

GROUP-EXTENDED MARKOV SYSTEMS, AMENABILITY, AND THE PERRON-FROBENIUS OPERATOR

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ABSTRACT. We characterise amenability of a countable group in terms of the spectral radius of the Perron-Frobenius operator associated to a group extension of a countable Markov shift and a Hölder continuous potential. This extends a result of Day for random walks and recent work of Stadlbauer for dynamical systems. Moreover, we show that if the potential satisfies a symmetry condition with respect to the group extension, then the logarithm of the spectral radius of the Perron-Frobenius operator is given by the Gurevič pressure of the potential.

1. INTRODUCTION AND STATEMENT OF RESULTS

Kesten ([Kes59a]) characterised amenability of a countable group G by the growth of the return probability of a symmetric random walk. Day ([Day64]) gave a criterion for amenability in terms of the spectrum of a convolution operator acting on the Banach space $\ell^p(G)$, $p \in \mathbb{N}$, without assuming the random walk to be symmetric. Recently, the relationship between amenability and dynamical systems was studied in the framework of the thermodynamic formalism ([Sha07, Sta13, Jae14, Jael4b]). Stadlbauer ([Sta13]) used the Gurevič pressure of a Hölder continuous potential ([Sar99]) to give an extension of Kesten's criterion for amenability to group extensions of Markov shifts. To characterise amenability via the Gurevič pressure, it is necessary to impose certain symmetry assumptions on the potential and on the Markov shift.

In this paper, we characterise amenability in terms of the spectral radius of the Perron-Frobenius operator associated to a group extension of a countable Markov shift and a Hölder continuous potential, where neither the group extension nor the potential is assumed to be symmetric. Another main result is to relate the spectral radius of the Perron-Frobenius operator to the Gurevič pressure.

Throughout this paper, let Σ denote a Markov shift with countable alphabet I and left shift map $\sigma : \Sigma \rightarrow \Sigma$ (see Section 2). For a countable group G and a semigroup homomorphism $\Psi : I^* \rightarrow G$, where I^* denotes the free semigroup generated by I , the *group-extended Markov system* $(\Sigma \times G, \sigma \times \Psi)$ is given by

$$\sigma \times \Psi : \Sigma \times G \rightarrow \Sigma \times G, (\sigma \times \Psi)(\omega, g) := (\sigma(\omega), g\Psi(\omega_1)), (\omega, g) \in \Sigma \times G.$$

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Let $\pi_1 : \Sigma \times G \rightarrow \Sigma$ denote the canonical projection. If Σ is topologically mixing and satisfies the big images and preimages (b.i.p.) property (see Definition 2.1) and if $\varphi : \Sigma \rightarrow \mathbb{R}$ is Hölder continuous with finite Gurevič pressure $\mathcal{P}(\varphi, \sigma)$ (see Definition 2.2), then the Perron-Frobenius operator $\mathcal{L}_{\varphi \circ \pi_1}$, given by $\mathcal{L}_{\varphi \circ \pi_1}(f)(x) := \sum_{(\sigma \rtimes \Psi)(y)=x} e^{\varphi \circ \pi_1(y)} f(y)$, acts as a bounded linear operator on a certain Banach space $(\mathcal{H}_\infty, |\cdot|_\infty)$ of functions $f : \Sigma \times G \rightarrow \mathbb{R}$ ([Sta13]). We refer to Section 3 for the definition of $(\mathcal{H}_\infty, |\cdot|_\infty)$ and denote by $\rho(\mathcal{L}_{\varphi \circ \pi_1})$ the spectral radius of $\mathcal{L}_{\varphi \circ \pi_1} : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$.

Our main result is the following characterisation of amenability for group-extended Markov systems.

Theorem 1.1. *Let Σ be a topologically mixing Markov shift with the b.i.p. property and let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system. Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be Hölder continuous with $\mathcal{P}(\varphi, \sigma) < \infty$. Then we have $\log \rho(\mathcal{L}_{\varphi \circ \pi_1}) = \mathcal{P}(\varphi, \sigma)$ if and only if G is amenable. In general, we have $\log \rho(\mathcal{L}_{\varphi \circ \pi_1}) \leq \mathcal{P}(\varphi, \sigma)$.*

Theorem 1.1 provides an extension of a result of Day for random walks on groups ([Day64, Theorem 1]). The Perron-Frobenius operator corresponds to the convolution operator acting on $\ell^2(G)$ in Day’s setting. Theorem 1.1 also generalises [Jae14, Theorem 3.21], where locally constant potentials were considered (see Remark 1.8 for details).

Stadlbauer ([Sta13]) gave a criterion for amenability in terms of the Gurevič pressure, which is more in the spirit of Kesten’s characterisation of amenability in terms of the growth rate of return probabilities (see [Kes59b, Kes59a]). More precisely, Stadlbauer proved the following two implications. Here, $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ refers to the Gurevič pressure of $\varphi \circ \pi_1$ with respect to $\sigma \rtimes \Psi$.

- (1) Under the assumptions of Theorem 1.1, if $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$, then G is amenable ([Sta13, Theorem 5.4]).
- (2) Let Σ be topologically mixing and let $(\Sigma \times G, \sigma \rtimes \Psi)$ be a symmetric group extension. If G is amenable and if φ is weakly symmetric with $\mathcal{P}(\varphi, \sigma) < \infty$, then $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$ ([Sta13, Theorem 4.1]).

Comparing Stadlbauer’s results with Theorem 1.1, we note that the spectral radius of the Perron-Frobenius operator can be used to characterise amenability for an arbitrary Hölder continuous potential. In contrast to this, the criterion in terms of the Gurevič pressure involves a certain symmetry condition on the Markov shift and the potential under consideration (see [Sta13, p. 455] for the definition of a symmetric group extension and a weakly symmetric potential). For an amenable group, it may happen that $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) < \mathcal{P}(\varphi, \sigma)$, if φ is not weakly symmetric. In fact, the gap between these pressures can be arbitrarily large, as the following example illustrates.

Example 1.2. Let $I := \{\pm 1\}$ and consider the full shift $\Sigma := I^{\mathbb{N}}$ and the group-extended Markov system given by the canonical semigroup homomorphism $\Psi : I^* \rightarrow (\mathbb{Z}, +)$. For a real number $\lambda > 0$, let $\varphi_\lambda : \Sigma \rightarrow \mathbb{R}$ be given by $\varphi_\lambda(x) := \lambda x_1$. Then we have $\mathcal{P}(\varphi_\lambda \circ \pi_1, \sigma \rtimes \Psi) = \log(2) < \log(e^\lambda + e^{-\lambda}) = \mathcal{P}(\varphi_\lambda, \sigma)$.

For a Markov shift with a finite alphabet, it is well known that the Gurevič pressure of a Hölder continuous potential coincides with the logarithm of the spectral radius of the Perron-Frobenius operator. However, if the Markov shift is constructed over an infinite alphabet, then the Gurevič pressure is less than or equal

to the logarithm of the spectral radius. The Gurevič pressure describes the growth of iterates of the Perron-Frobenius operator on functions supported on a cylindrical set, whereas for the spectral radius, we have to consider functions supported on the whole space (see Lemma 3.2). To relate the spectral radius of the Perron-Frobenius operator to the Gurevič pressure, we introduce the following notions of symmetry, generalising those given in [Jae14, Definition 3.10].

Definition 1.3. Let $(\Sigma \times G, \sigma \rtimes \Psi)$ be a group-extended Markov system and $\varphi : \Sigma \rightarrow \mathbb{R}$. Let $\alpha \geq 1$. We say that φ is *asymptotically α -symmetric with respect to Ψ* , if there exist $n_0 \in \mathbb{N}$ and sequences $(c_n) \in (\mathbb{R}^+)^{\mathbb{N}}$ and $(N_n) \in \mathbb{N}^{\mathbb{N}}$ with the property that $\lim_n (c_n)^{1/(2n)} = \alpha$, $\lim_n n^{-1}N_n = 0$ and such that, for each $g \in G$ and for all $n \geq n_0$, we have

$$(1.1) \quad \sum_{\omega \in \Sigma^n : \Psi(\omega) = g} e^{\sup S_n \varphi|_{[\omega]}} \leq c_n \sum_{\omega \in \Sigma^* : \Psi(\omega) = g^{-1}, n - N_n \leq |\omega| \leq n + N_n} e^{\sup S_{|\omega|} \varphi|_{[\omega]}}.$$

We say that φ is *compactly asymptotically α -symmetric with respect to Ψ* if there exists a sequence $(\Sigma_k)_{k \in \mathbb{N}}$ of topologically mixing subshifts of Σ with finite alphabet $I_k \subset I$, such that $\bigcup_{k \in \mathbb{N}} I_k = I$, the set $\Psi(\Sigma_k^*)$ is a subgroup of G , and $\varphi|_{\Sigma_k}$ is asymptotically α -symmetric with respect to $\Psi|_{I_k^*}$. If α is not specified, then we will tacitly assume that $\alpha = 1$.

Remark 1.4. If φ is (compactly) asymptotically α -symmetric with respect to Ψ , then so is $\varphi + \log h - \log h \circ \sigma + P$, for each $P \in \mathbb{R}$ and for each function $h : \Sigma \rightarrow \mathbb{R}^+$, which is bounded away from zero and infinity.

The following proposition generalises [Jae14, Corollary 3.17], where a locally constant potential φ on a finite state Markov shift was considered (see Remark 1.8 for details). The proposition relates Theorem 1.1 to Stadlbauer’s results and is also of independent interest, since it relates the Gurevič pressure of a not necessarily recurrent potential ([Sar01]) on $\Sigma \times G$ to the spectrum of the Perron-Frobenius operator.

Proposition 1.5. *Under the assumptions of Theorem 1.1, we have $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) \leq \log \rho(\mathcal{L}_{\varphi \circ \pi_1})$. If additionally φ is asymptotically α -symmetric with respect to Ψ , for some $\alpha \geq 1$, then*

$$\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) \geq \log \rho(\mathcal{L}_{\varphi \circ \pi_1}) - \log \alpha.$$

By combining Theorem 1.1 and the first assertion in Proposition 1.5, we obtain Stadlbauer’s result (1). Theorem 1.1 and the second assertion in Proposition 1.5 give the following implication, which is similar to Stadlbauer’s result (2) and [Jae11, Theorem 5.3.11].

Corollary 1.6. *Under the assumptions of Theorem 1.1, if G is amenable and if φ is asymptotically α -symmetric with respect to Ψ , for some $\alpha \geq 1$, then $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) \geq \mathcal{P}(\varphi, \sigma) - \log \alpha$.*

If φ is not assumed to be Hölder continuous, or if Σ does not satisfy the b.i.p. property as in (2), then a Gibbs measure for φ does not exist. To obtain an inequality as in the previous corollary, the existence of an approximating sequence of Hölder continuous potentials is sufficient. More precisely, suppose there exist a sequence of subgroups (G_j) of G , group-extended Markov systems $(\Sigma_j \times G_j, \sigma_j \rtimes \Psi_j)$ and potentials $\varphi_j : \Sigma_j \rightarrow \mathbb{R}$, such that φ_j is asymptotically α_j -symmetric with

respect to Ψ_j and the assumptions of Theorem 1.1 hold for each $j \in \mathbb{N}$. If moreover $\lim_j \mathcal{P}(\varphi_j, \sigma_j) \geq \mathcal{P}(\varphi, \sigma)$, $\lim_j \mathcal{P}(\varphi_j \circ \pi_1, \sigma_j \rtimes \Psi_j) \leq \mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ and $\lim_j \alpha_j \leq \alpha$, then we have $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) \geq \mathcal{P}(\varphi, \sigma) - \log \alpha$, provided that G is amenable. Using this approach, Stadlbauer derived the assertion in (2).

We make use of this approach to give a similar result for compactly asymptotically α -symmetric potentials of medium variation. Note that $\mathcal{P}(\varphi, \sigma)$ is allowed to be infinite in the following corollary.

Corollary 1.7. *Let Σ be a topologically mixing Markov shift and let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system. Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be of medium variation and suppose that φ is compactly asymptotically α -symmetric, for some $\alpha \geq 1$. If G is amenable, then $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) \geq \mathcal{P}(\varphi, \sigma) - \log \alpha$.*

The main ingredients in the proof of Theorem 1.1 are a recent result of Stadlbauer ([Sta13]) and a result of Day ([Day64, Theorem 1]). Proposition 1.5 extends [Jae14, Corollary 3.17] and goes back to work of Pier ([Pie84, pp. 196-202]), which was used by Gerl ([Ger88, p. 177]).

Remark 1.8. In [Jae14], the special case of a finite state Markov shift Σ and a locally constant potential $\varphi : \Sigma \rightarrow \mathbb{R}$ was considered. We have investigated the action of the Perron-Frobenius operator $\mathcal{L}_{\varphi \circ \pi_1}$ on the Banach space $V_k \subset L^2(\Sigma \times G, \mu_\varphi \times \lambda)$, for some $k \in \mathbb{N}$, where V_k consists of functions $f : \Sigma \times G \rightarrow \mathbb{R}$ which are constant on the cylindrical sets $[\omega] \times \{g\}$, $\omega \in \Sigma^k$, $g \in G$, and λ denotes the Haar measure on G . Note that $L^2(\Sigma \times G, \mu_\varphi \times \lambda)$ coincides with $(\mathcal{H}_2, |\cdot|_2)$ defined in Section 3. Moreover, since $\min \left\{ \sqrt{\mu_\varphi([\omega])} : \omega \in \Sigma^k \right\} |f|_\infty \leq |f|_2 \leq |f|_\infty$, for each $f \in V_k$, we have that $(V_k, |\cdot|_2)$ is canonically embedded in $(\mathcal{H}_\infty, |\cdot|_\infty)$. Furthermore, it follows from (3.2) in Lemma 3.2 that the norm of $\mathcal{L}_{\varphi \circ \pi_1}^n : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ is attained on V_k , for each $n \in \mathbb{N}$. Hence, the spectral radii of $\mathcal{L}_{\varphi \circ \pi_1}$ and $\mathcal{L}_{\varphi \circ \pi_1}|_{V_k}$ coincide and the results given in [Jae14, Theorem 3.21] follow from Proposition 1.5 and Theorem 1.1. Finally, we remark that, for a Hölder continuous potential φ , the logarithm of the spectral radius of $\mathcal{L}_{\varphi \circ \pi_1}$ acting on $L^2(\Sigma \times G, \mu_\varphi \times \lambda)$ is always equal to $\mathcal{P}(\varphi, \sigma)$ ([Jae14, Theorem 3.21]), in contrast to $\mathcal{L}_{\varphi \circ \pi_1} : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ considered in this paper.

2. PRELIMINARIES

Throughout, the state space for the symbolic thermodynamic formalism will be a Markov shift

$$\Sigma := \{ \omega := (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}} : a(\omega_i, \omega_{i+1}) = 1 \text{ for all } i \in \mathbb{N} \}$$

with finite or countable infinite alphabet $I \subset \mathbb{N}$, incidence matrix $A = (a(i, j)) \in \{0, 1\}^{I \times I}$ and left shift map $\sigma : \Sigma \rightarrow \Sigma$, given by $\sigma((\omega_1, \omega_2, \dots)) := (\omega_2, \omega_3, \dots)$. The set of A -admissible words of length $n \in \mathbb{N}$ is denoted by

$$\Sigma^n := \{ \omega \in I^n : a(\omega_i, \omega_{i+1}) = 1, \text{ for } 1 \leq i \leq n-1 \},$$

and the set of A -admissible words of arbitrary length by $\Sigma^* := \bigcup_{n \in \mathbb{N}} \Sigma^n$. Let us also define the word length function $|\cdot| : \Sigma^* \cup \Sigma \rightarrow \mathbb{N} \cup \{\infty\}$, where for $\omega \in \Sigma^*$ we set $|\omega|$ to be the unique $n \in \mathbb{N}$ such that $\omega \in \Sigma^n$, and for $\omega \in \Sigma$ we set $|\omega| := \infty$. For each $\omega \in \Sigma^* \cup \Sigma$ and $n \in \mathbb{N}$ with $n \leq |\omega|$, we define $\omega|_n := (\omega_1, \dots, \omega_n)$. For $\omega, \tau \in \Sigma$, we set $\omega \wedge \tau$ to be the longest common initial block of ω and τ , that is, $\omega \wedge \tau := \omega|_l$, where $l := \sup \{ n \in \mathbb{N} : \omega|_n = \tau|_n \}$. For $n \in \mathbb{N}$ and $\omega \in \Sigma^n$, we let $[\omega] := \{ \tau \in \Sigma : \tau|_n = \omega \}$ denote the cylindrical set given by ω .

If Σ is the Markov shift with alphabet I whose incidence matrix consists entirely of 1s, then we have that $\Sigma = I^{\mathbb{N}}$ and $\Sigma^n = I^n$ for all $n \in \mathbb{N}$. Then we set $I^* := \Sigma^*$. For $\omega, \tau \in I^*$ we denote by $\omega\tau \in I^*$ the *concatenation* of ω and τ , which is defined by $\omega\tau := (\omega_1, \dots, \omega_{|\omega|}, \tau_1, \dots, \tau_{|\tau|})$ for $\omega, \tau \in I^*$. Note that I^* forms a semigroup with respect to the concatenation operation. The semigroup I^* is the free semigroup over the set I and satisfies the following universal property: For each semigroup S and for every map $u : I \rightarrow S$, there exists a unique semigroup homomorphism $\widehat{u} : I^* \rightarrow S$ such that $\widehat{u}(i) = u(i)$, for all $i \in I$.

We equip $I^{\mathbb{N}}$ with the product topology of the discrete topology on I . The Markov shift $\Sigma \subset I^{\mathbb{N}}$ is equipped with the subspace topology. A countable basis of this topology on Σ is given by the cylindrical sets $\{[\omega] : \omega \in \Sigma^*\}$. We will make use of the following metric generating the topology on Σ . For $\beta > 0$, we define the metric d_β on Σ given by

$$d_\beta(\omega, \tau) := e^{-\beta|\omega \wedge \tau|}, \text{ for all } \omega, \tau \in \Sigma.$$

A function $\varphi : \Sigma \rightarrow \mathbb{R}$ is also called a *potential*. For $n \in \mathbb{N}$, we use $S_n\varphi : \Sigma \rightarrow \mathbb{R}$ to denote the *ergodic sum* of φ with respect to σ . In other words, $S_n\varphi := \sum_{i=0}^{n-1} \varphi \circ \sigma^i$. Also, we set $S_0\varphi := 0$. The function φ is called β -Hölder continuous, for some $\beta > 0$, if

$$V_\beta(\varphi) := \sup_{n \geq 1} \{V_{\beta,n}(\varphi)\} < \infty,$$

where

$$V_{\beta,n}(\varphi) := \sup \left\{ \frac{|\varphi(\omega) - \varphi(\tau)|}{d_\beta(\omega, \tau)} : \omega, \tau \in \Sigma, |\omega \wedge \tau| \geq n \right\}, \quad n \in \mathbb{N}.$$

We say that φ is *Hölder continuous* if there exists $\beta > 0$ such that φ is β -Hölder continuous. The function φ is of *medium variation*, if φ is continuous and if there exists a sequence $(D_n) \in \mathbb{R}^{\mathbb{N}}$ with $\lim_n (D_n)^{1/n} = 1$ such that, for all $n \in \mathbb{N}$, $\omega \in \Sigma^n$ and $x, y \in [\omega]$, we have $e^{S_n\varphi(x) - S_n\varphi(y)} \leq D_n$.

We need the following topological mixing properties for Markov shifts.

Definition 2.1. Let Σ be a Markov shift with alphabet $I \subset \mathbb{N}$.

- Σ is *irreducible* if, for all $i, j \in I$, there exists $\omega \in \Sigma^*$ such that $i\omega j \in \Sigma^*$.
- Σ is *topologically mixing* if, for all $i, j \in I$, there exists $n_0 \in \mathbb{N}$ with the property that, for all $n \geq n_0$, there exists $\omega \in \Sigma^n$ such that $i\omega j \in \Sigma^*$.
- Σ satisfies the *big images and preimages (b.i.p.) property* ([Sar03]) if there exists a finite set $\Lambda \subset I$ such that, for all $i \in I$, there exist $a, b \in \Lambda$ such that $aib \in \Sigma^3$.

The Gurevič pressure of a Hölder continuous potential on a topologically mixing Markov shift was introduced by Sarig ([Sar99, Definition 1]). One can easily verify that the Gurevič pressure is well defined also for a potential of medium variation on an irreducible Markov shift (cf. [Sta13, p. 453]). The Gurevič pressure extends the notion of the Gurevič entropy ([Gur69, Gur70]), which corresponds to the constant zero potential.

Definition 2.2. Let Σ be an irreducible Markov shift with alphabet I and left shift $\sigma : \Sigma \rightarrow \Sigma$. Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be of medium variation. For each $a \in I$ and $n \in \mathbb{N}$, we set

$$Z_n(\varphi, a, \sigma) := \sum_{\omega \in \Sigma^n, \omega_1 = a, \omega a \in \Sigma^*} e^{\sup S_n\varphi|_{[\omega]}}$$

and

$$Z_n^*(\varphi, a, \sigma) := \sum_{\omega \in \Sigma^n, \omega_1=a, \omega a \in \Sigma^*, \omega_2 \neq a, \dots, \omega_n \neq a} e^{\sup S_n \varphi|_{[\omega]}}.$$

The Gurevič pressure of φ with respect to σ is, for each $a \in I$, given by

$$\mathcal{P}(\varphi, \sigma) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\varphi, a, \sigma).$$

Remark 2.3. A group-extended Markov system $(\Sigma \times G, \sigma \rtimes \Psi)$ is a Markov shift with alphabet $I \times G$. If $(\Sigma \times G, \sigma \rtimes \Psi)$ is irreducible, $\varphi : \Sigma \rightarrow \mathbb{R}$ is of medium variation and $a \in I$, then

$$\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in \Sigma^n, \omega_1=a, \omega a \in \Sigma^*, \Psi(\omega)=\text{id}} e^{\sup S_n \varphi|_{[\omega]}}.$$

The next definition goes back to the work of Ruelle and Bowen (cf. [Rue69], [Bow75]).

Definition 2.4. Let Σ be a Markov shift and let $\varphi : \Sigma \rightarrow \mathbb{R}$ be Hölder continuous with $\mathcal{P}(\varphi, \sigma) < \infty$. We say that a Borel probability measure μ is a *Gibbs measure* for φ if there exists a constant $C_\mu \geq 1$ such that

$$(2.1) \quad C_\mu^{-1} \leq \frac{\mu([\omega])}{e^{S_{|\omega|} \varphi(\tau) - |\omega| \mathcal{P}(\varphi, \sigma)}} \leq C_\mu, \text{ for all } \omega \in \Sigma^* \text{ and } \tau \in [\omega].$$

The following criterion for the existence of Gibbs measures is taken from [Sar03, Theorem 1]. The uniqueness follows from [MU03, Theorem 2.2.4]. The existence of a σ -invariant Gibbs measure and a Hölder continuous eigenfunction on a topologically mixing Markov shift with the b.i.p. property follows from [MU03].

Theorem 2.5. *Let Σ be a topologically mixing Markov shift and let $\varphi : \Sigma \rightarrow \mathbb{R}$ be Hölder continuous. Then there exists a (unique) σ -invariant Gibbs measure μ_φ for φ if and only if Σ satisfies the b.i.p. property and $\mathcal{P}(\varphi, \sigma) < \infty$. Moreover, if Σ satisfies the b.i.p. property and $\mathcal{P}(\varphi, \sigma) < \infty$, then there exists a unique Hölder continuous function $h : \Sigma \rightarrow \mathbb{R}^+$, bounded away from zero and infinity, such that $\mathcal{L}_\varphi(h) = e^{\mathcal{P}(\varphi, \sigma)} h$ and $\int \psi d\mu_\varphi = \int \mathcal{L}_{\varphi + \log h - \log h \circ \sigma - \mathcal{P}(\varphi, \sigma)}(\psi) d\mu_\varphi$, for every bounded continuous function $\psi : \Sigma \rightarrow \mathbb{R}$.*

3. PROOFS

Let us first state the necessary definitions and notation which are needed for the proofs of our main results. If Σ is a topologically mixing Markov shift with the b.i.p. property, and if $\varphi : \Sigma \rightarrow \mathbb{R}$ is Hölder continuous with $\mathcal{P}(\varphi, \sigma) < \infty$, then there exists a unique σ -invariant Gibbs measure μ_φ for φ by Theorem 2.5. For $p \in \mathbb{N} \cup \{\infty\}$ and $\psi \in L^p(\Sigma, \mathcal{B}(\Sigma), \mu_\varphi)$, we denote by $\|\psi\|_p$ the L^p -norm of ψ , where $\mathcal{B}(\Sigma)$ is the Borel sigma algebra of Σ . For a group-extended Markov system $(\Sigma \times G, \sigma \rtimes \Psi)$, Stadlbauer ([Sta13]) introduced the Banach space $(\mathcal{H}_p, |\cdot|_p)$, given by

$$\mathcal{H}_p := \left\{ f : \Sigma \times G \rightarrow \mathbb{R} : |f|_p < \infty \right\}, \quad \text{where } |f|_p := \sqrt{\sum_{g \in G} (\|f(\cdot, g)\|_p)^2}.$$

Denote by $\mathbb{1}_\Omega$ the indicator function of a set $\Omega \subset \Sigma \times G$. Let \mathcal{H}_c be the closed subspace of \mathcal{H}_∞ generated by $\{\mathbb{1}_{\Sigma \times \{g\}} : g \in G\}$, which is isomorphic to the Hilbert

space $\ell^2(G)$. We use (\cdot, \cdot) to denote the inner product on \mathcal{H}_c , given by $(f_1, f_2) := \sum_{g \in G} \int f_1(\cdot, g) f_2(\cdot, g) d\mu_\varphi$, for all $f_1, f_2 \in \mathcal{H}_c$. For a closed subspace $\mathcal{V} \subset \mathcal{H}_\infty$, we set $\mathcal{V}^+ := \{f \in \mathcal{V} : f \geq 0\}$. For closed subspaces $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{H}_\infty$ and a bounded linear operator $T : \mathcal{V}_1 \rightarrow \mathcal{V}_2$, the operator norm of T is given by

$$\|T\| := \sup_{f \in \mathcal{V}_1, |f|_\infty = 1} |T(f)|_\infty.$$

We say that T is positive if $T(\mathcal{V}_1^+) \subset \mathcal{V}_2^+$. It is well known that, if T is positive and $\mathcal{V} = \mathcal{V}^+ - \mathcal{V}^+$, then $\|T\| = \sup_{f \in \mathcal{V}_1^+, |f|_\infty = 1} |T(f)|_\infty$.

It is shown in [Sta13, Proposition 5.2] that the Perron-Frobenius operator $\mathcal{L}_{\varphi \circ \pi_1} : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ is a bounded linear operator. Further, if $\mathcal{L}_\varphi(\mathbb{1}) = \mathbb{1}$, then

$$(3.1) \quad \|\mathcal{L}_{\varphi \circ \pi_1}^n\| \leq C, \quad n \in \mathbb{N},$$

where $C = C_{\mu_\varphi} \geq 1$ denotes the constant of the Gibbs measure μ_φ given by (2.1).

Definition 3.1. We define the linear operators $A : \mathcal{H}_\infty \rightarrow \mathcal{H}_c$ and $T_n : \mathcal{H}_c \rightarrow \mathcal{H}_c$, which are for $f \in \mathcal{H}_\infty$ and $n \in \mathbb{N}$ given by

$$A(f) := \sum_{g \in G} \left(\int f(\cdot, g) d\mu_\varphi \right) \mathbb{1}_{\Sigma \times \{g\}} \quad \text{and} \quad T_n := A \mathcal{L}_{\varphi \circ \pi_1}^n |_{\mathcal{H}_c}.$$

Clearly, the linear operators A and T_n are positive and bounded with $\|A\| = 1$ and $\|T_n\| \leq \|A\| \|\mathcal{L}_{\varphi \circ \pi_1}^n\| = \|\mathcal{L}_{\varphi \circ \pi_1}^n\|$, for each $n \in \mathbb{N}$. To state the next lemma, let us recall the definition of Λ_n ([Sta13, Proposition 5.2]), which is given by $\Lambda_n := \sup_{f \in \mathcal{H}_c^+, |f|_\infty = 1} |\mathcal{L}_{\varphi \circ \pi_1}^n(f)|_1$, for each $n \in \mathbb{N}$.

Lemma 3.2. *Under the assumptions of Theorem 1.1, suppose that $\mathcal{L}_\varphi(\mathbb{1}) = \mathbb{1}$. Then we have the following:*

- (1) $\|T_n\| = \Lambda_n, n \in \mathbb{N}$.
- (2) $\|T_n\| \leq \|\mathcal{L}_{\varphi \circ \pi_1}^n\| \leq C \|T_n\|, n \in \mathbb{N}$.
- (3) $\lim_{n \rightarrow \infty} \|T_n\|^{1/n} = \lim_{n \rightarrow \infty} (\Lambda_n)^{1/n} = \rho(\mathcal{L}_{\varphi \circ \pi_1})$.
- (4) $\limsup_{n \rightarrow \infty} n^{-1} \log (T_n \mathbb{1}_{\Sigma \times \{\text{id}\}}, \mathbb{1}_{\Sigma \times \{\text{id}\}}) = \mathcal{P}(\varphi \circ \pi_1, \sigma \times \Psi)$.

Proof. Let $n \in \mathbb{N}$. The assertion in (1) follows because T_n is positive and $|T_n(f)|_\infty = |\mathcal{L}_{\varphi \circ \pi_1}^n(f)|_1$, for each $f \in \mathcal{H}_c^+$. The first inequality in (2) holds since $\|A\| = 1$. To prove the second inequality we show that

$$(3.2) \quad \|\mathcal{L}_{\varphi \circ \pi_1}^n\| = \sup \left\{ |\mathcal{L}_{\varphi \circ \pi_1}^n(f)|_\infty : f \in \mathcal{H}_c^+, |f|_\infty = 1 \right\}.$$

For each $f \in \mathcal{H}_\infty^+$, there exists $\tilde{f} \in \mathcal{H}_c^+$, given by $\tilde{f}(x, g) := \|f(\cdot, g)\|_\infty, (x, g) \in \Sigma \times G$, which satisfies $|\tilde{f}|_\infty = |f|_\infty$ and $|\mathcal{L}_{\varphi \circ \pi_1}^n(f)|_\infty \leq |\mathcal{L}_{\varphi \circ \pi_1}^n(\tilde{f})|_\infty$. Hence, (3.2) follows. To prove $\|\mathcal{L}_{\varphi \circ \pi_1}^n\| \leq C \|T_n\|$, let $f \in \mathcal{H}_c^+$. By the Gibbs property (2.1) of μ_φ , there exists $C \geq 1$ such that, for each $g \in G$ and $x_0 \in \Sigma$,

$$\begin{aligned} \|\mathcal{L}_{\varphi \circ \pi_1}^n(f)(\cdot, g)\|_\infty &\leq \sum_{\omega \in \Sigma^n} \sup_{\tau \in \Sigma : \omega\tau \in \Sigma} e^{S_n \varphi(\omega\tau)} f(x_0, g\Psi(\omega)^{-1}) \\ &\leq C \sum_{\omega \in \Sigma^n} \mu_\varphi([\omega]) f(x_0, g\Psi(\omega)^{-1}). \end{aligned}$$

By Theorem 2.5 we have $\mu_\varphi([\omega]) = \int \mathcal{L}_\varphi^n(\mathbb{1}_{[\omega]}) d\mu_\varphi$, for each $\omega \in \Sigma^n$. Hence, we obtain that

$$(3.3) \quad \begin{aligned} & \sum_{\omega \in \Sigma^n} \mu_\varphi([\omega]) f(x_0, g\Psi(\omega)^{-1}) = \sum_{\omega \in \Sigma^n} \int \mathcal{L}_\varphi^n(\mathbb{1}_{[\omega]}) f(x_0, g\Psi(\omega)^{-1}) d\mu_\varphi \\ & = \sum_{\omega \in \Sigma^n} \int \mathcal{L}_{\varphi \circ \pi_1}^n(\mathbb{1}_{[\omega] \times G} f)(\cdot, g) d\mu_\varphi = \int \mathcal{L}_{\varphi \circ \pi_1}^n(f)(\cdot, g) d\mu_\varphi = \|\mathcal{L}_{\varphi \circ \pi_1}^n(f)(\cdot, g)\|_1, \end{aligned}$$

which proves $|\mathcal{L}_{\varphi \circ \pi_1}^n(f)|_\infty \leq C |\mathcal{L}_{\varphi \circ \pi_1}^n(f)|_1 = C |T_n(f)|_\infty$ and finishes the proof of (2).

The assertions in (3) follow by combining (1), (2) and Gelfand’s formula for the spectral radius (see e.g. [Rud73, Theorem 10.13]). Let us now turn to the proof of (4). By (3.3), we see that T_n is for each $n \in \mathbb{N}$, $f \in \mathcal{H}_c$ and $x_0 \in \Sigma$ given by

$$(3.4) \quad T_n(f) = \sum_{g \in G} \left(\sum_{\omega \in \Sigma^n} \mu_\varphi([\omega]) f(x_0, g\Psi(\omega)^{-1}) \right) \mathbb{1}_{\Sigma \times \{g\}}.$$

Consequently, by the Gibbs property (2.1) of μ_φ we have

$$C^{-1} \sum_{\omega \in \Sigma^n : \Psi(\omega) = \text{id}} e^{\sup S_n \varphi|_{[\omega]}} \leq (T_n \mathbb{1}_{\Sigma \times \{\text{id}\}}, \mathbb{1}_{\Sigma \times \{\text{id}\}}) \leq C \sum_{\omega \in \Sigma^n : \Psi(\omega) = \text{id}} e^{\inf S_n \varphi|_{[\omega]}}.$$

It follows that $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) \leq \limsup_{n \rightarrow \infty} n^{-1} \log (T_n \mathbb{1}_{\Sigma \times \{\text{id}\}}, \mathbb{1}_{\Sigma \times \{\text{id}\}})$ (cf. Remark 2.3). To prove equality, we fix $a \in I$. Since $(\Sigma \times G, \sigma \rtimes \Psi)$ is irreducible, there exists $\kappa(a) \in \Sigma^*$ such that $\Psi(a)\Psi(\kappa(a)) = \text{id}$. Since Σ has the b.i.p. property and $(\Sigma \times G, \sigma \rtimes \Psi)$ is irreducible, there exists a finite set $F \subset \Psi^{-1}(\text{id}) \cap \Sigma^*$ such that, for all $i, j \in I$ there exists $\gamma \in F$ with $i\gamma j \in \Sigma^*$. Set $l := \max_{\gamma \in F} |\gamma|$. For each $n \in \mathbb{N}$ and $\omega \in \Sigma^n$, there exist $\gamma_1, \gamma_2, \gamma_3 \in F$ such that $\tau(\omega) := a\gamma_1\kappa(a)\gamma_2\omega\gamma_3a \in \Sigma^*$. This defines a map from $\{\omega \in \Sigma^n : \Psi(\omega) = \text{id}\}$ to $\{\tau \in \Sigma^k : n \leq k \leq n + 3l + |\kappa(a)| + 1, \Psi(\tau) = \text{id}, \tau_1 = a, \tau a \in \Sigma^*\}$, which is at most $(3l + |\kappa(a)|)$ -to-one, for each $n \in \mathbb{N}$. Setting

$$M := e^{-\inf \varphi|_{[a]}} e^{-\inf S_{|\kappa(a)|} \varphi|_{[\kappa(a)]}} \max_{\gamma \in F} e^{-3 \inf S_{|\gamma|} \varphi|_{[\gamma]}},$$

we obtain that, for each $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{\omega \in \Sigma^n : \Psi(\omega) = \text{id}} e^{\inf S_n \varphi|_{[\omega]}} & \leq M (3l + |\kappa(a)|) \\ & \quad \times \sum_{k=n}^{n+3l+|\kappa(a)|+1} \sum_{\tau \in \Sigma^k : \Psi(\tau) = \text{id}, \tau_1 = a, \tau a \in \Sigma^*} e^{\sup S_k \varphi|_{[\tau]}}, \end{aligned}$$

which gives $\limsup_{n \rightarrow \infty} n^{-1} \log (T_n \mathbb{1}_{\Sigma \times \{\text{id}\}}, \mathbb{1}_{\Sigma \times \{\text{id}\}}) \leq \mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ and finishes the proof of (4). □

Proof of Theorem 1.1. Let us first verify that we may assume without loss of generality that $\mathcal{L}_\varphi(\mathbb{1}) = \mathbb{1}$ and hence, $\mathcal{P}(\varphi, \sigma) = 0$. Otherwise, there exists a Hölder continuous function $h : \Sigma \rightarrow \mathbb{R}^+$, bounded away from zero and infinity, such that $\mathcal{L}_\varphi(h) = e^{\mathcal{P}(\varphi, \sigma)} h$ by Theorem 2.5. Setting $\tilde{\varphi} := \varphi + \log h - \log h \circ \sigma - \mathcal{P}(\varphi, \sigma)$, we have $\mathcal{L}_{\tilde{\varphi}}(\mathbb{1}) = \mathbb{1}$ and $\mathcal{P}(\tilde{\varphi}, \sigma) = 0$. Since $\mathcal{L}_{\tilde{\varphi} \circ \pi_1}(f) = e^{-\mathcal{P}(\varphi, \sigma)} \frac{1}{h \circ \pi_1} \mathcal{L}_{\varphi \circ \pi_1}(f(h \circ \pi_1))$, for each $f \in \mathcal{H}_\infty$, and using that $h \circ \pi_1$ is bounded

away from zero and infinity, we obtain that $\mathcal{L}_{\tilde{\varphi} \circ \pi_1}$ and $e^{-\mathcal{P}(\varphi, \sigma)} \mathcal{L}_{\varphi \circ \pi_1}$ have the same spectrum. Hence, we have $\log \rho(\mathcal{L}_{\tilde{\varphi} \circ \pi_1}) = \log \rho(\mathcal{L}_{\varphi \circ \pi_1}) - \mathcal{P}(\varphi, \sigma)$. We have thus shown that we may assume without loss of generality that $\mathcal{L}_{\varphi}(\mathbb{1}) = \mathbb{1}$.

That $\rho(\mathcal{L}_{\varphi \circ \pi_1}) \leq 1$ follows from (3.1) and Gelfand’s formula for the spectral radius. We now turn to the proof of the amenability dichotomy. First suppose that $\rho(\mathcal{L}_{\varphi \circ \pi_1}) = 1$. By Lemma 3.2 (3) we obtain that $\lim_{n \rightarrow \infty} (\Lambda_n)^{1/n} = 1$. It follows from [Sta13, Lemma 5.3, Theorem 5.4] that G is amenable. In fact, the assumption that $e^{\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)} = 1$ in [Sta13, Lemma 5.3, Theorem 5.4] can be relaxed to $\lim_{n \rightarrow \infty} (\Lambda_n)^{1/n} = 1$ without affecting the proofs.

To prove the converse implication, suppose that G is amenable. It follows from a well-known result of Day ([Day64, Theorem 1(d)]) that $\|T_n\| = 1$, for each $n \in \mathbb{N}$. To prove this, observe that by (3.4), we have that T_n is the right convolution operator with respect to the probability density on G , given by

$$\mu_{\varphi}(\{x \in \Sigma : \Psi(x_1, \dots, x_n) = g\}),$$

for each $g \in G$. Finally, we conclude that $\rho(\mathcal{L}_{\varphi \circ \pi_1}) = 1$ by Lemma 3.2 (3). The proof is complete. \square

Proof of Proposition 1.5. By the arguments given at the beginning of the proof of Theorem 1.1 and by Remark 1.4, we may assume that $\mathcal{L}_{\varphi}(\mathbb{1}) = \mathbb{1}$. To prove $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) \leq \log \rho(\mathcal{L}_{\varphi \circ \pi_1})$, we observe that, by Lemma 3.2 (4), the Cauchy-Schwarz inequality and Lemma 3.2 (3), we have

$$\begin{aligned} \mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log (T_n \mathbb{1}_{\Sigma \times \{id\}}, \mathbb{1}_{\Sigma \times \{id\}}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|T_n\| \\ &= \log \rho(\mathcal{L}_{\varphi \circ \pi_1}). \end{aligned}$$

Now suppose that φ is asymptotically α -symmetric with respect to Ψ . The main task is to prove that $\lim_{n \rightarrow \infty} \|T_n\|^{1/n} \leq \alpha e^{\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)}$, from which the proposition follows by Lemma 3.2 (3). Since $T_n^* T_n$ is a self-adjoint operator on the Hilbert space \mathcal{H}_c , an argument of Pier ([Pie84, pp. 196-202]), which was used by Gerl ([Ger88, p. 177]), shows that for each $n \in \mathbb{N}$,

$$(3.5) \quad \|T_n^* T_n\| = \limsup_{k \rightarrow \infty} \left((T_n^* T_n)^k (\mathbb{1}_{\Sigma \times \{id\}}, \mathbb{1}_{\Sigma \times \{id\}}) \right)^{1/k}.$$

By (3.4) and the Gibbs property (2.1) of μ_{φ} , we have for all $g_1, g_2 \in G$ and $n \in \mathbb{N}$ that

$$(3.6) \quad \begin{aligned} C^{-1} \sum_{\omega \in \Sigma^n : g_1 \Psi(\omega) = g_2} e^{\sup S_n \varphi|_{[\omega]}} &\leq (T_n \mathbb{1}_{\Sigma \times \{g_1\}}, \mathbb{1}_{\Sigma \times \{g_2\}}) \\ &\leq C \sum_{\omega \in \Sigma^n : g_1 \Psi(\omega) = g_2} e^{\inf S_n \varphi|_{[\omega]}}. \end{aligned}$$

Since φ is asymptotically α -symmetric with respect to Ψ , there exist $n_0 \in \mathbb{N}$ and sequences $(c_n) \in (\mathbb{R}^+)^{\mathbb{N}}$ and $(N_n) \in \mathbb{N}^{\mathbb{N}}$ with the property that $\lim_n (c_n)^{1/(2n)} = \alpha$ and $\lim_n n^{-1} N_n = 0$, such that, for all $f_1, f_2 \in \mathcal{H}_c^+$ and $n > \max\{N_n, n_0\}$, we have

$$(3.7) \quad (T_n(f_1), f_2) \leq c_n C^2 \sum_{i=-N_n}^{N_n} (T_{n+i}^*(f_1), f_2).$$

Since Σ has the b.i.p. property and $(\Sigma \times G, \sigma \rtimes \Psi)$ is irreducible, there exists a finite set $F \subset \Psi^{-1}(\text{id}) \cap \Sigma^*$ such that, for all $i, j \in I$ there exists $\gamma \in F$ with $i\gamma j \in \Sigma^*$. Set $l := \max_{\gamma \in F} |\gamma|$ and $M := \max_{\gamma \in F} e^{-\inf S_{|\gamma|}\varphi_{|\gamma|}}$. It follows from (3.6) that, for all $t, u \in \mathbb{N}$ and for all $g_1, g_2 \in G$, we have

$$\begin{aligned} (T_t T_u \mathbb{1}_{\Sigma \times \{g_1\}}, \mathbb{1}_{\Sigma \times \{g_2\}}) &\leq C^2 \sum_{\omega_1 \in \Sigma^u, \omega_2 \in \Sigma^t: g_1 \Psi(\omega_1) \Psi(\omega_2) = g_2} e^{\inf S_u \varphi_{|\omega_1|}} e^{\inf S_t \varphi_{|\omega_2|}} \\ &\leq MC^2 \sum_{i=0}^l \sum_{\omega \in \Sigma^{u+t+i}: g_1 \Psi(\omega) = g_2} e^{\inf S_{u+t+i} \varphi_{|\omega|}} \\ &\leq MC^3 \sum_{i=0}^l (T_{u+t+i} \mathbb{1}_{\Sigma \times \{g_1\}}, \mathbb{1}_{\Sigma \times \{g_2\}}). \end{aligned}$$

Setting $D := MC^3$, we have thus shown that, for all $f_1, f_2 \in \mathcal{H}_c^+$,

$$(3.8) \quad (T_t T_u (f_1), f_2) \leq D \sum_{i=0}^l (T_{u+t+i} (f_1), f_2).$$

We now follow [Jae14, Proposition 3.11], which was motivated by [OW07]. Let $n > \max \{N_n, n_0\}$. Combining first (3.5) and (3.7), then applying $(2k - 1)$ -times the estimate in (3.8), we obtain that

$$\begin{aligned} \|T_n^* T_n\| &\leq \limsup_{k \rightarrow \infty} \left(c_n^k C^{2k} \left(\sum_{i_1 = -N_n}^{N_n} (T_{n+i_1} T_n) \cdots \sum_{i_k = -N_n}^{N_n} (T_{n+i_k} T_n) \right) \right. \\ &\quad \left. (\mathbb{1}_{\Sigma \times \{\text{id}\}}, \mathbb{1}_{\Sigma \times \{\text{id}\}}) \right)^{1/k} \\ &\leq c_n C^2 \limsup_{k \rightarrow \infty} \left((2N_n + 1)^k D^{2k-1} (l + 1)^{2k-1} \max_{r = -kN_n, \dots, k(N_n + 2l) - l} \right. \\ &\quad \left. \{(T_{2nk+r} (\mathbb{1}_{\Sigma \times \{\text{id}\}}), \mathbb{1}_{\Sigma \times \{\text{id}\}})\} \right)^{1/k}. \end{aligned}$$

Combining the previous estimate with $\|T_n^* T_n\| = \|T_n\|^2$, $\lim_n (c_n)^{1/(2n)} = \alpha$, $\lim_n n^{-1} N_n = 0$ and the fact that $\limsup_k (T_k \mathbb{1}_{\Sigma \times \{\text{id}\}}, \mathbb{1}_{\Sigma \times \{\text{id}\}})^{1/k} = e^{\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)}$ by Lemma 3.2 (4), we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|T_n\|^{1/n} &\leq \alpha \limsup_{n \rightarrow \infty} \left(\max \{e^{\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)(2n - N_n)}, e^{\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)(2n + N_n + 2l)}\} \right)^{1/2n} \\ &= \alpha e^{\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)}. \end{aligned}$$

The proof is complete. □

Lemma 3.3. *Let Σ be a Markov shift and let $\varphi : \Sigma \rightarrow \mathbb{R}$ be of medium variation. Suppose that φ is asymptotically α -symmetric with respect to Ψ , for some $\alpha \geq 1$. Then there exists a sequence of Hölder continuous functions $\varphi_j : \Sigma \rightarrow \mathbb{R}$ and $(D_j) \in \mathbb{R}^{\mathbb{N}}$, $D_j \geq 1$, $j \in \mathbb{N}$, such that φ_j is asymptotically $\alpha(D_j)^{1/(2j)}$ -symmetric, $\lim_j (D_j)^{1/(2j)} = 1$ and $\lim_j \varphi_j = \varphi$, where the convergence of (φ_j) is uniformly on compact subsets of Σ .*

Proof. Define $\varphi_j(x) := \inf \{\varphi(y) : y \in [x_1, \dots, x_j]\}$, for each $x \in \Sigma$ and $j \in \mathbb{N}$. Since φ is of medium variation, there exists a sequence $(D_j) \in \mathbb{R}^{\mathbb{N}}$, $D_j \geq 1$, such that, for all $j, n \in \mathbb{N}$, $\omega \in \Sigma^n$ and $x \in [\omega]$,

$$(3.9) \quad 1 \leq e^{S_n \varphi(x) - S_n \varphi_j(x)} \leq (D_j)^{\lfloor \frac{n}{j} \rfloor} \left(\sup_{k < j} D_k \right),$$

where $\lfloor u \rfloor$ denotes the largest integer not greater than u . Since φ is asymptotically α -symmetric, there exist $n_0 \in \mathbb{N}$ and sequences $(c_n) \in \mathbb{R}^{\mathbb{N}}$ and $(N_n) \in \mathbb{N}^{\mathbb{N}}$ with the property that $\lim_n (c_n)^{1/(2n)} = \alpha$, $\lim_n n^{-1} N_n = 0$ and such that (1.1) holds, for each $g \in G$ and for all $n \geq n_0$. Let $j \in \mathbb{N}$. By (1.1) and (3.9) we have for each $n \in \mathbb{N}$ with $n > \max \{N_n, n_0\}$ and $g \in G$,

$$\begin{aligned} \sum_{\omega \in \Sigma^n : \Psi(\omega) = g} e^{\sup S_n \varphi_j |[\omega]} &\leq \sum_{\omega \in \Sigma^n : \Psi(\omega) = g} e^{\sup S_n \varphi |[\omega]} \\ &\leq c_n \sum_{\omega \in \Sigma^* : \Psi(\omega) = g^{-1}, n - N_n \leq |\omega| \leq n + N_n} e^{\sup S_{|\omega|} \varphi |[\omega]} \\ &\leq c_n \max_{n - N_n \leq l \leq n + N_n} (D_j)^{\lfloor \frac{l}{j} \rfloor} \left(\sup_{k < j} D_k \right) \\ &\quad \times \sum_{\omega \in \Sigma^* : \Psi(\omega) = g^{-1}, n - N_n \leq |\omega| \leq n + N_n} e^{\sup S_{|\omega|} \varphi_j |[\omega]}. \end{aligned}$$

Since $\lim_n (c_n \max_{n - N_n \leq l \leq n + N_n} (D_j)^{\lfloor \frac{l}{j} \rfloor} \left(\sup_{k < j} D_k \right))^{1/(2n)} = \alpha (D_j)^{1/(2j)}$, we have that φ_j is asymptotically $\alpha (D_j)^{1/(2j)}$ -symmetric. By continuity of φ_j and φ , and using that $\varphi_j \leq \varphi_{j+1}$, for each $j \in \mathbb{N}$, we have that $\lim_j \varphi_j = \varphi$ locally uniformly by Dini's Theorem. \square

Proof of Corollary 1.7. First suppose that the alphabet of Σ is finite. Then $\mathcal{P}(\varphi, \sigma) < \infty$ and Σ satisfies the b.i.p. property. By Lemma 3.3, there exists a sequence of Hölder continuous functions $\varphi_j : \Sigma \rightarrow \mathbb{R}$, $j \in \mathbb{N}$, such that φ_j is asymptotically $\alpha (D_j)^{1/(2j)}$ -symmetric and $\lim_j \varphi_j = \varphi$ uniformly. By Corollary 1.6, we have $\mathcal{P}(\varphi_j \circ \pi_1, \sigma \rtimes \Psi) \geq \mathcal{P}(\varphi_j, \sigma) - \log(\alpha (D_j)^{1/(2j)})$. The claim follows by letting j tend to infinity, because we have $\lim_j (D_j)^{1/(2j)} = 1$, $\lim_j \mathcal{P}(\varphi_j \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ and $\lim_j \mathcal{P}(\varphi_j, \sigma) = \mathcal{P}(\varphi, \sigma)$.

Finally, suppose that φ is compactly asymptotically α -symmetric with respect to Ψ . For each $k \in \mathbb{N}$, define $\varphi_k := \varphi|_{J_k^*}$. Since $G_k := \Psi(\Sigma_k^*)$ is a subgroup of the amenable group G , we have that G_k is amenable. Moreover, φ_k is asymptotically α -symmetric with respect to $\Psi|_{J_k^*}$. Hence, by the first part of the proof, we have $\mathcal{P}(\varphi_k \circ \pi_1, \sigma \rtimes \Psi) \geq \mathcal{P}(\varphi_k, \sigma) - \log \alpha$. The result follows from the facts that $\lim_k \mathcal{P}(\varphi_k \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ and $\lim_k \mathcal{P}(\varphi_k, \sigma) = \mathcal{P}(\varphi, \sigma)$. This is proved for Hölder continuous functions in [Sar99, Theorem 2], and it is straightforward to extend the proof to functions of medium variation. \square

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REFERENCES

- [Bow75] Rufus Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, Vol. 470, Springer-Verlag, Berlin, 1975. MR0442989 (56 #1364)
- [Day64] Mahlon Marsh Day, *Convolutions, means, and spectra*, Illinois J. Math. **8** (1964), 100–111. MR0159230 (28 #2447)
- [Ger88] Peter Gerl, *Random walks on graphs with a strong isoperimetric property*, J. Theoret. Probab. **1** (1988), no. 2, 171–187, DOI 10.1007/BF01046933. MR938257 (89g:60216)
- [Gur69] B. M. Gurevič, *Topological entropy of a countable Markov chain* (Russian), Dokl. Akad. Nauk SSSR **187** (1969), 715–718. MR0263162 (41 #7767)
- [Gur70] B. M. Gurevič, *Shift entropy and Markov measures in the space of paths of a countable graph* (Russian), Dokl. Akad. Nauk SSSR **192** (1970), 963–965. MR0268356 (42 #3254)
- [Jae14] Johannes Jaerisch, *Fractal models for normal subgroups of Schottky groups*, Trans. Amer. Math. Soc., to appear. DOI 10.1090/S0002-9947-2014-06095-9.
- [Jae14b] ———, *Recurrence and pressure for group extensions*, Ergodic Theory Dynam. Systems, to appear.
- [Jae11] ———, *Thermodynamic formalism for group-extended Markov systems with applications to Fuchsian groups*, Doctoral Dissertation at the University Bremen (2011).
- [Kes59a] Harry Kesten, *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156. MR0112053 (22 #2911)
- [Kes59b] Harry Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354. MR0109367 (22 #253)
- [MU03] R. D. Mauldin and M. Urbański, *Graph directed Markov systems*, Cambridge Tracts in Mathematics, vol. 148, Cambridge University Press, Cambridge, 2003. MR2003772 (2006e:37036)
- [OW07] Ronald Ortner and Wolfgang Woess, *Non-backtracking random walks and cogrowth of graphs*, Canad. J. Math. **59** (2007), no. 4, 828–844, DOI 10.4153/CJM-2007-035-1. MR2338235 (2008h:05057)
- [Pie84] Jean-Paul Pier, *Amenable locally compact groups*, Pure and Applied Mathematics (New York), A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1984. MR767264 (86a:43001)
- [Rud73] Walter Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York, 1973. MR0365062 (51 #1315)
- [Rue69] David Ruelle, *Statistical mechanics: Rigorous results*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0289084 (44 #6279)
- [Sar99] Omri M. Sarig, *Thermodynamic formalism for countable Markov shifts*, Ergodic Theory Dynam. Systems **19** (1999), no. 6, 1565–1593, DOI 10.1017/S0143385799146820. MR1738951 (2000m:37009)
- [Sar01] Omri M. Sarig, *Thermodynamic formalism for null recurrent potentials*, Israel J. Math. **121** (2001), 285–311, DOI 10.1007/BF02802508. MR1818392 (2001m:37059)
- [Sar03] Omri Sarig, *Existence of Gibbs measures for countable Markov shifts*, Proc. Amer. Math. Soc. **131** (2003), no. 6, 1751–1758 (electronic), DOI 10.1090/S0002-9939-03-06927-2. MR1955261 (2004b:37056)
- [Sha07] Richard Sharp, *Critical exponents for groups of isometries*, Geom. Dedicata **125** (2007), 63–74, DOI 10.1007/s10711-007-9137-9. MR2322540 (2008f:20107)
- [Sta13] Manuel Stadlbauer, *An extension of Kesten’s criterion for amenability to topological Markov chains*, Adv. Math. **235** (2013), 450–468, DOI 10.1016/j.aim.2012.12.004. MR3010065

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