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CONVEX HULLS OF PLANAR RANDOM WALKS WITH DRIFT

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ABSTRACT. Denote by L_n the perimeter length of the convex hull of an n-step planar random walk whose increments have finite second moment and non-zero mean. Snyder and Steele showed that $n^{-1}L_n$ converges almost surely to a deterministic limit and proved an upper bound on the variance $\mathbb{V}\mathrm{ar}[L_n] = O(n)$. We show that $n^{-1}\mathbb{V}\mathrm{ar}[L_n]$ converges and give a simple expression for the limit, which is non-zero for walks outside a certain degenerate class. This answers a question of Snyder and Steele. Furthermore, we prove a central limit theorem for L_n in the non-degenerate case.

1. Introduction and main results

On each of n unsteady steps, a drunken gardener drops a seed. Once the flowers have bloomed, what is the minimum length of fencing required to enclose the garden?

Let $Z_1, Z_2, \ldots$ be a sequence of independent, identically distributed (i.i.d.) random vectors on $\mathbb{R}^2$. Write $\mathbf{0}$ for the origin in $\mathbb{R}^2$. Define the random walk $(S_n; n \in \mathbb{Z}_+)$ by $S_0 := \mathbf{0}$ and for $n \geq 1$, $S_n := \sum_{i=1}^{n} Z_i$. Let $\mathcal{H}_n := \mathrm{hull}(S_0, \ldots, S_n)$, the convex hull of positions of the walk up to and including the nth step, and let $L_n := |\partial \mathcal{H}_n|$ denote the length of the perimeter of $\mathcal{H}_n$. Assume that the increments of the random walk have finite mean $\mathbb{E}\|Z_1\| < \infty$.

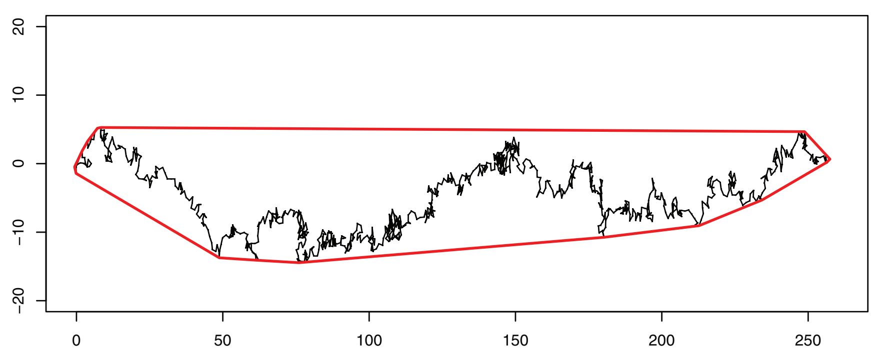

FIGURE 1. Example with mean drift $\mathbb{E}[Z_1]$ of magnitude $\mu = 1/4$ and $n = 10^3$ steps.

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Convex hulls of random points have received much attention over the last several decades: see [3] for an extensive survey, including more than 150 bibliographic references, and sources of motivation more serious than our drunken gardener, such as modelling the 'home-range' of animal populations. An important tool in the study of random convex hulls is provided by a result of Cauchy in classical convex geometry. Spitzer and Widom [5], using Cauchy's formula, and later Baxter [1], using a combinatorial argument, showed that

$$\mathbb{E}[L_n] = 2 \sum_{i=1}^{n} \frac{1}{i} \mathbb{E}\|S_i\|.$$

Note that $\mathbb{E}[L_n]$ thus scales like n in the case where the one-step mean drift vector $\mathbb{E}[Z_1] \neq \mathbf{0}$ but like $n^{1/2}$ in the case where $\mathbb{E}[Z_1] = \mathbf{0}$ (provided $\mathbb{E}[\|Z_1\|^2] < \infty$). The Spitzer–Widom–Baxter result, in common with much of the literature, is concerned with first-order properties of L_n; see [3] for a summary of results in this direction for various random convex hulls, with a specific focus on (driftless) planar Brownian motion. See Figure 1 for a simulation.

Much less is known about higher-order properties of L_n. Assuming that $\mathbb{E}[\|Z_1\|^2] < \infty$, Snyder and Steele [4] obtained an upper bound for $\mathbb{V}\mathrm{ar}[L_n]$ using Cauchy's formula together with a version of the Efron–Stein inequality. Snyder and Steele's result (Theorem 2.3 of [4]) can be expressed as

$$n^{-1} \mathbb{V}\mathrm{ar}[L_n] \leq \frac{\pi^2}{2} \left(\mathbb{E}[\|Z_1\|^2] - \|\mathbb{E}[Z_1]\|^2 \right), \ (n \in \mathbb{N} := \{1, 2, \ldots\}). \tag{1.1}$$

As far as we are aware, there are no lower bounds for $\mathbb{V}\mathrm{ar}[L_n]$ in the literature. According to the discussion in [4, §5], Snyder and Steele had "no compelling reason to expect that $O(n)$ is the correct order of magnitude" in their upper bound for $\mathbb{V}\mathrm{ar}[L_n]$, and they speculated that perhaps $\mathbb{V}\mathrm{ar}[L_n] = o(n)$ (maybe with a distinction between the cases of zero and non-zero drift). Our first main result settles this question under minimal conditions, confirming that (1.1) is indeed of the correct order, apart from in certain degenerate cases, while demonstrating that the constant on the right-hand side of (1.1) is not, in general, sharp.

Theorem 1.1. *Suppose that* $\mathbb{E}[\|Z_1\|^2] < \infty$ *and* $\|\mathbb{E}[Z_1]\| \neq 0$. *Then*

$$\lim_{n \to \infty} n^{-1} \mathbb{V}\mathrm{ar}[L_n] = \frac{4\mathbb{E}[((Z_1 - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1])^2]}{\|\mathbb{E}[Z_1]\|^2} =: \sigma^2 \in [0, \infty). \tag{1.2}$$

Remarks 1.1. (i) The assumptions $\mathbb{E}[\|Z_1\|^2] < \infty$ and $\|\mathbb{E}[Z_1]\| \neq 0$ ensure $\sigma^2 < \infty$.

(ii) To compare the limit result (1.2) with Snyder and Steele's upper bound (1.1), observe that

$$\sigma^2 = 4 \left(\frac{\mathbb{E}[(Z_1 \cdot \mathbb{E}[Z_1])^2] - \|\mathbb{E}[Z_1]\|^4}{\|\mathbb{E}[Z_1]\|^2} \right) \leq 4 \left(\mathbb{E}[\|Z_1\|^2] - \|\mathbb{E}[Z_1]\|^2 \right).$$

(iii) The limit σ^2 is zero if and only if $(Z_1 - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1] = 0$ with probability 1, i.e., if $Z_1 - \mathbb{E}[Z_1]$ is always orthogonal to $\mathbb{E}[Z_1]$. In such a degenerate case, (1.2) says that $\mathbb{V}\mathrm{ar}[L_n] = o(n)$. This is the case, for example, if Z_1 takes values $(1, 1)$ and $(1, -1)$ each with probability $1/2$. Note that the Snyder–Steele bound (1.1) applied in this example says only that $\mathbb{V}\mathrm{ar}[L_n] \leq (\pi^2/2)n$, which is not the correct order. Here, the two-dimensional trajectory can be viewed as a space-time trajectory of a *one-dimensional* simple symmetric random walk. We conjecture that in fact $\mathbb{V}\mathrm{ar}[L_n] = O(\log n)$. Steele [6] obtains variance results for the *number* *of faces* of the convex hull of one-dimensional simple random walk, and comments that such results for L_n seem "far out of reach" [6, p. 242].

In the case where $\mathbb{E}[\|Z_1\|^2] < \infty$ and $\|\mathbb{E}[Z_1]\| = \mu > 0$, Snyder and Steele deduce from their bound (1.1) a strong law of large numbers for L_n, namely $\lim_{n\to\infty} n^{-1} L_n = 2\mu$, a.s. (see [4, p. 1168]). Given this and the variance asymptotics of Theorem 1.1, it is natural to ask whether there is an accompanying central limit theorem. Our next result gives a positive answer in the non-degenerate case, again with essentially minimal assumptions.

Theorem 1.2. *Suppose that* $\mathbb{E}[\|Z_1\|^2] < \infty$ *and* $\|\mathbb{E}[Z_1]\| \neq 0$. *Suppose that* σ^2 *as defined in* (1.2) *satisfies* $\sigma^2 > 0$. *Then for any* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{L_n - \mathbb{E}[L_n]}{\sqrt{\mathbb{V}\mathrm{ar}[L_n]}} \leq x\right] = \lim_{n\to\infty} \mathbb{P}\left[\frac{L_n - \mathbb{E}[L_n]}{\sqrt{\sigma^2 n}} \leq x\right] = \Phi(x), \tag{1.3}$$

where Φ *is the standard normal distribution function.*

Our Theorems 1.1 and 1.2 will be deduced as consequences of the following result, which shows, perhaps surprisingly, that $L_n - \mathbb{E}[L_n]$ can be well-approximated by a sum of i.i.d. random variables.

Theorem 1.3. *Suppose that* $\mathbb{E}[\|Z_1\|^2] < \infty$ *and* $\|\mathbb{E}[Z_1]\| \neq 0$. *Then, as* $n \to \infty$,

$$n^{-1/2}\left| L_n - \mathbb{E}[L_n] - \sum_{i=1}^{n} \frac{2(Z_i - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1]}{\|\mathbb{E}[Z_1]\|} \right| \to 0, \text{ in } L^2.$$

The subsequent sections of the paper present the proofs of these theorems. The main ingredients, which we present in turn, include a martingale difference representation, Cauchy's formula from convex geometry, and an analysis of the geometry of the convex hull via extrema (the strong law of large numbers with the non-zero drift provides much of the regularity that we need).

To finish this section we discuss some simulations. We considered a specific form of random walk with increments $Z_i - \mathbb{E}[Z_i] = (\cos\Theta_i, \sin\Theta_i)$, where Θ_i was uniformly distributed on $[0, 2\pi)$, corresponding to a uniform distribution on a unit circle centred at $\mathbb{E}[Z_i] = (\mu, 0)$, say. We took one example with $\mu = 0$ and two examples with $\mu \neq 0$ of different magnitudes. In these latter cases, the results above take the form: $\lim_{n\to\infty} n^{-1}\mathbb{V}\mathrm{ar}[L_n] = 4\mathbb{E}[\cos^2\Theta_1] = 2$ (Theorem 1.1) and $(2n)^{-1/2}(L_n - \mathbb{E}[L_n])$ converges in distribution to a standard normal distribution (Theorem 1.2). The corresponding pictures in Figures 2 and 3 show an agreement between the simulations and theory.

The results of this paper do not cover the case where $\|\mathbb{E}[Z_1]\| = 0$. The simulations in this case suggest that, for the example we considered, $\lim_{n\to\infty} n^{-1}\mathbb{V}\mathrm{ar}[L_n]$ exists (see the leftmost plot in Figure 2), but Figure 3 does not appear to be consistent with a normal distribution as a limiting distribution. The method of the present paper provides a promising approach to the zero-drift case, but a new idea will be needed to gain control over the geometry in that case.

2. Martingale difference representation

The first step in the proofs is a martingale difference argument, based on resampling members of the sequence $Z_1, \ldots, Z_n$, to get an expression for $\mathbb{V}\mathrm{ar}[L_n]$ amenable to analysis. Let $\mathcal{F}_0$ denote the trivial σ-algebra, and for $n \in \mathbb{N}$ set

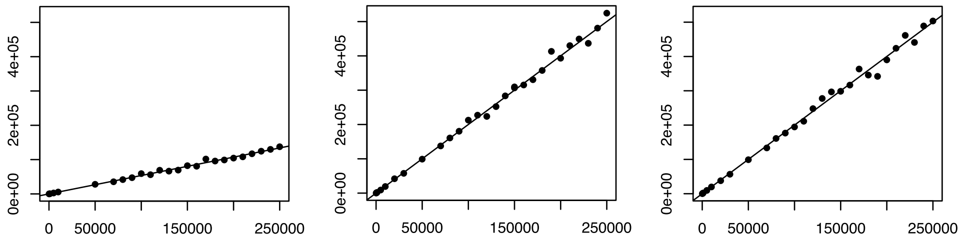

FIGURE 2. Plots of $y = \mathbb{V}\mathrm{ar}[L_n]$ estimates against $x = n$ for about 25 values of n in the range 10^2 to 2.5×10^5 for 3 examples with $\mu =$ (left to right) 0, 0.2, 0.36. Each point is estimated from the sample variance of 10^3 repeated simulations. Also plotted are straight lines $y = 0.536x$ (leftmost plot) and $y = 2x$ (other two plots).

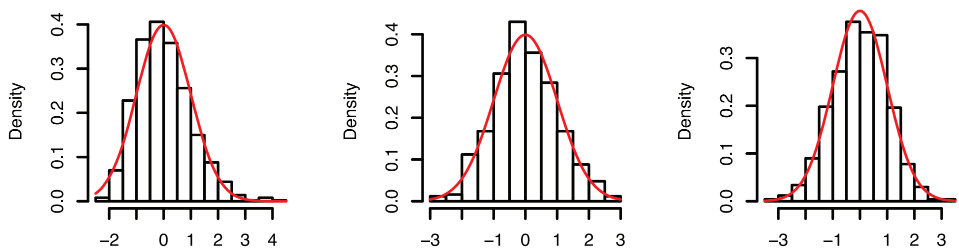

FIGURE 3. Simulated histogram estimates for the distribution of $\frac{L_n - \mathbb{E}[L_n]}{\sqrt{\mathbb{V}\mathrm{ar}[L_n]}}$ with $n = 5 \times 10^3$ in the three examples described in Figure 2. Each histogram is compiled from 10^3 samples.

$\mathcal{F}_n := \sigma(Z_1, \ldots, Z_n)$, the σ-algebra generated by the first n steps of the random walk. Then S_n is $\mathcal{F}_n$-measurable, and for $n \in \mathbb{N}$ we can write $L_n = \Lambda_n(Z_1, \ldots, Z_n)$ for $\Lambda_n : \mathbb{R}^{2n} \to [0, \infty)$ a measurable function.

Let $Z_1', Z_2', \ldots$ be an independent copy of the sequence $Z_1, Z_2, \ldots$. Fix $n \in \mathbb{N}$. For $i \in \{1, \ldots, n\}$, we 'resample' the ith increment, replacing Z_i with Z_i', as follows. Set

$$S_j^{(i)} := \begin{cases} S_j & \text{if } j < i, \\ S_j - Z_i + Z_i' & \text{if } j \geq i; \end{cases} \tag{2.1}$$

then $(S_j^{(i)}; 0 \leq j \leq n)$ is the random walk $(S_j; 0 \leq j \leq n)$ but with the ith step independently resampled. We let $L_n^{(i)}$ denote the perimeter length of the corresponding convex hull for this modified walk, namely $\mathrm{hull}(S_0^{(i)}, \ldots, S_n^{(i)})$, i.e.,

$$L_n^{(i)} := \Lambda_n(Z_1, \ldots, Z_{i-1}, Z_i', Z_{i+1}, \ldots, Z_n).$$

For $i \in \{1, \ldots, n\}$, define

$$D_{n,i} := \mathbb{E}[L_n - L_n^{(i)} \mid \mathcal{F}_i]; \tag{2.2}$$

in words, $-D_{n,i}$ is the expected change in the perimeter length of the convex hull, given $\mathcal{F}_i$, on replacing Z_i by Z_i'. The point of this construction is the following result.

Lemma 2.1. *Let $n \in \mathbb{N}$. Then (i) $L_n - \mathbb{E}[L_n] = \sum_{i=1}^n D_{n,i}$, and (ii) $\mathbb{V}\mathrm{ar}[L_n] = \sum_{i=1}^n \mathbb{E}[D_{n,i}^2]$, whenever the latter sum is finite.*

Proof. The idea is well known. Since $L_n^{(i)}$ is independent of Z_i, $\mathbb{E}[L_n^{(i)} \mid \mathcal{F}_i] = \mathbb{E}[L_n^{(i)} \mid \mathcal{F}_{i-1}] = \mathbb{E}[L_n \mid \mathcal{F}_{i-1}]$, so that (2.2) may be written as

$$D_{n,i} = \mathbb{E}[L_n \mid \mathcal{F}_i] - \mathbb{E}[L_n \mid \mathcal{F}_{i-1}]. \tag{2.3}$$

Then (2.3) yields the representation $\sum_{i=1}^n D_{n,i} = \mathbb{E}[L_n \mid \mathcal{F}_n] - \mathbb{E}[L_n \mid \mathcal{F}_0]$, giving (i). Here $(D_{n,i}; 1 \le i \le n)$ is a martingale difference sequence, and orthogonality (see e.g. [2, p. 218]) readily yields (ii). □

Remark 2.1. Lemma 2.1 with the conditional Jensen's inequality gives the bound

$$\mathbb{V}\mathrm{ar}[L_n] \le \sum_{i=1}^n \mathbb{E}[(L_n^{(i)} - L_n)^2],$$

which is a factor of 2 larger than the upper bound obtained from the Efron–Stein inequality; see equation (2.3) in [4].

3. Cauchy formula

Let $\mathbf{e}_\theta = (\cos\theta, \sin\theta)$ be the unit vector in direction $\theta \in (-\pi, \pi]$. For $\theta \in [0, \pi]$, define

$$M_n(\theta) := \max_{0\le j\le n} (S_j \cdot \mathbf{e}_\theta), \text{ and } m_n(\theta) := \min_{0\le j\le n} (S_j \cdot \mathbf{e}_\theta).$$

Note that since $S_0 = \mathbf{0}$, we have $M_n(\theta) \ge 0$ and $m_n(\theta) \le 0$, a.s. In the present setting (see [4], formula (2.1)), Cauchy's formula for convex sets yields

$$L_n = \int_0^\pi (M_n(\theta) - m_n(\theta))\,\mathrm{d}\theta = \int_0^\pi R_n(\theta)\mathrm{d}\theta,$$

where $R_n(\theta) := M_n(\theta) - m_n(\theta) \ge 0$ is the *parametrized range function*. Similarly, when the ith increment is resampled as described in Section 2,

$$L_n^{(i)} = \int_0^\pi \left(M_n^{(i)}(\theta) - m_n^{(i)}(\theta)\right)\mathrm{d}\theta = \int_0^\pi R_n^{(i)}(\theta)\mathrm{d}\theta,$$

where $R_n^{(i)}(\theta) = M_n^{(i)}(\theta) - m_n^{(i)}(\theta)$, defining

$$M_n^{(i)}(\theta) := \max_{0\le j\le n} (S_j^{(i)} \cdot \mathbf{e}_\theta), \text{ and } m_n^{(i)}(\theta) := \min_{0\le j\le n} (S_j^{(i)} \cdot \mathbf{e}_\theta).$$

Thus to study $D_{n,i} = \mathbb{E}[L_n - L_n^{(i)} \mid \mathcal{F}_i]$ we will consider

$$L_n - L_n^{(i)} = \int_0^\pi \left(R_n(\theta) - R_n^{(i)}(\theta)\right)\mathrm{d}\theta = \int_0^\pi \Delta_n^{(i)}(\theta)\mathrm{d}\theta, \tag{3.1}$$

where $\Delta_n^{(i)}(\theta) := R_n(\theta) - R_n^{(i)}(\theta)$. For $\theta \in [0, \pi]$, let

$$\underline{J}_n(\theta) := \arg\min_{0\le j\le n}(S_j \cdot \mathbf{e}_\theta), \text{ and } \bar{J}_n(\theta) := \arg\max_{0\le j\le n}(S_j \cdot \mathbf{e}_\theta),$$

so $m_n(\theta) = S_{\underline{J}_n(\theta)} \cdot \mathbf{e}_\theta$ and $M_n(\theta) = S_{\bar{J}_n(\theta)} \cdot \mathbf{e}_\theta$. Similarly, recalling (2.1), define

$$\underline{J}_n^{(i)}(\theta) := \arg\min_{0\le j\le n}(S_j^{(i)} \cdot \mathbf{e}_\theta), \text{ and } \bar{J}_n^{(i)}(\theta) := \arg\max_{0\le j\le n}(S_j^{(i)} \cdot \mathbf{e}_\theta).$$

(Apply the following conventions in the event of ties: arg min takes the maximum argument among tied values, and arg max the minimum.)

We will use the following simple bound repeatedly in the arguments that follow. In fact, with a little more work one can reduce the bound on the right-hand side of (3.2) by a factor of 2 (cf. [4], Lemma 2.1), but the form given here is good enough for us.

Lemma 3.1. *Almost surely, for any $\theta \in [0, \pi]$ and any $i \in \{1, 2, \dots, n\}$,*

$$|\Delta_n^{(i)}(\theta)| \le 2\|Z_i\| + 2\|Z_i'\|. \tag{3.2}$$

Proof. The triangle inequality implies that

$$|\Delta_n^{(i)}(\theta)| \le |M_n^{(i)}(\theta) - M_n(\theta)| + |m_n^{(i)}(\theta) - m_n(\theta)|.$$

For some $\bar{J}_n(\theta) \in \{0, 1, \dots, n\}$, we have $M_n(\theta) = S_{\bar{J}_n(\theta)} \cdot \mathbf{e}_\theta$ and, by definition, $M_n^{(i)}(\theta) \ge S^{(i)}_{\bar{J}_n(\theta)} \cdot \mathbf{e}_\theta$. If $\bar{J}_n(\theta) < i$, then, by (2.1), $S^{(i)}_{\bar{J}_n(\theta)} = S_{\bar{J}_n(\theta)}$, and so $M_n^{(i)}(\theta) \ge M_n(\theta)$. Otherwise, if $\bar{J}_n(\theta) \ge i$, then, by (2.1), $S^{(i)}_{\bar{J}_n(\theta)} = S_{\bar{J}_n(\theta)} - Z_i + Z_i'$, and so

$$\begin{aligned} M_n^{(i)}(\theta) &\ge S_{\bar{J}_n(\theta)} \cdot \mathbf{e}_\theta - Z_i \cdot \mathbf{e}_\theta + Z_i' \cdot \mathbf{e}_\theta \\ &\ge M_n(\theta) - \|Z_i\| - \|Z_i'\|. \end{aligned}$$

Hence we conclude that, a.s., $M_n^{(i)}(\theta) \ge M_n(\theta) - \|Z_i\| - \|Z_i'\|$. The analogous argument in the other direction shows that $|M_n^{(i)}(\theta) - M_n(\theta)| \le \|Z_i\| + \|Z_i'\|$. Moreover, a similar argument shows that the same bound holds for $|m_n^{(i)}(\theta) - m_n(\theta)|$, and (3.2) follows. ☐

4. Control of extrema

For the remainder of the paper, without loss of generality, we suppose that $\mathbb{E}[Z_1] = \mu \mathbf{e}_{\pi/2}$ with $\mu \in (0, \infty)$. Observe that $(S_j \cdot \mathbf{e}_\theta; 0 \le j \le n)$ is a one-dimensional random walk: indeed, $S_j \cdot \mathbf{e}_\theta = \sum_{k=1}^j Z_k \cdot \mathbf{e}_\theta$. The mean drift of this one-dimensional random walk is

$$\mathbb{E}[Z_1 \cdot \mathbf{e}_\theta] = \mathbb{E}[Z_1] \cdot \mathbf{e}_\theta = \mu \sin \theta. \tag{4.1}$$

Note that the drift $\mu \sin \theta$ is positive if $\theta \in (0, \pi)$. This crucial fact gives us control over the behaviour of the extrema such as $M_n(\theta)$ and $m_n(\theta)$ that contribute to (3.1), and this will allow us to estimate the conditional expectation of the final term in (3.1) (see Lemma 5.1 below).

For $\gamma \in (0, 1/2)$ and $\delta \in (0, \pi/2)$ (two constants that will be chosen to be suitably small later in our arguments), we denote by $E_{n,i}(\delta, \gamma)$ the event that the following occur:

- for all $\theta \in [\delta, \pi - \delta]$, $\underline{J}_n(\theta) < \gamma n$ and $\bar{J}_n(\theta) > (1 - \gamma)n$;
- for all $\theta \in [\delta, \pi - \delta]$, $\underline{J}_n^{(i)}(\theta) < \gamma n$ and $\bar{J}_n^{(i)}(\theta) > (1 - \gamma)n$.

We write $E^{\mathrm{c}}_{n,i}(\delta, \gamma)$ for the complement of $E_{n,i}(\delta, \gamma)$. The idea is that $E_{n,i}(\delta, \gamma)$ will occur with high probability, and on this event we have good control over $\Delta_n^{(i)}(\theta)$. The next result formalizes these assertions. For $\gamma \in (0, 1/2)$, define $I_{n,\gamma} := \{1, \dots, n\} \cap [\gamma n, (1 - \gamma)n]$.

Lemma 4.1. *For any $\gamma \in (0, 1/2)$ and any $\delta \in (0, \pi/2)$, the following hold.*

(i) *If $i \in I_{n,\gamma}$, then, a.s., for any $\theta \in [\delta, \pi - \delta]$,*

$$\Delta_n^{(i)}(\theta) \mathbf{1}(E_{n,i}(\delta, \gamma)) = (Z_i - Z_i') \cdot \mathbf{e}_\theta \mathbf{1}(E_{n,i}(\delta, \gamma)). \tag{4.2}$$

(ii) *If* $\mathbb{E}\|Z_1\| < \infty$ *and* $\|\mathbb{E}[Z_1]\| \neq 0$, *then* $\min_{1\leq i\leq n} \mathbb{P}[E_{n,i}(\delta,\gamma)] \to 1$ *as* $n \to \infty$.

Proof. First we prove part (i). Suppose that $i \in I_{n,\gamma}$, so $\gamma n \leq i \leq (1-\gamma)n$. Suppose that $\theta \in [\delta, \pi-\delta]$. Then on $E_{n,i}(\delta,\gamma)$, we have $\underline{J}_n(\theta) < i < \bar{J}_n(\theta)$ and $\underline{J}_n^{(i)}(\theta) < i < \bar{J}_n^{(i)}(\theta)$. Then from (2.1) it follows that in fact $\underline{J}_n(\theta) = \underline{J}_n^{(i)}(\theta)$ and $\bar{J}_n(\theta) = \bar{J}_n^{(i)}(\theta)$. Hence $m_n(\theta) = m_n^{(i)}(\theta)$ and $M_n^{(i)}(\theta) = S^{(i)}_{\bar{J}_n(\theta)} \cdot \mathbf{e}_\theta = M_n(\theta) + (Z_i' - Z_i)\cdot \mathbf{e}_\theta$, by (2.1). Equation (4.2) follows.

Next we prove part (ii). Suppose that $\mu = \|\mathbb{E}[Z_1]\| > 0$. Since $\mathbb{E}\|Z_1\| < \infty$, the strong law of large numbers implies that $\|n^{-1}S_n - \mathbb{E}[Z_1]\| \to 0$, a.s., as $n \to \infty$. In other words, for any $\varepsilon_1 > 0$, there exists $N := N(\varepsilon_1)$ such that $\mathbb{P}[N < \infty] = 1$ and $\|n^{-1}S_n - \mathbb{E}[Z_1]\| < \varepsilon_1$ for all $n \geq N$. In particular, for $n \geq N$, by (4.1),

$$\left|n^{-1}S_n \cdot \mathbf{e}_\theta - \mu \sin\theta\right| = \left|n^{-1}S_n \cdot \mathbf{e}_\theta - \mathbb{E}[Z_1]\cdot \mathbf{e}_\theta\right| \leq \left\|n^{-1}S_n - \mathbb{E}[Z_1]\right\| < \varepsilon_1, \tag{4.3}$$

for all $\theta \in [0, 2\pi)$.

Take $\varepsilon_1 < \mu \sin\delta$. If $n \geq N$, then, by (4.3),

$$S_n \cdot \mathbf{e}_\theta > (\mu\sin\theta - \varepsilon_1)n \geq (\mu \sin\delta - \varepsilon_1)n,$$

provided $\theta \in [\delta, \pi-\delta]$. By choice of ε_1, the last term in the previous display is strictly positive. Hence, for $n \geq N$, for any $\theta \in [\delta, \pi-\delta]$, $S_n \cdot \mathbf{e}_\theta > 0$. But $S_0 \cdot \mathbf{e}_\theta = 0$. So $\underline{J}_n(\theta) < N$ for all $\theta \in [\delta, \pi-\delta]$, and

$$\mathbb{P}\left[\bigcap_{\theta\in[\delta,\pi-\delta]} \{\underline{J}_n(\theta) < \gamma n\}\right] \geq \mathbb{P}[N < \gamma n] \to 1,$$

as $n \to \infty$, since $N < \infty$ a.s.

Now,

$$\max_{0\leq j\leq(1-\gamma)n} S_j \cdot \mathbf{e}_\theta \leq \max\left\{\max_{0\leq j\leq N} S_j\cdot \mathbf{e}_\theta, \max_{N\leq j\leq (1-\gamma)n} S_j \cdot \mathbf{e}_\theta\right\}. \tag{4.4}$$

For the final term on the right-hand side of (4.4), (4.3) implies that

$$\max_{N\leq j\leq(1-\gamma)n} S_j \cdot \mathbf{e}_\theta \leq \max_{0\leq j\leq(1-\gamma)n} (\mu\sin\theta + \varepsilon_1)j \leq (\mu \sin\theta + \varepsilon_1)(1-\gamma)n.$$

On the other hand, if $n \geq N$, then (4.3) implies that $S_n \cdot \mathbf{e}_\theta \geq (\mu\sin\theta - \varepsilon_1)n$. Here $\mu\sin\theta - \varepsilon_1 \geq (\mu\sin\theta + \varepsilon_1)(1-\gamma)$ if $\varepsilon_1 < \frac{\gamma\mu\sin\theta}{2-\gamma}$. Now we choose $\varepsilon_1 < \frac{\gamma\mu\sin\delta}{2}$. Then, for any $\theta \in [\delta, \pi-\delta]$, we have that, for $n \geq N$,

$$S_n \cdot \mathbf{e}_\theta > \max_{N\leq j\leq(1-\gamma)n} S_j \cdot \mathbf{e}_\theta.$$

Hence, by (4.4),

$$\begin{aligned}\mathbb{P}\left[\bigcap_{\theta\in[\delta,\pi-\delta]} \{\bar{J}_n(\theta) > (1-\gamma)n\}\right] &\geq \mathbb{P}\left[\bigcap_{\theta\in[\delta,\pi-\delta]} \left\{S_n\cdot \mathbf{e}_\theta > \max_{0\leq j\leq(1-\gamma)n} S_j \cdot \mathbf{e}_\theta\right\}\right] \\ &\geq \mathbb{P}\left[N \leq n, \bigcap_{\theta\in[\delta,\pi-\delta]} \left\{S_n \cdot \mathbf{e}_\theta > \max_{0\leq j\leq N} S_j\cdot \mathbf{e}_\theta\right\}\right].\end{aligned}$$

Also, for $n \geq N$, $S_n \cdot \mathbf{e}_\theta > (1 - \frac{\gamma}{2})\mu n \sin\delta$, so we obtain

$$\mathbb{P}\left[\bigcap_{\theta\in[\delta,\pi-\delta]} \{\bar{J}_n(\theta) > (1-\gamma)n\}\right] \geq \mathbb{P}\left[N \leq n,\ \max_{0\leq j\leq N} \|S_j\| \leq \left(1 - \frac{\gamma}{2}\right)\mu n \sin\delta\right],$$

using the fact that $\max_{0\leq j\leq N} S_j \cdot \mathbf{e}_\theta \leq \max_{0\leq j\leq N} \|S_j\|$ for all θ.

Now, as $n \to \infty$, $\mathbb{P}[N > n] \to 0$ and

$$\mathbb{P}\left[\max_{0\leq j\leq N} \|S_j\| > \left(1 - \frac{\gamma}{2}\right)\mu n \sin\delta\right] \to 0,$$

since $N < \infty$ a.s. Thus we conclude that

$$\mathbb{P}\left[\bigcap_{\theta\in[\delta,\pi-\delta]} \{\underline{J}_n(\theta) < \gamma n,\ \bar{J}_n(\theta) > (1-\gamma)n\}\right] \to 1,$$

as $n \to \infty$, and the same result holds for $\underline{J}_n^{(i)}(\theta)$ and $\bar{J}_n^{(i)}(\theta)$, uniformly in $i \in \{1, \ldots, n\}$, since resampling Z_i does not change the distribution of the trajectory. □

5. Approximation lemma

The following result is a key component to our proof. Recall that $D_{n,i} = \mathbb{E}[L_n - L_n^{(i)} \mid \mathcal{F}_i]$.

Lemma 5.1. *Suppose that $\mathbb{E}\|Z_1\| < \infty$, $\gamma \in (0, 1/2)$, and $\delta \in (0, \pi/2)$. For any $i \in I_{n,\gamma}$,*

$$\begin{aligned}\left|D_{n,i} - \frac{2(Z_i - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1]}{\|\mathbb{E}[Z_1]\|}\right| &\leq 6\delta\|Z_i\| + 6\delta\mathbb{E}\|Z_1\| + 3\pi\|Z_i\|\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i] \\ &\quad + 3\pi\mathbb{E}[\|Z_i'\|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i], \text{ a.s.}\end{aligned} \tag{5.1}$$

Proof. Taking (conditional) expectations in (3.1), we obtain

$$D_{n,i} = \int_0^\pi \mathbb{E}[\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta + \int_0^\pi \mathbb{E}[\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta. \tag{5.2}$$

For the second term on the right-hand side of (5.2), we have

$$\left|\int_0^\pi \mathbb{E}[\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta\right| \leq \int_0^\pi \mathbb{E}[|\Delta_n^{(i)}(\theta)|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta. \tag{5.3}$$

Applying the bound (3.2), we obtain

$$\begin{aligned}\int_0^\pi \mathbb{E}[|\Delta_n^{(i)}(\theta)|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta &\leq 2\pi\mathbb{E}[(\|Z_i\| + \|Z_i'\|)\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i] \\ &= 2\pi\|Z_i\|\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i] + 2\pi\mathbb{E}[\|Z_i'\|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i],\end{aligned} \tag{5.4}$$

since Z_i is $\mathcal{F}_i$-measurable with $\mathbb{E}\|Z_i\| < \infty$.

We decompose the first integral on the right-hand side of (5.2) as $I_1 + I_2 + I_3$, where

$$I_1 := \int_0^\delta \mathbb{E}[\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta,$$
$$I_2 := \int_\delta^{\pi-\delta} \mathbb{E}[\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta,$$
$$I_3 := \int_{\pi-\delta}^{\pi} \mathbb{E}[\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta.$$

First we deal with I_1 and I_3. We have

$$|I_1| \le \int_0^\delta \mathbb{E}[|\Delta_n^{(i)}(\theta)| \mid \mathcal{F}_i]\mathrm{d}\theta \le 2\delta\mathbb{E}[\|Z_i\| + \|Z_i'\| \mid \mathcal{F}_i], \text{ a.s.},$$

by another application of (3.2). Here $\mathbb{E}[\|Z_i\| \mid \mathcal{F}_i] = \|Z_i\|$, since Z_i is $\mathcal{F}_i$-measurable, and, since Z_i' is independent of $\mathcal{F}_i$, $\mathbb{E}[\|Z_i'\| \mid \mathcal{F}_i] = \mathbb{E}\|Z_i'\| = \mathbb{E}\|Z_1\|$. A similar argument applies to I_3, so that

$$|I_1 + I_3| \le 4\delta\|Z_i\| + 4\delta\mathbb{E}\|Z_1\|, \text{ a.s.} \tag{5.5}$$

We now consider I_2. From (4.2), since $i \in I_{n,\gamma}$, we have

$$\begin{aligned} I_2 &= \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta \\ &= \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta - \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta. \end{aligned}$$

Here, by the triangle inequality,

$$\begin{aligned} &\left| \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta \right| \\ &\le \int_0^\pi \mathbb{E}[(\|Z_i\| + \|Z_i'\|)\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta \\ &= \pi\|Z_i\|\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i] + \pi\mathbb{E}[\|Z_i'\|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i], \end{aligned} \tag{5.6}$$

similarly to (5.4). Finally, similarly to (5.5),

$$\begin{aligned} &\left| \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta - \int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta \right| \\ &\le 2\delta\mathbb{E}[\|Z_i\| + \|Z_i'\| \mid \mathcal{F}_i] = 2\delta\left(\|Z_i\| + \mathbb{E}\|Z_1\|\right). \end{aligned} \tag{5.7}$$

We combine (5.2) with (5.3) and the bounds in (5.4)–(5.7) to give

$$\begin{aligned} \left| D_{n,i} - \int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta \right| &\le 6\delta\|Z_i\| + 6\delta\mathbb{E}\|Z_1\| + 3\pi\|Z_i\|\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i] \\ &\quad + 3\pi\mathbb{E}[\|Z_i'\|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i], \text{ a.s.} \end{aligned} \tag{5.8}$$

To complete the proof of the lemma, we compute the integral on the left-hand side of (5.8). First note that $\mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i] = (Z_i - \mathbb{E}[Z_i']) \cdot \mathbf{e}_\theta$, since Z_i is $\mathcal{F}_i$-measurable and Z_i' is independent of $\mathcal{F}_i$, so that

$$\int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta = \int_0^\pi (Z_i - \mathbb{E}[Z_i]) \cdot \mathbf{e}_\theta \mathrm{d}\theta.$$

To evaluate the last integral, it is convenient to introduce the notation $Z_i - \mathbb{E}[Z_i] = R_i \mathbf{e}_{\Theta_i}$ where $R_i = \|Z_i - \mathbb{E}[Z_i]\| \geq 0$ and $\Theta_i \in [0, 2\pi)$. Then

$$\int_0^\pi (Z_i - \mathbb{E}[Z_i]) \cdot \mathbf{e}_\theta \mathrm{d}\theta = \int_0^\pi R_i \mathbf{e}_{\Theta_i} \cdot \mathbf{e}_\theta \mathrm{d}\theta = R_i \int_0^\pi \cos(\theta - \Theta_i) \mathrm{d}\theta$$
$$= 2R_i \sin \Theta_i = 2R_i \mathbf{e}_{\Theta_i} \cdot \mathbf{e}_{\pi/2}.$$

Now (5.1) follows from (5.8), and the proof is complete. □

6. Completing the proofs of the theorems

For ease of notation, we write $Y_i := 2\|\mathbb{E}[Z_1]\|^{-1}(Z_i - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1]$, and define

$$W_{n,i} := D_{n,i} - Y_i.$$

The upper bound for $|W_{n,i}|$ in Lemma 5.1 together with Lemma 4.1(ii) will enable us to prove the following result, which will be the basis of our proof of Theorem 1.3.

Lemma 6.1. *Suppose that* $\mathbb{E}[\|Z_1\|^2] < \infty$ *and* $\|\mathbb{E}[Z_1]\| \neq 0$. *Then*

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n \mathbb{E}[W_{n,i}^2] = 0.$$

Proof. Fix $\varepsilon > 0$. We take $\gamma \in (0, 1/2)$ and $\delta \in (0, \pi/2)$, to be specified later. We divide the sum of interest into two parts, namely $i \in I_{n,\gamma}$ and $i \notin I_{n,\gamma}$. Now from (3.1) with (3.2) we have $|L_n^{(i)} - L_n| \leq 2\pi(\|Z_i\| + \|Z_i'\|)$, a.s., so that

$$|D_{n,i}| \leq 2\pi\mathbb{E}[\|Z_i\| + \|Z_i'\| \mid \mathcal{F}_i] = 2\pi(\|Z_i\| + \mathbb{E}\|Z_i\|).$$

It then follows from the triangle inequality that

$$|W_{n,i}| \leq |D_{n,i}| + 2\|Z_i - \mathbb{E}[Z_i]\| \leq (2\pi + 2)(\|Z_i\| + \mathbb{E}\|Z_i\|).$$

So provided $\mathbb{E}[\|Z_1\|^2] < \infty$, we have $\mathbb{E}[W_{n,i}^2] \leq C_0$ for all n and all i, for some constant $C_0 < \infty$, depending only on the distribution of Z_1. Hence

$$\frac{1}{n} \sum_{i \notin I_{n,\gamma}} \mathbb{E}[W_{n,i}^2] \leq \frac{1}{n} 2\gamma n C_0 = 2\gamma C_0,$$

using the fact that there are at most $2\gamma n$ terms in the sum. From now on, choose $\gamma > 0$ small enough so that $2\gamma C_0 < \varepsilon$.

Now consider $i \in I_{n,\gamma}$. For such i, (5.1) shows that, for some constant $C_1 < \infty$,

$$|W_{n,i}| \leq C_1(1 + \|Z_i\|)\delta + C_1\|Z_i\|\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i] + C_1\mathbb{E}[\|Z_i'\|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i], \text{ a.s.} \tag{6.1}$$

Here, for any $B_1 \in (0, \infty)$, a.s.,

$$\mathbb{E}[\|Z_i'\|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i] \leq \mathbb{E}[\|Z_i'\|\mathbf{1}\{\|Z_i'\| > B_1\} \mid \mathcal{F}_i] + B_1\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i]$$
$$= \mathbb{E}[\|Z_i'\|\mathbf{1}\{\|Z_i'\| > B_1\}] + B_1\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i],$$

since Z_i' is independent of $\mathcal{F}_i$. Here, since $\mathbb{E}\|Z_i'\| = \mathbb{E}\|Z_1\| < \infty$, the dominated convergence theorem implies that $\mathbb{E}[\|Z_i'\|\mathbf{1}\{\|Z_i'\| > B_1\}] \to 0$ as $B_1 \to \infty$. So we can choose $B_1 = B_1(\delta)$ large enough so that

$$\mathbb{E}[\|Z_i'\|\mathbf{1}(E_{n,i}^{\mathrm{c}}(\delta,\gamma)) \mid \mathcal{F}_i] \leq \delta + B_1\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i], \text{ a.s.}$$

Combining this with (6.1) we see that there is a constant $C_2 < \infty$ for which

$$|W_{n,i}| \leq C_2(1 + \|Z_i\|)\left(\delta + B_1\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i]\right), \text{ a.s.}$$

Hence

$$\begin{aligned} W_{n,i}^2 &\leq C_2^2(1+\|Z_i\|)^2\left(\delta^2+2B_1\delta\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)\mid\mathcal{F}_i]+B_1^2\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)\mid\mathcal{F}_i]^2\right)\\ &\leq C_3^2(1+\|Z_i\|)^2\left(\delta+B_1^2\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)\mid\mathcal{F}_i]\right),\end{aligned}$$

for some constant $C_3 < \infty$, using the facts that $\delta < \pi/2 < 2$ and $\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma) \mid \mathcal{F}_i] \leq 1$. Taking expectations we get

$$\mathbb{E}[W_{n,i}^2] \leq C_3^2\delta\mathbb{E}[(1+\|Z_i\|)^2] + C_3^2B_1^2\mathbb{E}\left[(1+\|Z_i\|)^2\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)\mid\mathcal{F}_i]\right].$$

Provided $\mathbb{E}[\|Z_1\|^2] < \infty$, there is a constant $C_4 < \infty$ such that the first term on the right-hand side of the last display is bounded by $C_4\delta$. Now fix $\delta > 0$ small enough so that $C_4\delta < \varepsilon$; this choice also fixes B_1. Then

$$\mathbb{E}[W_{n,i}^2] \leq \varepsilon + C_3^2B_1^2\mathbb{E}\left[(1+\|Z_i\|)^2\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)\mid\mathcal{F}_i]\right]. \tag{6.2}$$

For the final term in (6.2), observe that, for any $B_2 \in (0,\infty)$, a.s.,

$$\begin{aligned}(1+\|Z_i\|)^2\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)\mid\mathcal{F}_i] \leq{}& (1+B_2)^2\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)\mid\mathcal{F}_i]\\ &+(1+\|Z_i\|)^2\mathbf{1}\{\|Z_i\|>B_2\}.\end{aligned} \tag{6.3}$$

Here $\mathbb{E}[(1+\|Z_i\|)^2\mathbf{1}\{\|Z_i\|>B_2\}] \to 0$ as $B_2 \to \infty$, provided $\mathbb{E}[\|Z_1\|^2] < \infty$, by the dominated convergence theorem. Hence, since δ and B_1 are fixed, we can choose $B_2 = B_2(\varepsilon) \in (0,\infty)$ such that $C_3^2B_1^2\mathbb{E}[(1+\|Z_i\|)^2\mathbf{1}\{\|Z_i\|>B_2\}] < \varepsilon$. Then taking expectations in (6.3) we obtain from (6.2) that

$$\mathbb{E}[W_{n,i}^2] \leq 2\varepsilon + C_3^2B_1^2(1+B_2)^2\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)].$$

Now choose n_0 such that $C_3^2B_1^2(1+B_2)^2\mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)] < \varepsilon$ for all $n \geq n_0$, which we may do by Lemma 4.1(ii). So for the given $\varepsilon > 0$ and $\gamma \in (0,1/2)$, we can choose n_0 such that for all $i \in I_{n,\gamma}$ and all $n \geq n_0$, $\mathbb{E}[W_{n,i}^2] \leq 3\varepsilon$. Hence

$$\frac{1}{n}\sum_{i\in I_{n,\gamma}}\mathbb{E}[W_{n,i}^2] \leq 3\varepsilon,$$

for all $n \geq n_0$.

Combining the estimates for $i \in I_{n,\gamma}$ and $i \notin I_{n,\gamma}$, we see that

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[W_{n,i}^2] \leq 2\gamma C_0 + 3\varepsilon \leq 4\varepsilon,$$

for all $n \geq n_0$. Since $\varepsilon > 0$ was arbitrary, the result follows. □

Now we can complete the proofs of our main theorems.

Proof of Theorem 1.3. First note that

$$\mathbb{E}[W_{n,i}\mid\mathcal{F}_{i-1}] = \mathbb{E}[D_{n,i}\mid\mathcal{F}_{i-1}] - \mathbb{E}[Y_i\mid\mathcal{F}_{i-1}] = 0 - \mathbb{E}[Y_i],$$

since $D_{n,i}$ is a martingale difference sequence and Y_i is independent of $\mathcal{F}_{i-1}$. Here, by definition, $\mathbb{E}[Y_i] = 0$, and so $W_{n,i}$ is also a martingale difference sequence. Therefore, by orthogonality, $n^{-1}\mathbb{E}[(\sum_{i=1}^n W_{n,i})^2] = n^{-1}\sum_{i=1}^n \mathbb{E}[W_{n,i}^2] \to 0$ as $n \to \infty$, by Lemma 6.1. In other words, $n^{-1/2}\sum_{i=1}^n W_{n,i} \to 0$ in L^2, which, with Lemma 2.1(i), implies the statement in the theorem. □

Proof of Theorem 1.1. Write

$$\xi_n = \frac{L_n - \mathbb{E}[L_n]}{\sqrt{n}}; \text{ and } \zeta_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i, \text{ where } Y_i = \frac{2(Z_i - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1]}{\|\mathbb{E}[Z_1]\|}. \tag{6.4}$$

Then Theorem 1.3 shows that $|\xi_n - \zeta_n| \to 0$ in L^2 as $n \to \infty$. Also, with σ^2 as given by (1.2), $\mathbb{E}[\zeta_n^2] = \sigma^2$. Then a computation shows that

$$n^{-1}\mathbb{V}\text{ar}[L_n] = \mathbb{E}[\xi_n^2] = \mathbb{E}[(\xi_n - \zeta_n)^2] + \mathbb{E}[\zeta_n^2] + 2\mathbb{E}[(\xi_n - \zeta_n)\zeta_n].$$

Here, by the L^2 convergence, $\mathbb{E}[(\xi_n - \zeta_n)^2] \to 0$ and, by the Cauchy–Schwarz inequality, $|\mathbb{E}[(\xi_n - \zeta_n)\zeta_n]| \leq \left(\mathbb{E}[(\xi_n - \zeta_n)^2]\mathbb{E}[\zeta_n^2]\right)^{1/2} \to 0$ as well. Thus $\mathbb{E}[\xi_n^2] \to \sigma^2$ as $n \to \infty$. □

In the proof of Theorem 1.2 we will use two facts about convergence in distribution that we now recall (see e.g. [2, p. 73]). First, if sequences of random variables ξ_n and ζ_n are such that $\zeta_n \to \zeta$ in distribution for some random variable ζ and $|\xi_n - \zeta_n| \to 0$ in probability, then $\xi_n \to \zeta$ in distribution (this is *Slutsky's theorem*). Second, if $\zeta_n \to \zeta$ in distribution and $\alpha_n \to \alpha$ in probability, then $\alpha_n\zeta_n \to \alpha\zeta$ in distribution.

Proof of Theorem 1.2. Suppose σ^2 as given by (1.2) satisfies $\sigma^2 > 0$. Again use the notation for ξ_n and ζ_n as given by (6.4). Then, by Theorem 1.3, $|\xi_n - \zeta_n| \to 0$ in L^2, and hence in probability.

In the sum ζ_n, the Y_i are i.i.d. random variables with mean 0 and variance $\mathbb{E}[Y_i^2] = \sigma^2$. Hence the classical central limit theorem (see e.g. [2, p. 93]) shows that ζ_n converges in distribution to a normal random variable with mean 0 and variance σ^2. Slutsky's theorem then implies that ξ_n has the same distributional limit. Hence, for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\xi_n}{\sqrt{\sigma^2}} \leq x\right] = \lim_{n\to\infty} \mathbb{P}\left[\frac{L_n - \mathbb{E}[L_n]}{\sqrt{\sigma^2 n}} \leq x\right] = \Phi(x),$$

where Φ is the standard normal distribution function. Moreover,

$$\mathbb{P}\left[\frac{L_n - \mathbb{E}[L_n]}{\sqrt{\mathbb{V}\text{ar}[L_n]}} \leq x\right] = \mathbb{P}\left[\frac{\xi_n \alpha_n}{\sqrt{\sigma^2}} \leq x\right],$$

where $\alpha_n = \sqrt{\frac{\sigma^2 n}{\mathbb{V}\text{ar}[L_n]}} \to 1$ by Theorem 1.1. Thus we verify the limit statements in (1.3). □

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