', L') \to (V, K, L)$.

Let $\mathcal{T}$ be a Haken system for M such that $T \in \mathcal{T}$. We will call a torus $J \in \mathcal{T}$ *innermost with respect to* K if M cut along $\mathcal{T}$ has a component C such that ∂C contains $\partial\eta(K)$ and a copy of J. In other words, $J \in \mathcal{T}$ is innermost with respect to K if there are no other tori in $\mathcal{T}$ separating J from $\eta(K)$. Choose T to be innermost with respect to K.

Let $W = \overline{V - \eta(K \cup L)}$. We first wish to show that W is atoroidal. By way of contradiction, suppose that there is an essential torus $F \subset W$. Then F bounds a solid torus in V which we will denote by $\widehat{F}$. Since T is innermost with respect to K, either F is parallel to T in M or $K \subset V - \widehat{F}$. By assumption, F is essential in W and hence not parallel to $T \subset \partial W$. So $K \subset V - \widehat{F}$ and, since F is incompressible, $L \subset \widehat{F}$. But then F must be unknottted and parallel to $\partial\eta(L) \subset \partial W$, which is a contradiction. Hence W is indeed atoroidal, and $W' = \overline{V' - \eta(K' \cup L')}$ must be atoroidal as well.

To finish the proof, we must consider two cases depending on whether T is compressible in $V - \eta(K(q))$.

Case 1. $K(q)$ is essential in V.

We wish to show that there is an isotopy $\Phi : S^3 \to S^3$ such that $\Phi(K(q)) = K(0)$ and $\Phi(V) = V$. First, suppose $K(q)$ is the core of V. By Lemma 2.5, K is also the core of V. Since L is cosmetic, there is an ambient isotopy $\psi : S^3 \to S^3$ taking $K(q)$ to $K(0)$. Since $K(q)$ and $K(0)$ are both the core of V, we may choose ψ so that $\psi(V) = V$ and let $\Phi = \psi$.

If $K(q)$ is not the core of V, then T is a companion torus for $K(q)$. Since $K(0) = (K(q))_L(-q)$, we may apply Lemma 2.5 to $K(q)$ to see that T is also a companion torus for $K(0)$. Again, there is an ambient isotopy $\psi : S^3 \to S^3$ taking $K(q)$ to $K(0)$ such that V and $\psi(V)$ are both solid tori containing $K(0) = \psi(K(q)) \subset S^3$. If $\psi(V) = V$, we once more let $\Phi = \psi$. If $\psi(V) \neq V$, we may apply Lemma 2.6 to V and $\psi(V)$. If part (2) of Lemma 2.6 were satisfied, then $\psi(V) \cap V$ would give rise to a knotted 3-ball contained in either V or $\psi(V)$. This contradicts the fact that W, and hence $\psi(W)$, are atoroidal. Hence part (1) of Lemma 2.6 holds, and there is an isotopy $\phi : S^3 \to S^3$ fixing $K(0)$ such that $(\phi \circ \psi)(T) \cap T = \emptyset$. Let $\Phi = (\phi \circ \psi) : S^3 \to S^3$. Recall that by Lemma 3.1, $M(q)$ contains no Type 2 tori. Hence T remains innermost with respect to $K(q)$ in S^3, and therefore $\Phi(T)$ is also innermost with respect to $K(0)$. The fact that T and $\Phi(T)$ are disjoint and both innermost with respect to K implies that T and $\Phi(T)$ are in fact parallel in M_K. So, after an isotopy which fixes $K(0) \subset S^3$, we may assume that $\Phi(V) = V$.

Now let $h = (f^{-1} \circ \Phi \circ f) : V' \to V'$. Note that h preserves the fixed longitudes of $\partial V'$ (up to sign). Since h maps $K'(q)$ to $K'(0)$, $K'(q)$ and $K'(0)$ are isotopic in S^3. So either L' gives an order-q cosmetic generalized crossing change for the pattern knot K' or L' is a nugatory crossing circle for K'.

Suppose L' is nugatory. Then L' bounds a crossing disk D' and another disk $D'' \subset M_{K'}$. We may assume $D' \cap D'' = L'$. Let $A = D' \cup (D'' \cap V')$. Since $\partial V'$ is incompressible in $V' - \eta(K')$, by surgering along components of $D'' \cap \partial V'$ which bound disks or co-bound annuli in $\partial V'$, we may assume A is a properly embedded annulus in V' and each component of $A \cap \partial V'$ is a longitude of V'. Since $L' \subset V'$, we can extend the homeomorphism h on $V' \subset S^3$ to a homeomorphism H on all of S^3. Since V' is unknotted, let C be the core of the solid torus $S^3 - \text{int}(V')$. We may assume that H fixes C. Since $D' \cup D''$ gives the same (trivial) connect sum decomposition of $K'(0) = K'(q)$ and H preserves canonical longitudes on $\partial V'$, we may assume $H(D')$ is isotopic to D' and $H(D'')$ is isotopic to D''. In fact, this isotopy may be chosen so that $H(C)$ and $H(V')$ remain disjoint throughout the isotopy and $H(V') = V'$ still holds after the isotopy. Thus, we may assume $h(A) = A$ and A cuts V' into two solid tori V_1' and V_2', as shown in Figure 4.

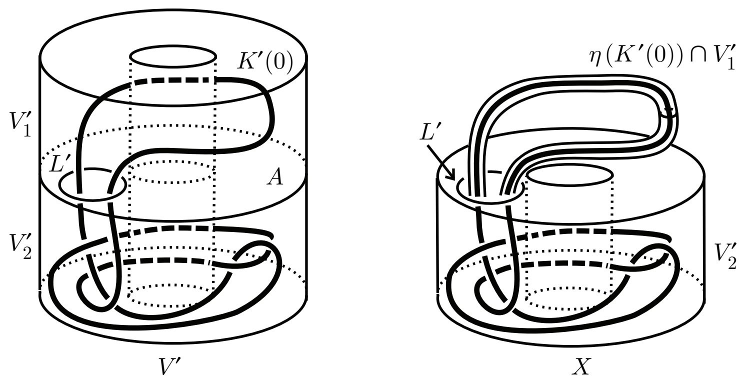

FIGURE 4. On the left is the solid torus V', cut into two solid tori by the annulus A. On the right is a diagram depicting the construction of X from Subcase 1.1 of the proof of Theorem 3.3.

We now consider two subcases, depending on how h acts on V_1' and V_2'.

Subcase 1.1. $h : V_i' \to V_i'$ for $i = 1, 2$.

Up to ambient isotopy, we may assume the following:

(1) $K'(q) \cap V_1' = K'(0) \cap V_1'$.
(2) $K'(q) \cap V_2'$ is obtained from $K'(0) \cap V_2'$ via q full twists at L'.

Let X be the 3-manifold obtained from $V_2' - \eta(V_2' \cap K'(0))$ by attaching to $A \subset \partial V_2'$ a thickened neighborhood of $\partial\eta(K'(0)) \cap V_1'$. (See Figure 4.) Then $h|_X$ fixes X away from V_2' and acts on $X \cap V_2'$ by twisting $\partial\eta(K'(0)) \subset \partial X$ q times at L'. Hence there is a homeomorphism from X to $h(X)$ given by q Dehn twists at $L' \subset \partial X$. So, by Theorem 2.7, L' bounds a disk in $X \subset (V' - \eta(K'))$. But this means L bounds a disk in $(V - \eta(K)) \subset M_K$ and hence L is nugatory, contradicting our initial assumptions.

Subcase 1.2. h maps $V_1' \to V_2'$ and $V_2' \to V_1'$.

Again, we may assume the following:

(1) $K'(q) \cap V_1' = K'(0) \cap V_2'$.
(2) $K'(q) \cap V_2'$ is obtained from $K'(0) \cap V_1'$ via q full twists at L'.

This time we construct X from $V_1' - \eta(V_1' \cap K'(0))$ by attaching a thickened neighborhood of $\partial\eta(K'(0)) \cap V_2'$ to $A \subset \partial V_1'$. Then the argument of Subcase 1.1 once again shows that L must have been nugatory, giving a contradiction.

Hence, in Case 1, we have a pattern knot K' for K admitting an order-q cosmetic generalized crossing change, as desired.

Case 2. T is compressible in $V - \eta(K(q))$.

In this case, $K(q)$ is contained in a 3-ball $B \subset V$. Since $K(q)$ is not essential in V, by Lemma 2.5, $K(0)$ is also not essential in V and $K(0) = f(K'(0))$ can be isotoped to $K(q) = f(K'(q))$ via an isotopy contained in the 3-ball $B \subset V$. This means that, once again, $K'(0)$ is isotopic to $K'(q)$ in S^3, and either L' gives an order-q cosmetic generalized crossing change for the pattern knot K' or L' is a nugatory crossing circle for K'. Applying the arguments of each of the subcases above, we see that L' cannot be nugatory, and hence K' is a pattern knot for K admitting an order-q cosmetic generalized crossing change. □

In [15], Thurston shows that any knot falls into exactly one of three categories: torus knots, hyperbolic knots and satellite knots. Theorem 3.3 gives obstructions as to when cosmetic generalized crossing changes can occur in satellite knots. This leads us to several useful corollaries.

Corollary 3.4. *Let K be a satellite knot admitting a cosmetic generalized crossing change of order q with $|q| \geq 6$. Then K admits a pattern knot K' which is hyperbolic.*

Proof. Applying Theorem 3.3, repeatedly if necessary, we know K admits a pattern knot K' which is not a satellite knot and which also admits an order-q cosmetic generalized crossing change. Kalfagianni has shown that fibered knots do not admit cosmetic generalized crossing changes of any order [7], and it is well-known that all torus knots are fibered. Hence, by Thurston's classification of knots [15], K' must be hyperbolic. □

Corollary 3.5. *Suppose K' is a torus knot. Then no prime satellite knot with pattern K' admits an order-q cosmetic generalized crossing change with $|q| \geq 6$.*

Proof. Let K be a prime satellite knot which admits a cosmetic generalized crossing change of order q with $|q| \geq 6$. By way of contradiction, suppose T is a companion torus for K corresponding to a pattern torus knot K'. Since K is prime, Lemma 3.1 implies that T is Type 1 and hence corresponds to a torus in $M = M_{K \cup L}$, which we will also denote by T. If T is not essential in M, then T must to be parallel in M to $\partial\eta(L)$. But then T would be compressible in M_K, which cannot happen since T is a companion torus for $K \subset S^3$ and is thus essential in M_K. So T is essential in M, and there is a Haken system $\mathcal{T}$ for M with $T \in \mathcal{T}$. Since the pattern knot K' is a torus knot and hence not a satellite knot, T must be innermost with respect to K. Then the arguments in the proof of Theorem 3.3 show that K' admits an order-q cosmetic generalized crossing change. However, torus knots are fibered and hence admit no cosmetic generalized crossing changes of any order, giving us our desired contradiction. □

Note that if K' is a torus knot which lies on the surface of the unknotted solid torus V', then (K', V') is a pattern for satellite knot which is by definition a cable knot. Since any cable of a fibered knot is fibered, it was already known by [7] that these knots do not admit cosmetic generalized crossing changes. However, Corollary 3.5 applies not only to cables of non-fibered knots but also to pattern torus knots embedded in *any* unknotted solid torus V' and hence gives us a new class of knots which do not admit a cosmetic generalized crossing change of order q with $|q| \geq 6$.

The proof of Corollary 3.5 leads us to the following.

Corollary 3.6. *Let K' be a knot such that $g(K') = 1$ and K' is not a satellite knot. If there is a prime satellite knot K such that K' is a pattern knot for K and K admits a cosmetic generalized crossing change of order q with $|q| \geq 6$, then K' is hyperbolic and algebraically slice.*

Proof. By Corollary 3.5 and its proof, K' is hyperbolic and admits a cosmetic generalized crossing change of order q. Then by Lemma 2.2, K' is algebraically slice. □

Finally, the following corollary summarizes the progress we have made in this paper towards answering Problem 1.3.

Corollary 3.7. *If there exists a knot admitting a cosmetic generalized crossing change of order q with $|q| \geq 6$, then there must be such a knot which is hyperbolic.*

Proof. Suppose there is a knot K with a cosmetic generalized crossing change of order q with $|q| \geq 6$. Since K cannot be a fibered knot, K is not torus knot and either K itself is hyperbolic or K is a satellite knot. If K is a satellite knot, then by Corollary 3.4 and its proof, K admits a pattern knot K' which is hyperbolic and has an order-q cosmetic generalized crossing change. □

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References

[1] *Problems in low-dimensional topology,* edited by Rob Kirby, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 35–473. MR1470751

[2] Cheryl Balm, Stefan Friedl, Efstratia Kalfagianni, and Mark Powell, *Cosmetic crossings and Seifert matrices*, Comm. Anal. Geom. **20** (2012), no. 2, 235–253. MR2928712

[3] Cheryl Balm and Efstratia Kalfagianni, *Cosmetic crossings of twisted knots*, preprint, arXiv:1301.6369 [math.GT].

[4] David Gabai, *Foliations and the topology of* 3*-manifolds. II*, J. Differential Geom. **26** (1987), no. 3, 461–478. MR910017 (89a:57014a)

[5] C. McA. Gordon, *Boundary slopes of punctured tori in* 3*-manifolds*, Trans. Amer. Math. Soc. **350** (1998), no. 5, 1713–1790, DOI 10.1090/S0002-9947-98-01763-2. MR1390037 (98h:57032)

[6] John Hempel, *3-manifolds*, AMS Chelsea Publishing, Providence, RI, 2004. Reprint of the 1976 original. MR2098385 (2005e:57053)

[7] Efstratia Kalfagianni, *Cosmetic crossing changes of fibered knots*, J. Reine Angew. Math. **669** (2012), 151–164. MR2980586

[8] Efstratia Kalfagianni and Xiao-Song Lin, *Knot adjacency, genus and essential tori*, Pacific J. Math. **228** (2006), no. 2, 251–275, DOI 10.2140/pjm.2006.228.251. MR2274520 (2007k:57010)

[9] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR1472978 (98f:57015)

[10] Darryl McCullough, *Homeomorphisms which are Dehn twists on the boundary*, Algebr. Geom. Topol. **6** (2006), 1331–1340 (electronic), DOI 10.2140/agt.2006.6.1331. MR2253449 (2007f:57043)

[11] Kimihiko Motegi, *Knotting trivial knots and resulting knot types*, Pacific J. Math. **161** (1993), no. 2, 371–383. MR1242205 (94h:57012)

[12] V. V. Prasolov and A. B. Sossinsky, *Knots, links, braids and 3-manifolds*, An introduction to the new invariants in low-dimensional topology. Translated from the Russian manuscript by Sossinsky [Sosinskiĭ], Translations of Mathematical Monographs, vol. 154, American Mathematical Society, Providence, RI, 1997. MR1414898 (98i:57018)

[13] Martin Scharlemann and Abigail Thompson, *Link genus and the Conway moves*, Comment. Math. Helv. **64** (1989), no. 4, 527–535, DOI 10.1007/BF02564693. MR1022995 (91b:57006)
[14] Horst Schubert, *Knoten und Vollringe* (German), Acta Math. **90** (1953), 131–286. MR0072482 (17,291d)
[15] William P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 357–381, DOI 10.1090/S0273-0979-1982-15003-0. MR648524 (83h:57019)
[16] Ichiro Torisu, *On nugatory crossings for knots*, Topology Appl. **92** (1999), no. 2, 119–129, DOI 10.1016/S0166-8641(97)00238-1. MR1669827 (2000b:57015)
[17] Chad Wiley, *Nugatory crossings in closed 3-braid diagrams*, Thesis (Ph.D.)–University of California, Santa Barbara, 2008, ProQuest LLC, Ann Arbor, MI. MR2712035

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